1 Introduction

This handout is about *heuristics*, rather than solutions. Most other handouts tell you about inequalities you can apply (e.g. AM-GM, Cauchy, ...), but here we focus on **information you can obtain without before solving the inequality**.

Knowledge of the standard tools is assumed and not specifically covered here. Proficiency with the standard tools is ideal but not required (and honestly a waste of time).

1.1 Caveat: a word on philosophy

There are typically two philosophies of dealing with inequalities (and problems in general): (1) a "heuristics" and intuition based approach, as per this handout; (2) a "form"-based approach, by studying how terms move around in manipulations.

So the disclaimer is that **using the wrong approach can end horribly**. For instance, IMO 2012 Q2 will lead you on a wild goose chase if you try to "motivate" it, but falls to a two-line weighted AM-GM. In general, problems that are not "tight" will not be friendly for Approach 1.

1.2 Summary of Examples

Just in case you want to try them before you read the solutions.

1. Prove that for $x, y, z > 0$ with product 1,

$$\sum_{cyc} \frac{x^3}{(1+y)(1+z)} \geq \frac{3}{4}$$

2. (ISL 2016 A1/NTST 2017) Let a, b, c be positive real numbers such that $\min(ab, bc, ca) \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

3. (Secrets in Inequalities, vol. 1) If $a, b, c \in [1, 2]$, show that

$$a^3 + b^3 + c^3 \leq 5abc$$

4. (CGMO 2011) Positive reals a, b, c, d satisfy $abcd = 1$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{9}{a+b+c+d} \geq \frac{25}{4}$$

5. (MEMO 2017) Determine the smallest possible real constant C such that the inequality

$$|x^3 + y^3 + z^3 + 1| \leq C|x^5 + y^5 + z^5 + 1|$$

holds for all real numbers x, y, z satisfying $x + y + z = -1$.

6. (ISL 2015 A8) Find the largest real constant a such that for all $n \geq 1$ and for all real numbers $x_0, x_1, ..., x_n$ satisfying $0 = x_0 < x_1 < x_2 < \cdots < x_n$ we have

$$\frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1} + \cdots + \frac{1}{x_n - x_{n-1}} \geq a\left(\frac{2}{x_1} + \frac{3}{x_2} + \cdots + \frac{n+1}{x_n}\right)$$

1.3 Quickfire General Advice

- Small cases. Build intuition.
- Easy stuff first.
- Extract as much information as possible from the problem (i.e. apply heuristics).
- Take note of coincidences (like symmetries), most of them can be exploited somehow.
- Keep things neat. Symmetry is valuable, but feel free to force a WLOG ordering if that solves the problem
- Be aware of your "algebraic tolerance" if you have to get your hands dirty. Same advice also holds for coordinate geometry/complex bash.
- Don't be afraid to ditch your current approach. This is however much easier said than done.

2 Equality Cases

Just so you don't miss it:

Equality cases are really important.

Why are equality cases important? Simply because any proof necessarily *asserts* certain equality cases. If you use AM-GM anywhere in your proof, then when you trace the equality case later on those terms that you AM-GMed will be forced to be equal. Here's a cautionary tale:

Example 1 ('Vasc')

Show that for all reals a, b, c,

$$(a^2 + b^2 + c^2)^2 \geq 3(a^3b + b^3c + c^3a)$$

This attracts many fakesolves (for instance, this is not "trivial by Muirhead"), and the obvious reason why they are fakesolves is due to the additional equality case at

$$(a : b : c) = \left(\sin^2\left(\frac{4\pi}{7}\right), \sin^2\left(\frac{2\pi}{7}\right), \sin^2\left(\frac{\pi}{7}\right)\right).$$

Needless to say, you never saw that coming, and there is no way that a fast proof by Cauchy or AM-GM will reproduce this strange equality case.

Solution. The inequality is in fact equivalent to

$$\sum_{cyc}(a^2 - 2ab + bc - c^2 + ca)^2 \geq 0. \qquad \blacksquare$$

While we don't expect to pull such magical identities from mid-air[1], knowing a strange equality case is often the biggest hint towards the right answer.

