1 Isogonal Conjugates: The lowdown

1.1 Where we're going

In this article, we're going to develop a theory of isogonal conjugation. As a reminder, isogonal conjugates are formally defined as follows:

Definition 1

With respect to a polygon $A_1...A_k$ (where $A_{k+1} = A_1$), the **isogonal conjugate** of a point P is another point Q satisfying

$$\angle(A_iA_{i+1}, PA_{i+1}) = \angle(QA_{i+1}, A_{i+1}A_{i+2}) \qquad \text{for each } i = 1, 2, ..., k$$

One way to remember is that viewed from every vertex, a pair of isogonal conjugates should be the same angle away from each side. It's fairly obvious that isogonal conjugation is **involutive** (i.e the isogonal conjugate of the isogonal conjugate is the original point).

In particular, we're most interested here with isogonal conjugates within a *quadrilateral*[1]. Unlike the case for triangles, an isogonal conjugate does not always exist, and there is a neat criterion for when it does:

Fact 1 (Isogonal Criterion for Quadrilaterals)

Point P inside quadrilateral $ABCD$ has an isogonal conjugate if and only if $\angle APB + \angle CPD = 180°$. More generally, we require $\angle(AP, BP) = \angle(DP, CP)$.

Because points must satisfy this property, there aren't as many "well-known" pairs of isogonal conjugates. For a quadrilateral $ABCD$, we do have:

- Opposite vertices A and C (or B and D) are isogonal conjugates
- Intersections $AD \cap BC$ and $AB \cap CD$
- If an incenter exists, it would be self-conjugated
- The Miquel point has an isogonal conjugate (where?)

If we think of isogonal conjugation as a function f, then this function can only make sense on the locus

$$\mathcal{C} = \mathcal{C}(ABCD) := \{P : \angle(AP, BP) = \angle(DP, CP)\}$$

where it has an isogonal conjugate, and we also expect the output to be on $\mathcal{C}$. Furthermore, applying f twice we should recover the original point.

The surprising thing is that any two values uniquely determine the entire isogonal conjugation map $f : \mathcal{C} \to \mathcal{C}$! This is explicitly worded as the following fact:

Fact 2 (Main Fact)

Let $\mathcal{C} = \mathcal{C}(ABCD)$, and let $f : \mathcal{C} \to \mathcal{C}$ be the isogonal conjugation with respect to the quadrilateral $ABCD$. (That is, f sends X to its isogonal conjugate $f(X)$ in $ABCD$.)

Suppose $B' = f(A')$ and $D' = f(C')$. Then f is the isogonal conjugation map on $A'B'C'D'$. (We will write this as $f_{ABCD} = f_{A'B'C'D'}$.)

Stated another way: there is a single isogonal conjugation swapping $A \leftrightarrow C, B \leftrightarrow D, A' \leftrightarrow C', B' \leftrightarrow D'$, which will also swap any pair of isogonal conjugates on both $ABCD$ and $A'B'C'D'$.

[1] Why not for larger polygons? A later problem asks you to show that pentagons generally only have a unique pair of isogonal conjugates...

This might feel abstract and confusing at first. But here's an example application to really see how it plays out in action:

Example 3 (Yufei Zhao's notes)

Let $ABCD$ be a convex quadrilateral (not necessarily cyclic). Let AB, CD intersect at E and AD, BC intersect at F. Suppose X is a point inside the quadrilateral such that $\angle AXE = \angle CXF$. Prove that $\angle AXB + \angle CXD = 180°$.

Solution. The quadrilateral $AECF$ has an isogonal conjugation map $f : \mathcal{C}(AECF) \to \mathcal{C}(AECF)$. This map sends X somewhere but also swaps B and D, so $f_{AECF} = f_{ABCD}$. Therefore, X has isogonal conjugate $f(X)$ with respect to $ABCD$, and thus it satisfies that angle condition.

Here's an even more mind-boggling example:

Example 4 (2012 Training)

Let $ABCD$ be a quadrilateral with an inscribed circle centered at I. Suppose there are points P, Q such that quadrilateral $APCQ$ has an inscribed circle centered at I. Prove that quadrilateral $BPDQ$ has an inscribed circle centered at I.

