this notebook

Linear Recurrences

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$\{X_n\}$: $\boxed{X_{n+k} = a_{k-1} X_{n+k-1} + \cdots + a_0 X_n}$

① Linearity (of sequences)

$$X_{n+k} = a_{k-1} X_{n+k-1} + \cdots + a_0 X_n \quad \forall$$

$$y_{n+k} = a_{k-1} y_{n+k-1} + \cdots + a_0 y_n \quad \forall$$

$$(X_{n+k} + y_{n+k}) = a_{k-1} (X_{n+k-1} + y_{n+k-1}) + \cdots + a_0 (X_n + y_n)$$

$\{X_n\}, \{y_n\}$ satisfy $\boxed{}$, so will the sum $\{X_n$

$\{\lambda x_n\}$

② $k=1$. $X_{n+1} = r x_n$. Sol: $x_n = A \cdot r^n$

When is this a solution for $\boxed{}$?

$$r^k = a_{k-1} r^{k-1} + \cdots + a_0$$

(characteristic
(LHS - RHS
poly - in
"characteristic

$$X_n = A_1 r_1^n + \cdots + A_m r_m^m$$

are solutions!

(3) Elementary way to solve this.

E.g. $x_{n+2} = 5x_{n+1} - 6x_n$. $\Big[\ r^2 - 5r + 6 =$

$r = 2$ is a roo

$$\underbrace{x_{n+2} - 3x_{n+1}}_{y_{n+1}} = \underbrace{2x_{n+1} - 6x_n}_{2y_n}$$

$\times(1-3x)$?

$y_{n+1} = 2y_n \qquad y_n = A \cdot 2^n$

$$\begin{array}{rl} & x_{n+2} - 3x_{n+1} = A\cdot 2^n \\ - & x'_{n+2} - 3x'_{n+1} = A\cdot 2^n \\ \hline & z_{n+2} - 3z_{n+1} = 0 \end{array}$$

$x'_n = (-\frac{1}{2}) A \cdot 2$

$z_n = x_n - x'_n$

$= B \cdot 3^n$

Think about

$x_{n+2} = 4x_{n+1} - 4x_n$, $\quad (r-2)^2 =$

$x_n = (An+B$

(4) Finite differences *.

char Eq: $(r-1)^k$

$\Leftrightarrow \Delta^k \{x_n\} = \{0\}$

$\Leftrightarrow x_n$ is a deg $k-1$ poly

$\{x_n\} \xrightarrow{\Delta} \{x_n - x$

deg k poly (in n) iff deg (k- poly

$k \geq 1$

$\{x_n\}$ 1, 4, 9, 16, 25, 36

$\Delta\{x_n\}$ 3 5 7 4

$\Delta^2\{x_n\}$ 2 2 2

(5) "Continuation" ($\boxed{}$ is "deg k")

$x_0, \ldots, x_{k-1}$ known, Then rest of seq. is

Can continue this backwards. x_{-15} is well-defin

Think about: $\mathbb{Z}$, $a_i \in \mathbb{Z}$

(6) Main Thm.

Fact (General Formula for Linear Recurrences)

Suppose $\{x_i\}$ is a sequence satisfying

$$x_{n+k} = a_{k-1}x_{n+k-1} + a_{k-2}x_{n+k-2} + \ldots + a_0 x_n$$

for all integers $n \geq 0$. Then x_n has the following general formula:

$$x_n = P_1(n)\alpha_1^n + P_2(n)\alpha_2^n + \ldots + P_m(n)\alpha_m^n$$

where the α_i are the roots of the **characteristic polynomial**:

$$x^k - a_{k-1}x^{k-1} - a_{k-2}x^{k-2} - \ldots - a_0 = (x-\alpha_1)^{\beta_1}(x-\alpha_2)^{\beta_2}\ldots(x-\alpha_m)^{\beta_m}$$

and P_i are polynomials with degree at most $\beta_i - 1$

$(x-2)^2(x+1)$

$x_n = A \cdot 2^n + B \cdot n 2^n + C \cdot (-1)$

(7) Some Tricks

"$\sum a_i \gamma_i^n$"

- Rec $\rightleftarrows$ Sum of Exp.