In fact, here's an example where the equality case not only warns you about what not to do, but also tells you what to do:

Example 2 (Old ISL)

Prove that for $x, y, z > 0$ with product 1,

$$\sum_{cyc} \frac{x^3}{(1+y)(1+z)} \geq \frac{3}{4}$$

If you're naive you might try:

$$\frac{x^3}{(1+y)(1+z)} + (1+y) + (1+z) \geq 3x$$

However, it won't work:

$$\sum_{cyc} \frac{x^3}{(1+y)(1+z)} \geq x + y + z - 6 \not\geq \frac{3}{4}$$

If we consider the expected equality case at $(x, y, z) = (1, 1, 1)$, then we will know that this was doomed to begin with, since in the first step we applied AM-GM to $(\frac{1}{4}, 2, 2)$. Instead, a slight modification proves to be successful:

$$\frac{x^3}{(1+y)(1+z)} + \frac{1+y}{8} + \frac{1+z}{8} \geq \frac{3x}{4}$$

[1] there is a "slightly more motivatable" method based on an ad-hoc discriminant bash

Now, summing cyclically,

$$\sum_{cyc} \frac{x^3}{(1+y)(1+z)} \geq \frac{x+y+z}{2} - \frac{3}{4} \geq \frac{3}{4}$$

where we used $x+y+z \geq 3\sqrt[3]{xyz} = 3$.

I really can't overstate the importance of the equality case. If anything, the next two methods both rely on knowledge of what the equality case is.

3 Smoothing

A proof by smoothing starts from the general case and attempts to "adjust" the variables in a way that makes the inequality tighter. As a (very simple) example:

Example 3 (AM-GM)

Show that for positive reals $a_1, a_2, ..., a_n$,

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n}$$

Solution. For $x, y > 0$ with fixed sum, xy becomes larger as x and y are pushed closer together. We will apply this idea on some pairs of variables.

Denote $A = \frac{a_1+a_2+...+a_n}{n}$, then we select $a_i \leq A < a_j$ (if i or j does not exist then all $a_i = A$ and the conclusion is obvious). Replace the two variables (a_i, a_j) with $(A, a_i + a_j - A)$. Clearly, the new pair has a larger product. So RHS increases while LHS stays the same, and thus the inequality has become tighter.

Repeating this step, we find that the number of i where $a_i = A$ strictly increases, so eventually $a_i = A$ for all i, where the inequality is true. Hence before this, the inequality must been true as well (since it only became tighter after each operation). ■

In practice however, smoothing is never this clean. In the above example:

- For any fixed variable, there are only two separate terms containing it. This is rarely the case.
- $a_i a_j$ changes predictably, despite the fact that we parked a_i and a_j at very weird places. In practice this is much much worse, and usually we will be lucky to even map them both to $\frac{a_i+a_j}{2}$ or $\sqrt{a_i a_j}$.

  (In an n-variable setting, this step often comes from the $n = 2$ case.)

Depending on what smoothing step you come up with, you may be in for a rough bash or a smooth (heh) time.

3.1 Example - Mount Inequality erupts, NTST edition

Nobody likes it when Mount Inequality erupts[2].

Example 4 (ISL 2016 A1/NTST 2017)

Let a, b, c be positive real numbers such that $\min(ab, bc, ca) \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

Solution. Let's solve the two-variable case: for $x, y > 0, xy \geq 1$,

$$(x^2+1)(y^2+1) \leq \left(\left(\frac{x+y}{2}\right)^2 + 1\right)^2$$

This is fairly easy because we expect $(x-y)^2$ to be a factor[3]. A fairly cute way is to note

$$(x^2+1)(y^2+1) = (x-y)^2 + (xy+1)^2 \leq \left(\left(\frac{x+y}{2}\right)^2 + 1\right)^2$$

[2] a reference to the AoPS thread title for USAMO 2017 Q6. Also folks got destroyed on NTST.

[3] Why? Stay tuned!

or $(x-y)^2 \le \left(\frac{x-y}{2}\right)^2 \left(\left(\frac{x+y}{2}\right)^2 + xy + 2\right)$.

This makes us want to do things like $(a,b) \to \left(\frac{a+b}{2}, \frac{a+b}{2}\right)$. We quickly check that the min condition still holds under this operation. Now there are two options:

Option 1. WLOG $a \ge b \ge c$. Then we smooth a and b together ($(a,b,c) \to (x,x,c)$ where $x = \frac{a+b}{2}$) and attempt to directly expand/factorize the resulting 2-variable expression. This works but is pretty ugly.