Solution. We know how this starts: take the isogonal conjugation map of $ABCD$: this swaps $A \leftrightarrow C, B \leftrightarrow D, I \leftrightarrow I$, so $f_{ABCD} = f_{AICI}$. (We're using I twice?) However, f_{APCQ} also maps $A \leftrightarrow C$ and $I \leftrightarrow I$, so $f_{AICI} = f_{APCQ}$. Using the same logic again, $f_{APCQ} = f_{PIQI}$ but because this swaps $B \leftrightarrow D$ we conclude that $BPDQ$ has incenter I. (As an added bonus, we also have P, Q are isogonal conjugates!)

In the sections below, we build up to the main fact above (that allows us to do these crazy short but powerful proofs). I suggest skipping directly to the problems (or going through just subsection 2.1) if it's your first time seeing things of this flavor.

2 The main story

2.1 Isogonal Conjugates refresher

You might already know

Fact 5
In a triangle, every point has an isogonal conjugate. (Try to prove this without trig Ceva!)

However, life is not so easy for larger polygons. Even in a quadrilateral, not every point has an isogonal conjugate (construct one yourself). There is, however, a nice criteria:

Fact 6 (Isogonal Criteria for Quadrilaterals)
Point P inside quadrilateral $ABCD$ has an isogonal conjugate if and only if $\angle APB+\angle CPD = 180^\circ$. More generally, we require $\angle(AP, BP) = \angle(DP, CP)$.

Here's a sketch of the proof (work through it if you've never seen it before):

1. In the $\Rightarrow$ direction:
   (a) $\angle APB + \angle CPD = 180^\circ$ implies that P has a pedal circle (i.e. the projections from P to each side are concyclic).
   (b) There exists a point Q whose projections to each side are the second intersection of P's pedal circle.
   (c) For two adjacent sides, if the projections of P, Q onto each side are concyclic, then P, Q are isogonal at this angle. Thus P, Q are isogonal conjugates.
2. In the $\Leftarrow$ direction:
   (a) For two adjacent sides, if P, Q are isogonal at this angle then their projections to these two sides are concyclic.
   (b) Show that P, Q share the same pedal circle. [2]
   (c) Given that P has a pedal circle, conclude that $\angle APB + \angle CPD = 180^\circ$.

From this we also see that

Fact. In *any* polygon, P has an isogonal conjugate if and only if P has a pedal circle.

One possibly useful interpretation is the following:

Fact. If P, Q are isogonal conjugates in a convex polygon, then they are the foci of an ellipse inscribed in the polygon.

Sketch. It suffices to show that the reflections of Q about each side $(Q_1, ..., Q_n)$ are equidistant from P (call this distance d), and this is easiest argued "at each angle". This implies that for X along the polygon, $PX + QX$ has minimal value d and this happens precisely at the intersection of PQ_i with the i-th side.

Here's a problem that blatantly references this:

[2] This is nontrivial. In particular, given circles $(AA'BB'), (BB'CC'), (CC'AA')$, it is generally false that A, A', B, B', C, C' are concyclic.

Example 7 (ISL 2000 G3)
Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that

$$OD + DH = OE + EH = OF + FH$$

and the lines AD, BE, and CF are concurrent.

Sketch. Let D, E, F be the tangency points for the inscribed ellipse with foci O, H. Then by denegerate Brianchon $AFBDCE$ around this ellipse, AD, BE, CF are concurrent.

2.2 Coharmonic hexagons

The story begins with a seemingly unrelated object. The motivation is two fold:

Coharmonic Ranges. The theory of projective involutions[3] tells us that there is an involution swapping $A \leftrightarrow D, B \leftrightarrow E, C \leftrightarrow F$ if and only if

$$(A, D; B, E; C, F) := (A, C; B, E) \cdot (C, E; D, A) \cdot (E, A; F, C) = -1$$

This is called the **generalized cross-ratio**, and it is also preserved under projective transformations, since each cross-ratio is preserved. Such A, B, C, D, E, F is called a **coharmonic range**.