$(\gamma - x)(\gamma - y) = 0$

$ax^n + by^n$

$z_{n+1} = (x+y)z_n - (xy)z_{n-1}$

- Linearity is so so so impt!

$\{X_n - \lambda X_{n-1}\}$

- Rec $\longleftarrow$ Combi

Ex $a_1 = 1$, $\sum_{d|n} a_d = 2^n$

Show $n | a_n$.

Hints

Q1 $\alpha > 1$, $\lfloor \alpha^n \rfloor$, $\frac{1}{\alpha^n} < 1$

$\underbrace{\alpha^n + \frac{1}{\alpha^n}}_{\text{force to } \equiv 1 \bmod p\ ?} = \lfloor \alpha^n \rfloor + 1$.

$n = 0$

$2 \not\equiv 1 \bmod p$

Q2 $f(0) = 0$, f odd. $\boxed{\checkmark}$

$a+b+c=0$ is strong.

$c = 0$

f non-decr.

$x > 0$

$\dots f(\frac{x}{2}), f(x), f(2x), f(4x), \dots \geq 0$

8 "Cheat way"

Suppose $\gamma^k - a_{k-1}\gamma^{k-1} - \cdots - a_0 = 0$

has $\underline{\text{distinct}}$ roots $\gamma_1, \dots, \gamma_k$

Sol'ns: $\gamma_1^n, \gamma_2^n, \dots, \gamma_k^n$

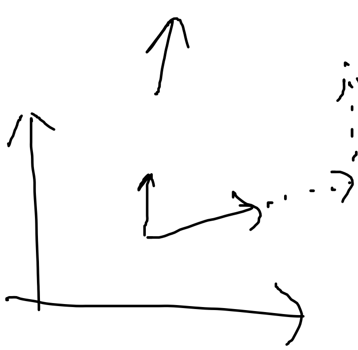
$\boxed{\text{Claim}}$: $\exists\, c_1, \dots, c_k$ s.t.

$$\sum c_i \gamma_i^n = X_n \quad \text{for } n = 0, 1, ..$$

9 Vectors.

$$\vec{V}_i = (1, \gamma_i, \gamma_i^2, \dots, \gamma_i^{k-1}) \quad ✓$$

$$\sum c_i \vec{V}_i = \vec{X} = (X_0, \dots, X_{k-1})$$



"span" $\left\{ \sum_{i=1}^{k} c_i \vec{V}_i : c_i \in \mathbb{R} \right\} \overset{?}{=}$

Suppose otherwise. Then one of v_i's be "redundant"

"v_i's are line

c_i not all zero. $\sum c_i v_i = 0$ | "dependent"

[Linear algebra machinery.

$$\det\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{cases} = 0 & \text{if dep.} \\ \neq 0 & \text{if indep.} \end{cases}$$

" poly of γ_i's with total deg. $\frac{n(n-1)}{2}$

$$\begin{pmatrix} 1 & 1 & & 1 \\ \gamma_1 & \gamma_2 & & \gamma_k \\ \vdots & \vdots & \cdots & \\ \gamma_1^{k-1} & \gamma_2^{k-1} & & \gamma_k^{k-1} \end{pmatrix}$$

homogeneous.

$$\prod_{i<j}(\gamma_i - \gamma_j)$$

"Vandermonde matrix"

(10) Analysis (?)

$\sum c_i \vec{v}_i = 0.$ $\rightarrow$ $\sum c_i \gamma_i^n = 0$ for $n = 0, \ldots$

also satisfies the deg k recurrence.