Option 2. The only reason stopping us from spamming the smoothing step $(a,b,c) \to \left(\frac{a+b}{2}, \frac{a+b}{2}, c\right)$ is that it doesn't terminate. But there is a sense that we get "close enough" to be correct.

This calls for an analysis-flavored continuity argument. First, define $f(a,b,c) = RHS - LHS$.

Claim — Fix $a,b,c > 0$. For any a',b',c' where $|a-a'|, |b-b'|, |c-c'| < \varepsilon$ where ε is sufficiently small in terms of a,b,c,

$$|f(a,b,c) - f(a',b',c')| < 100\varepsilon$$

Now we perform the algorithm that we wanted to do. For a given $a,b,c > 0$, fix a small enough $\varepsilon > 0$ and start with $(a_0,b_0,c_0) = (a,b,c)$, then alternately apply $(a,b,c) \to \left(\frac{a+b}{2}, \frac{a+b}{2}, c\right)$ and $(a,b,c) \to \left(a, \frac{b+c}{2}, \frac{b+c}{2}\right)$, i.e.

$$(a_{n+1}, b_{n+1}, c_{n+1}) = \begin{cases} \left(\frac{a_n+b_n}{2}, \frac{a_n+b_n}{2}, c_n\right) & n \text{ even} \\ \left(a_n, \frac{b_n+c_n}{2}, \frac{b_n+c_n}{2}\right) & n \text{ odd} \end{cases}$$

Both operations preserve the min condition, and by the two-variable case we have $f(a_n,b_n,c_n) \le f(a_{n+1},b_{n+1},c_{n+1})$. Moreover, by using a monovariant ($S_n = |a_n - b_n| + |b_n - c_n| + |c_n - a_n|$, which halves each step after the first), eventually all three variables converge to the mean $m = \frac{a+b+c}{3}$ (i.e. $|a_n - m| < \varepsilon$ for large enough n) because the mean is preserved.

Applying the claim, $f(a,b,c) \le f(a_n,b_n,c_n) < f(m,m,m) + 100\varepsilon$ for large enough n, but since this worked for any $\varepsilon > 0$, $f(a,b,c) \le f(m,m,m) = 0$. ■

Remark.

1. What the claim is really trying to say is that f is continuous. IYKYK.
2. If you need an archaic name to remember this by, in the pre-2010 era this was known as *strong variable mixing*. But don't cite this; your SMO/NTST graders won't know what you're talking about.

3.2 Normalization

A polynomial P is *homogeneous* if $P(ka,kb,kc) = k^{\deg P} P(a,b,c)$. Intuitively, this means that each term in P should have the same total degree (i.e. sum of degree in each variable). Similarly, we call an inequality *homogeneous* if $LHS - RHS$ is homogeneous.

This gives us the seemingly useless smoothing step $(a,b,c) \to (ka,kb,kc)$, but what we can do now is to set $k = \frac{1}{a+b+c}$, which means that we can assume $a+b+c = 1$ and preserve full generality. This is sometimes useful:

Example 5 (Cauchy-Schwarz)

Show that for non-negative reals $a_1, a_2, ..., a_n; b_1, b_2, ..., b_n$,

$$(a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2) \ge (a_1b_1 + a_2b_2 + ... + a_nb_n)^2$$

Solution. Not only is the above inequality homogeneous, it is homogeneous if we just consider $\{a_i\}$ alone (or $\{b_i\}$ alone). So we can assume $a_1^2 + a_2^2 + ... + a_n^2 = 1$ and $b_1^2 + b_2^2 + ... + b_n^2 = 1$, then the conclusion immediately follows from:

$$\sum a_ib_i \le \sum \frac{1}{2}(a_i^2 + b_i^2) = 1.$$ ■

Another application is if you're trying to do some expanding-bash, you should toss away extra conditions (e.g. "positive reals a, b, c *with sum 1...*") by substituting copies of it into the equation until it is homogeneous. For instance, if a, b, c have sum 1, then expressions like $a^2 + 1$ should be re-written as $a^2 + (a + b + c)^2$.

Disclaimer: Normalization is rare in practice, but it's a good demonstration of how symmetry can be used to your advantage. Homogenization doesn't work all the time, you will have to judge whether it's helpful or not. Also, if the condition itself is not homogeneous you're screwed anyway.