Harmonic Quadrilaterals. Put the usual coordinates on a line, letting a be the coordinate for point A and so on. Then A, B, C, D harmonic is equivalent to

$$\frac{a-b}{b-c} \Bigg/ \frac{a-d}{d-c} = -1$$

Rather interestingly, interpreting this equation in $\mathbb{C}$ gives us harmonic quadrilaterals. (What are the projective involutions in $\mathbb{C}$?)

The natural thing to do now is to put both things together:

Definition 2
Hexagon $ABCDEF$ is a **coharmonic hexagon** if

$$(A, D; B, E; C, F)_{\mathbb{C}} = \frac{a-b}{b-c} \cdot \frac{c-d}{d-e} \cdot \frac{e-f}{f-a} = -1$$

where a, b, c, d, e, f are the complex numbers corresponding to A, B, C, D, E, F respectively.

Geometrically, for a convex hexagon, this is equivalent to these two conditions combined:

- $\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1$
- $\angle ABC + \angle CDE + \angle EFA = 360^\circ$

In analogy with the situation of ranges on a line, this condition is in fact equivalent to the existence of an involution swapping $A \leftrightarrow D, B \leftrightarrow E, C \leftrightarrow F$. More concretely:

Fact 8
Hexagon $ABCDEF$ is coharmonic if and only if there is an inverflection swapping $A \leftrightarrow D, B \leftrightarrow E, C \leftrightarrow F$.

Here's another blatant problem you can demolish with this fact:

[3] f is an projective involution if it preserves cross-ratio and satisfy $f \circ f = \mathrm{id}$

Example 9 (ISL 1998 G6)
Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360^\circ$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

Prove that

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1.$$

Sketch. By fact 8, $ABCDEF$ coharmonic is equivalent to $DBCAEF$ coharmonic, so we are done.

2.3 Examples of Coharmonic Hexagons

1. On a circle: involutions are perspectivities (why?), so a coharmonic hexagon on a circle has concurrent diagonals. (Check this manually!) This is also true for general conics.
2. A *complete quadrilateral* is made up of the 6 pairwise intersections between 4 lines. Note that the "opposite points" (points that don't lie on the same line) form an involution: in particular, the Miquel point is the center of inverflection
3. You can invert the above configuration! I'll put this as a separate fact below.
4. ... any more? A lot more, actually.

Fact 10 (Inverted Miquel)
In a quadrilateral $ABCD$ and a point X, define

$$X_1 = (ABX) \cap (CDX), \qquad X_2 = (ADX) \cap (BCX)$$

then $ABXCDX_2$ is coharmonic.

2.4 Isogonal hexagons

I actually left out the most important example of a coharmonic hexagon. This is important enough that it gets its own fact:

Fact 11
If P and Q are isogonal conjugates in $ABCD$, then $ABPCDQ$ is coharmonic.

As you might expect, this isn't true in reverse (since for any P there is some Q that makes $ABPCDQ$ coharmonic). There is however a nifty criterion for the reverse:

Fact 12
If $ABPCDQ$ is coharmonic and the midpoints of AC, BD, PQ are collinear, then P, Q are isogonal conjugates in $ABCD$.

Then it makes sense to define:

Definition 3
Hexagon $ABCDEF$ is an **isogonal hexagon** if it is coharmonic and the midpoint of its diagonals are collinear. In particular, this implies that any pair of opposite vertices are isogonal conjugates with respect to the other four vertices.

The key takeaway is that

Isogonal conjugation is just inverflection restricted to the locus (of points with an isogonal conjugate).

2.5 Finishing up the main fact

Now we do have enough firepower to prove this:

Fact (Main Fact)
Let $\mathcal{C} = \mathcal{C}(ABCD)$, and let $f : \mathcal{C} \to \mathcal{C}$ be the isogonal conjugation with respect to the quadrilateral $ABCD$. (That is, f sends X to its isogonal conjugate $f(X)$ in $ABCD$.)

Suppose $B' = f(A')$ and $D' = f(C')$. Then f is the isogonal conjugation map on $A'B'C'D'$. (We will write this as $f_{ABCD} = f_{A'B'C'D'}$.)