$\Rightarrow$ holds for all n

Impossible if $\gamma_i \in \mathbb{R}_{>0}$

$$\sum c_i \gamma_i^n = -\sum c_i \gamma_i^n$$

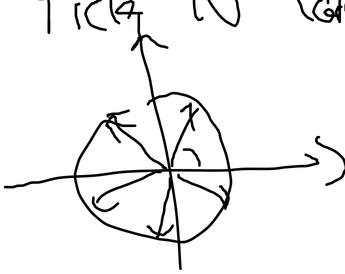
γ_i: max. size $\qquad\qquad$ γ_j: others

$|\gamma_i| = M$ $\qquad\qquad$ $|\gamma_j| < M$

$$\underbrace{\sum c_i \left(\frac{\gamma_i}{M}\right)^n}_{1\cdot 1 = 1} = \underbrace{-\sum c_j \left(\frac{\gamma_j}{M}\right)^n}_{\to\, 0}$$

Pick N larger s.t. $\left(\frac{\gamma_i}{M}\right)^N \simeq 1$ $\quad \forall\ |\gamma_i| = M$ sim

and larger



$$\sum c_i \left(\frac{\gamma_i}{M}\right)^N \simeq \boxed{\sum c_i = 0}$$

$\downarrow$

0

$$\sum (c_i\gamma_i)\gamma_i^n \ . \longrightarrow \sum_{|\gamma_i| \max} c_i\gamma_i = 0$$

$\vdots$

$$\therefore \sum_{|\gamma_i| \max} c_i\gamma_i^n = 0 \quad \forall n. \quad \Rightarrow \quad c_i =$$

?

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(1) Linearity (of recurrence relations)

rec. rel. $\longleftrightarrow$ characteristic poly

$\{x_n\}$: P, Q $\Rightarrow$ $P+Q$

$\underline{P \cdot R}$, R a

Euclidean deg!

"minimal char. poly"

$A_n = r_1^n + r_2^n + \cdots + r_k^n$

$(r - r_1)(r - r_2) \cdots (r - r_k)$

$\vec{A}_0 =$

$\vec{A}_1 =$

$\sum c_i \vec{v}_i \Rightarrow \sum c_i' \vec{A}_i$ can cover everyth

$\vec{A}_i = (A_i, A_{i+1}, \ldots,$

$\sum c_i'' \vec{A}_i = 0 \Rightarrow \sum c_i'' A_{i+n} = 0$

for $n = 0, \ldots,$

$\Rightarrow \sum_{i=1}^{k} c_i'' A_{i+n} = 0 \ \forall$

rec. rel. of deg $\leq k$

(root of unity filter

12) Periodic sequences.

$$X_{n+k} = X_n . \qquad X_n = \sum c_i (\omega^i)^n$$

$$(X_0, \dots, X_{n-1}) \Rightarrow (c_1, \dots, c_n)$$

$$(c_1, \dots, c_n) \Rightarrow (\quad ? \quad)$$

Hint: it's very close to the original

BB: "Skolem's Thm".

Fix prime $p \geq 5$. Let $a_n = 3^n - 2^n$.

What does this seq. look like?

$$v_p(a_0), v_p(a_1), v_p(a_2), \dots$$

$0, 0, \dots, 1, \dots, 1, \dots, 1, \dots, 1, \dots, 2$

$\quad d, \dots d, \dots \quad , d+1$

<u>Thm</u>. If you have a $\mathbb{R}$-linear recurrence, then the zeroes are eventually periodic

Ref: Evan Chen "Napkin"

Also,

Recurrence relations $\cong$ Matrix multiplication

$$\begin{pmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & 1 \\ a_0 & a_1 & - & - & a_{k-1} \end{pmatrix} \begin{pmatrix} x_n \\ \vdots \\ x_{n+k-1} \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ \vdots \\ x_{n+k} \end{pmatrix}$$

γ_i — eigenvalues

More hints

Q5 "Indicator function".

$F_n = \begin{pmatrix} 0 \text{ if } n \text{ is} \\ \text{in AP \#1} \end{pmatrix} \begin{pmatrix} 0 \text{ if } n \text{ is} \\ \text{in AP \#2} \end{pmatrix} ($

Q2 Find another a s.t. $f(ax) = af(x)$.

(or many other)

Q4 $a_n = \gamma_1^n + \dots + \gamma_k^n$.

B Get to ① first!

Q3 $\{f(n)\}$ is a recurrence (in n

what else?