3.3 Convexity

Sums of convex/concave functions can be smoothed:

Fact 6 (Fixed-Sum Convex Smoothing)
Let $f(x)$ be convex on the interval I. Suppose $a < a + \varepsilon < b - \varepsilon < b$ are all in I. Then,

$$f(a) + f(b) \geq f(a + \epsilon) + f(b - \epsilon)$$

This means that

- For **convex** functions f, we can decrease the sum $f(a)+f(b)$ by "smoothing" a and b together, and increase the sum by "unsmoothing" a and b apart.
- For **concave** functions f, we can increase the sum $f(a) + f(b)$ by "smoothing" a and b together, and decrease the sum by "unsmoothing" a and b apart.

This quickly implies things like Jensen's inequality (or even Karamata's if you're feeling brave), but much more is possible. Suppose $x_1 + ... + x_n = c$, and you have a function f made up of concave and convex parts. To maximize $f(x_1) + ... + f(x_n)$, we simply smooth along the concave parts and unsmooth along the convex parts, this gives:

Fact 7 (Fixed Sum Smoothing, general version)
In the above scenario, we can WLOG assume that (1) there is at most one variable within each convex part and (2) all variables within a concave part are equal.

This is very messy to execute in practice, but usually the following special case is (typically) more than sufficient:

Fact 8 ($n - 1$ "Eevee" Principle)
Let $a_1, a_2, ..., a_n$ be reals with fixed sum, and f be a function with exactly one inflection point (i.e. $f''(x) = 0$ has exactly one root).

If $f(a_1) + f(a_2) + ... + f(a_n)$ achieves maximal or minimal value, then (at least) $n - 1$ of them are equal to each other[a].

[a] it's actually EV for Equal Variables, hehe

If there are no extra conditions like fixed sum/product, then we can still think about single-variable smoothing. If the equality case lies at the endpoints of an interval, sometimes this is the reason why:

Fact 9 (Convex Extrema)

If $f(x)$ is **convex** on the interval $a \le x \le b$, then $f(x)$ attains a maximum, and that value is either $f(a)$ or $f(b)$.

If $f(x)$ is **concave** on the interval $a \le x \le b$, then $f(x)$ attains a minimum, and that value is either $f(a)$ or $f(b)$.

Example 10 (Secrets in Inequalities, vol. 1)

If $a, b, c \in [1, 2]$, show that

$$a^3 + b^3 + c^3 \le 5abc$$

Solution. The function $f(a, b, c) = a^3 + b^3 + c^3 - 5abc$ is convex in each variable, so we just need to check the 4 cases (up to symmetry) where $a, b, c \in \{1, 2\}$. ■

> **Remark.** Note that this has *the opposite sign* as compared to normal AM-GM! Any time you see inequalities with the sign going the wrong way, it's usually convexity at work.

3.4 Other ways to smooth

Traditionally, smoothing refers specifically just to push variables together, but it can done in a much more generic that.

Philosophically, you should think of smoothing as a kind of WLOG (in parallel to what you sometimes do for combinatorics). You are reducing all your cases down to a smaller subset, which are hopefully easier to deal with.

In general, anything that changes a small part of the inequality predictably qualifies as a smoothing step. The more synergistic your smoothing steps are, the less algebra you have to do.

Here are some nice examples:

Example 11

If $a, b, c \ge 0$, then we have

$$(a + b + c)^3 \ge 6\sqrt{3}|(a - b)(b - c)(c - a)|$$

Solution. Turns out this is tight. God knows what the equality case on this is.

But WLOG $a \ge b \ge c$, and now we simultaneously lower a, b, c keeping $a - b, b - c$ fixed (i.e. shift $(a, b, c) \to (a - \epsilon, b - \epsilon, c - \epsilon)$. This keeps RHS constant but decreases LHS, and this only stops when c hits 0. Then if we write $t = a/b$, we get the single variable

$$(t + 1)^3 \ge 6\sqrt{3}t(t - 1)$$

which factors as $\left(t - \sqrt{3} - 2\right)^2 (t + 7 - 4\sqrt{3}) \ge 0$. ■

> **Remark.**
>
> 1. This method is also known as *elementary mixing variables* (again, Vietnamese technique).
> 2. How might you find that double root by hand? Well, if $(x - \alpha)^2$ divides $P(x)$ it must also divide $P'(x)$, then you can run the Euclidean algorithm.