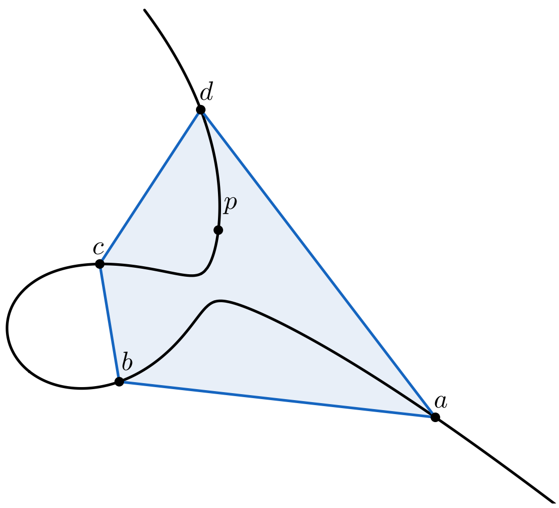
Stated another way: there is a single isogonal conjugation swapping $A \leftrightarrow C, B \leftrightarrow D, A' \leftrightarrow C', B' \leftrightarrow D'$, which will also swap any pair of isogonal conjugates on both $ABCD$ and $A'B'C'D'$.

Sketch. Let X be any point with an isogonal conjugate in $ABCD$ (concisely written as $X \in \mathcal{C}(ABCD)$), and let $Y = f_{ABCD}(X)$. The inverflection $\mathcal{I}$ swapping $A \leftrightarrow C, B \leftrightarrow D$ also swaps $X \leftrightarrow Y, A' \leftrightarrow C', B' \leftrightarrow D'$, so the inverflection $\mathcal{I}'$ swapping $A' \leftrightarrow C', B' \leftrightarrow D'$ also swaps $X \leftrightarrow Y$.

It remains to check that $A'B'XC'D'Y$ is isogonal, but the midpoints of $A'C', B'D', XY$ all lie on the line connecting the midpoints of AC and BD, so this follows from fact 12.

2.6 Loci

What *exactly* is the locus of points that have an isogonal conjugate in a quadrilateral? It's this funky animal:



By grinding the condition $\angle(AP, PB) = \angle(DP, PC)$ with complex numbers, we get that the loci of p satisfies $P(p, \bar{p}) = 0$ where P is a polynomial of degree 3. This means that in usual (x, y) coordinates it should also be the zero locus of a degree 3 polynomial in x and y.[4]

There isn't much else to say about this curve, except that it is invariant under some inverflection. (The center of inverflection lies on this curve too!)

For bored people. This curve also passes through the circular points at infinity. Why?

[4] Compare this with conics, which are the zero locus of a degree 2 polynomial. Here when we say degree, we refer the total degree.

2.7 Perspective Involutions

There is a slightly grander picture of things that this is part of. First, let's look at this theorem:

Fact 13 (Dual Desargues Involution Theorem)
Let $ABCDEF$ be a complete quadrilateral (i.e. ACE, ADF, BCF, BDE are lines). Then for any point P, $(PA, PB; PC, PD; PE; PF) = -1$.

(In other words, from any perspective on the plane, a complete quadrilateral is an involution.)

Sketch. Trig Menelaus, or projective chase.

We can ask the reverse question: for which hexagons $ABCDEF$ is this true? This feels like a really exciting question to ask, but the answer is disappointing: the only such hexagons are lines and complete quadrilaterals.

However, we can start asking what the locus of points P for which hexagons look like an involution, and we get a much larger class of answers:

1. For coharmonic hexagons on a circle (i.e. concurrent diagonals), every other point on the circle is in this locus. (Is the locus precisely the circle?) Of course, what's projectively true for circles is also true for conics.
2. For our isogonal cubic locus, the involution looks like a reflection from every point on the (isogonal) locus! (WHY?) [5]
3. ...what else?

2.8 Grand Finale

What's the point if we don't use this to crack open some IMO problems?

Example 14 (IMO 2018 Q6)
A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180°$.

Solution. Of course, we immediately recognize the condition as showing that X has an isogonal conjugate. The hard part is figuring out where the length condition comes in.

