1 Big Sums

1.1 Recap

1. Let n be a positive integer. Let $x_1, x_2, ..., x_n$ be a sequence of n real numbers. Say that a sequence $a_1, a_2, ..., a_n$ is *unimodular* if each a_i is ± 1. Prove that

$$\sum a_1 a_2 ... a_n (a_1 x_1 + a_2 x_2 + ... + a_n x_n)^n = 2^n n! x_1 x_2 ... x_n$$

where the sum is over all 2^n unimodular sequences $a_1, a_2, ..., a_n$.

2. (a) Show that if

$$\sum_{(x_1, x_2, ..., x_n) \in \mathbb{F}_p^n} x_1^{a_1} x_2^{a_2} ... x_n^{a_n} \not\equiv 0 \pmod p$$

then $\min\{a_i\} \geq p - 1$. (We can pretend $\mathbb{F}_p = \{0, 1, 2, ..., p - 1\}$.)

(b) Show that for any d, n, there exists integers $x_1, x_2, ..., x_d$ such that $x_1^d + x_2^d + ... + x_d^d - n$ is a multiple of p.

1.2 Problems

1. (ELMO 2014) Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be real numbers strictly between 0 and 1. For any subset S of $\{1, 2, ..., n\}$, define

$$f(S) = \prod_{i \in S} a_i \cdot \prod_{j \notin S} (1 - a_j).$$

Suppose that $\sum_{|S| \text{ odd}} f(S) = \frac{1}{2}$. Prove that $a_k = \frac{1}{2}$ for some k. (Here the sum ranges over all subsets of $\{1, 2, ..., n\}$ with an odd number of elements.)

2. (a) Given a set of $n \geq 3$ points in general position (i.e. no three collinear), a subset K of S is called *good* if the points in K are the vertices of a convex polygon with no other points of S in its interior. For each $3 \leq k \leq n$, let c_k be the number of good subsets of S of size k. Prove that the sum

$$\sum_{k=3}^{n} (-1)^k c_k$$

is dependent only on n and not on S.

(b) (ISL 2006) Let S be a finite set of points in the plane such that no three of them are on a line. For each convex polygon P whose vertices are in S, let $a(P)$ be the number of vertices of P, and let $b(P)$ be the number of points of S which are outside P. A line segment, a point, and the empty set are considered as convex polygons of 2, 1, and 0 vertices respectively. Prove that for every real number x

$$\sum_{P} x^{a(P)} (1 - x)^{b(P)} = 1,$$

where the sum is taken over all convex polygons with vertices in S.

3. (239 Olympiad 2008) $x_1, x_2, ..., x_n$ are natural numbers such that the sums of any subset of them are distinct. Prove that

$$x_1^2 + x_2^2 + ... + x_n^2 \geq \frac{4^n - 1}{3}$$

4. (RMM 2012) Given a finite number of boys and girls, a sociable set of boys is a set of boys such that every girl knows at least one boy in that set; and a sociable set of girls is a set of girls such that every boy knows at least one girl in that set. Prove that the number of sociable sets of boys and the number of sociable sets of girls have the same parity. (Acquaintance is assumed to be mutual.)

5. Given some natural number $m > 1$, let $\omega = e^{\frac{2\pi i}{m}}$. Suppose f is a polynomial with complex coefficients such that

$$x^2 \mid f(x) - \omega x$$

Show that

$$x^{m+1} \mid f^m(x) - x$$

6. (239 Olympiad 2016) A finite family of finite sets $\mathcal{F}$ satisfies the following: (i) if $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$. (ii) if $A \in \mathcal{F}$, then $|A|$ is not divisible by 3. Prove that we can specify 2 elements such that each set contains at least one of them.

7. (239 Olympiad 2017) In a circle, n triangles are inscribed such that all $3n$ vertices are distinct. Prove that we may put, in each triangle, a boy and a girl such that along the circumference are alternating boys and girls.

8. (HMIC 2017) Let S be the set $\{-1, 1\}^n$, that is, n-tuples such that each coordinate is either -1 or 1. For

$$s = (s_1, s_2, \ldots, s_n), t = (t_1, t_2, \ldots, t_n) \in \{-1, 1\}^n$$

define $s \odot t = (s_1 t_1, s_2 t_2, \ldots, s_n t_n)$.

Let c be a positive constant and let $f : S \to \{-1, 1\}$ be a function such that there are at least $(1 - c) \cdot 2^{2n}$ pairs (s, t) with $s, t \in S$ such that $f(s \odot t) = f(s)f(t)$. Show that there exists a function f' such that $f'(s \odot t) = f'(s)f'(t)$ for all $s, t \in S$ and $f(s) = f'(s)$ for at least $(1 - 10c) \cdot 2^n$ values of $s \in S$.