Example 12 (Iran TST 1996)

Show that for $a, b, c > 0$,

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)}^2 \geq \frac{9}{4(ab+bc+ca)}$$

Solution. Note the double equality cases at $(a, b, c) = (k, k, k), (k, k, 0)$, so none of the usual stuff works.

Let's try something else. Since the inequality is symmetric, define

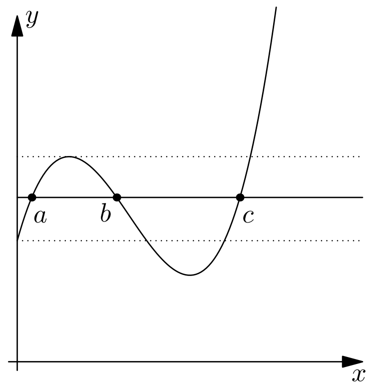
$$\begin{cases} p = a + b + c \\ q = ab + bc + ca \\ r = abc \end{cases}$$

then we can express the inequality as

$$4q(4pr + p^4 - 2p^2q + q^2) - 9(pq - r)^2 \geq 0$$

and now we smooth p, q, r instead of a, b, c. We have to be careful here though, because we still need an underlying a, b, c to exist.

To help us visualize this, we draw the curve $f(x) = x^3 - px^2 + qx$:



Then, a, b, c show up as the intersections between $y = f(x)$ and $y = r$. So we can wriggle the line $y = r$ up and down as long as: (1) there are three intersections (counting multiplicity); (2) all intersections are ≥ 0.

Hence, we can adjust r upwards or downwards until one of the following happens (WLOG $a \leq b \leq c$):

1. $a = 0$ (e.g. the bottom dotted line)
2. $a = b$ or $b = c$ (e.g. the top dotted line)

In either case we end up with a single variable inequality, which is super easy.

Returning to the problem, note that the expression is a quadratic in r with a negative leading coefficient. Hence, the expression is concave in r and attains the minimum at one of the endpoints. We get one of the above scenarios, which is easily checked by plugging a, b, c in the original inequality. ■

Remark.

1. This is sometimes called the *pqr*-method. Of course, we can adjust p or q instead if we really wanted to.
2. This approach definitely works on any symmetric three-variable inequality of degree at most 5, since then r is guaranteed to be linear.
3. The usual solution is to expand this and rely on Schur's inequality to produce the multiple

> equality cases.

3.5 Example - What does a smoothing bash look like?

Here is a "classic example" for smoothing. Realistically, in this problem the smoothing step isn't as clean as the other examples we've seen so far. (If you know a better way to do this problem please let me know.)

Example 13

(CGMO 2011) Positive reals a, b, c, d satisfy $abcd = 1$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{9}{a+b+c+d} \geq \frac{25}{4}.$$

Solution. The usual tools fail spectacularly here, especially since nothing can be done about $abcd = 1$ without the inequality melting into an intractable mess.

Let's try to do something like $(a, b, c, d) \to (\sqrt{ab}, \sqrt{ab}, c, d)$ which preserves the condition $abcd = 1$. For this to work for any a, b, c, d would be too much to hope for, but let's just try simplifying the statement anyway to see what we need to assume. We want:

$$\frac{2}{\sqrt{ab}} + \frac{9}{2\sqrt{ab}+c+d} \leq \frac{1}{a} + \frac{1}{b} + \frac{9}{a+b+c+d}$$

It's actually not too bad if we group $\frac{1}{a} + \frac{1}{b}$ with $\frac{2}{\sqrt{ab}}$. For cleanliness, we do $(a, b, c, d) = (w^2, x^2, y^2, z^2)$, then:

$$\begin{aligned}
&\frac{2}{wx} + \frac{9}{2wx+y^2+z^2} \leq \frac{1}{w^2} + \frac{1}{x^2} + \frac{9}{w^2+x^2+y^2+z^2} \\
\Leftrightarrow\ &\frac{9}{2wx+y^2+z^2} - \frac{9}{w^2+x^2+y^2+z^2} \leq \frac{(w-x)^2}{w^2x^2} \\
\Leftrightarrow\ &\frac{9(w-x)^2}{(2wx+y^2+z^2)(w^2+x^2+y^2+z^2)} \leq \frac{(w-x)^2}{w^2x^2} \\
\Leftrightarrow\ &\frac{9}{(2wx+y^2+z^2)(w^2+x^2+y^2+z^2)} \leq \frac{1}{w^2x^2} \\
\Leftrightarrow\ &(2wx+y^2+z^2)(w^2+x^2+y^2+z^2) \geq 9w^2x^2
\end{aligned}$$