The one synthetic observation you need is that $X_1 = (XAB) \cap (XCD)$ lies on AC, and similarly $X_2 = (XBC) \cap (XDA)$ lies on BD. If you remember fact 10 earlier, ABX_1CDX_2 are coharmonic, and further more we easily see that

$$\angle BXA + \angle DXC = 180° \quad \Leftrightarrow \quad \angle BX_1A + \angle DX_1C = 180°$$

so we really just need to show that X_1 and X_2 are isogonal conjugates (and it *has* to be true, because it's equivalent to the problem).

This is really easy, because it suffices to show that BX_1, BD are isogonal at $\angle B$ and DX_1, BD are isogonal at $\angle D$. Another way to say is that we need the isogonals to BD at $\angle B$ and $\angle D$ to meet on AC. This is fairly simple to do (and where the ratio condition is finally used).

[5] In fact, I have a sneaking suspicion that if it does look like a reflection along a locus, then it must pass through the circular points.

2.9 Problems

0. Let's say you were doing a geometry problem and you were using Geogebra (naughty!); what would be the fastest way to check if six points on a line formed an involution?

1. Prove facts 8, 10, 11, 12. (Hint: with suitable translation, rotation and scaling, all inverflections look like $z \to 1/z$.)

2. Given a convex quadrilateral $ABCD$, prove that there exists a point P inside the quadrilateral such that

$$\angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA = \angle PDA + \angle PCB = 90^\circ$$

   if and only if $AC \perp BD$.

3. Show that every convex pentagon has at most one pair of isogonal conjugates.

4. In triangle ABC, let P and Q be two interior points such that $\angle ABP = \angle QBC$ and $\angle ACP = \angle QCB$. Point D lies on segment BC. Prove that $\angle APB + \angle DPC = 180^\circ$ if and only if $\angle AQC + \angle DQB = 180^\circ$.

5. In a cyclic quadrilateral $ABCD$, the diagonals meet at point E, the midpoint of side AB is F, and the feet of perpendiculars from E to the lines DA, AB and BC are P, Q and R, respectively. Prove that the points P, Q, R and F are concyclic.

6. Given a convex quadrilateral $ABCD$, let I_A, I_B, I_C and I_D denote the centers of $\omega_A, \omega_B, \omega_C$and ω_D, which are the incircles of triangles DAB, ABC, BCD and CDA respectively. If $\angle BI_AA + \angle I_CI_AI_D = 180^\circ$, prove that $\angle BI_BA + \angle I_CI_BI_D = 180^\circ$.

7. In triangle ABC, D and E are points on sides AB and AC respectively such that $BE \perp CD$. Let X be a point inside the triangle such that $\angle XBC = \angle EBA$ and $\angle XCB = \angle DCA$. Show that $\angle EXD = 90^\circ - \angle A$.

8. Let P and Q be a pair of isogonal conjugates in triangle ABC and let I be the incenter of the triangle. Consider the tangents from the points P and Q to the circle centered at I and tangent to lines AP, AQ. Prove that these tangents meet on BC.

9. Suppose that a point P inside convex hexagon $A_1A_2A_3A_4A_5A_6$ has an isogonal conjugate. Show that $\angle A_1PA_2 + \angle A_3PA_4 + \angle A_5PA_6 = 180^\circ$.

10. A hexagon $ABCDEF$ is inscribed in a circle satisfies $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$. Let the reflections of B, D, F about AC, CE, EA be B_1, D_1, F_1 respectively. Prove that $\triangle BDF \sim \triangle B_1D_1F_1$.

11. Given a triangle $ABCD$ with incenter I and Miquel point M, show that $MI^2 = MA \cdot MC$.

12. In triangle ABC, let D, E be isogonal conjugates and F, G be isogonal conjugates.
    (a) Show that $X = DF \cap EG$ and $Y = DG \cap EF$ are isogonal conjugates.
    (b) Show that the Miquel point of lines DF, DG, EF, EG lies on the circumcircle of ABC.

13. Given a $\triangle ABC$ with distinct pairs of isogonal conjugates $(P, Q), (P_1, Q_1), (P_2, Q_2)$, (P_3, Q_3) such that

$$\overrightarrow{PQ} = \overrightarrow{P_1Q_1} = \overrightarrow{P_2Q_2} = \overrightarrow{P_3Q_3}$$

   as *vectors*. Prove that P is the orthocenter of $\triangle P_1P_2P_3$.