Of course, the last bit works as long as $y^2 + z^2 \geq wx$. This pretty much means that we can always smooth a and b together if c or d is the largest among a, b, c, d. So if $a \leq b \leq c \leq d$, we can perform this on a, b and a, c repeatedly which will converge to $a = b = c$.

At this point, we can just bash out the last bit: the original inequality is equivalent to

$$\frac{(a-1)^2(12a^6 + 24a^5 + 36a^4 - 27a^3 - 14a^2 - a + 12)}{4a(3a^4+1)} \geq 0$$

(where naturally we predicted that $(a-1)^2$ was a factor[4]). It remains to show that $f(a) = 12a^6 + 24a^5 + 36a^4 - 27a^3 - 14a^2 - a + 12 \geq 0$. Ordinarily you would graph this in WolframAlpha, but check out the appendix to see how you might do this in by pen and paper. ■

> **Remark.** If your f had another root you didn't know about, you're screwed (by equality case logic). Factor it out first.

[4] What is this sorcery? Stay tuned...

4 Double Roots

4.1 Some motivation

Inequalities (with equality cases) are closely related to double roots. Think about this:

Fact 14

Let P be a polynomial with real coefficients. If $P(x) \geq 0$ for all x and $P(\alpha) = 0$, then $(x-\alpha)^2$ divides $P(x)$.

If you draw a picture, this is simple enough: if you draw a line tangent to the graph of $P(x)$ at $x = \alpha$, then $P'(\alpha)$ is precisely the gradient of the tangent. We must have $P'(\alpha) = 0$, otherwise P will dip below 0 either right before or right after α.

Algebraically, we can set $y = x - \alpha$ and expand $P(x) = a_0 + a_1 y + O(y^2)$[5] where $x \approx \alpha$. Then it's clear that when x moves in the range $[\alpha - \varepsilon, \alpha + \varepsilon]$ we can ignore the $O(y^2)$ part and treat f as linear (which means it has to be identically 0).

In fact, we only need $P(x) \geq 0$ around the neighbourhood of α, so conditions like $x > 0$ are generally not a problem.

Perhaps more interestingly, does this work in more variables?

Fact 15

Let $P(x, y)$ be a bivariate polynomial. Suppose that $P(x, y) \geq 0$ for all x, y and $P(x, x) = 0$ for all x. Then $(x - y)^2$ divides $P(x, y)$.

Actually we expect something very similar to work. Treating $\mathbb{R}[x, y]$ as $\mathbb{R}[y][x - y]$ (read: polynomials in $x - y$ with coefficients which are polynomials in y), we write:

$$P(x, y) = P_0(y) + (x - y)P_1(y) + (x - y)^2 \cdot (\cdots)$$

But for a fixed value of y, $(x-y)^2$ divides $P(x, y)$. Taking large x, we get that $P_0(y)+(x-y)P_1(y) \equiv 0$ for fixed y. So for all y, $P_1(y) \equiv 0$ and $P_0(y) - yP_1(y) \equiv 0$, and hence the fact is proven.

This is an *extremely* useful heuristic to have! Broadly speaking, this essentially means that whenever we have an equality case, we should expect a double root.

Unfortunately there's nothing similar for x, y, z, since we expect expressions like $\sum k_a(b - c)^2$.

4.2 Example - Absolute value, absolute nightmare?

Here's an example where reasoning about roots helps:

Example 16

(MEMO 2017) Determine the smallest possible real constant C such that the inequality

$$|x^3 + y^3 + z^3 + 1| \leq C|x^5 + y^5 + z^5 + 1|$$

holds for all real numbers x, y, z satisfying $x + y + z = -1$.

Since the inequality varies over reals (instead of non-negative reals), we can throw away most of the standard stockpile and instead try to use our "bare hands". It's very likely this ends up as a factorization/sum of squares-type deal.

[5] $O(y^2) = y^2(...)$. More generally, $O(y^2)$ is a statement on size (e.g. $cy^2 \leq O(y^2) \leq Cy^2$), in this case for when y is near 0.

The -1 clearly isn't going to help, so let's homogenize:

$$|(x+y+z)^2 \cdot (x^3+y^3+z^3-(x+y+z)^3)| \le C|x^5+y^5+z^5-(x+y+z)^5|$$

But, now note that $RHS^2 - LHS^2$ does have an equality case at $x = -y$, so we expect $(x+y)^2$ to divide it, or at least $(x+y)|RHS - LHS$. In addition, $LHS = 0$ whenever $x = -y$. Hence it's also reasonable to think that $(x+y)$ divides it (and by symmetry, so must $(y+z)$ and $(z+x)$). Indeed, it's not hard to verify that

$$(x+y+z)^3 - (x^3+y^3+z^3) = 3(x+y)(y+z)(z+x)$$

Similarly, we should expect

$$(x+y+z)^5 - (x^5+y^5+z^5) = (A(x^2+y^2+z^2) + B(xy+yz+zx))(x+y)(y+z)(z+x)$$

for some constants A, B which are easily determined by plugging two sets of values in (x, y, z). I'll save you the trouble and tell you $A = B = 5$.

Anyway, this means that after throwing away $(x+y)(y+z)(z+x)$, the remaining terms are non-negative (so we can toss the absolute values). Moreover, the remaining degree is 2. Easy peasy.

4.3 Generalized double roots = partial derivatives?

Of course, not all inequalities look like polynomials. In general, the best that we have is $f'(\alpha) = 0$, or for multivariate f:

$$\left.\frac{\partial f}{\partial x_i}\right|_{x_i=\alpha_i} = 0$$

where $(x_i) = (\alpha_i)$ is the equality case.

If you're analyzing the behaviour of each variable, you can try to solve for where the minima happens, but it's not strictly necessary.

Note: derivatives are good as a heuristic but bad as a proof technique. Even ignoring boundary cases, we only know that (minimum at $\alpha \Rightarrow f'(\alpha) = 0$) but not ($f'(\alpha) = 0 \Rightarrow$ minimum at α).

4.4 Undetermined coefficients

Some methods attempt to bound each term in the inequality by a term of some predetermined form, like the tangent line trick or isolated fudging. In either case, the variable coefficients must be determined, and assuming a "double root" can help us make an educated guess.

Example 17 (Nesbitt)

Show that for $a, b, c > 0$,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2}$$

Solution. Perhaps we predict that for some r:

$$\frac{a}{b+c} \ge \frac{3a^r}{2(a^r+b^r+c^r)}$$

Let's cross-multiply and do a check at $b = c = 1$:

$$f(a) = 2a(a^r+b^r+c^r) - 3a^r(b+c) = 2a^{r+1} + 4a - 6a^r \ge 0$$

We expect $f'(1) = 0$, so $f'(1) = 2(r+1) + 4 - 6r = 0$. Hence $r = \frac{3}{2}$. Indeed:

$$a^{\frac{5}{2}} + ab^{\frac{3}{2}} + ab^{\frac{3}{2}} \ge 3a(ab^2)^{\frac{1}{2}} = 3a^{\frac{3}{2}}b$$

and a similar inequality when c is replaced with b. Adding them together, we are done! ■

While this may seem like a lot of work just for Nesbitt (why not do a one-line Cauchy?), this does have a comparative advantage sometimes.

4.5 Example - OH MY GOD IT'S AN A8

By the way, I couldn't solve this, but this is the perfect example of how surprisingly far you could get using just heuristics.

Example 18 (ISL 2015 A8)

Find the largest real constant a such that for all $n \geq 1$ and for all real numbers $x_0, x_1, ..., x_n$ satisfying $0 = x_0 < x_1 < x_2 < \cdots < x_n$ we have

$$\frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1} + \cdots + \frac{1}{x_n - x_{n-1}} \geq a\left(\frac{2}{x_1} + \frac{3}{x_2} + \cdots + \frac{n+1}{x_n}\right)$$

Solution. This is going to be me rattling away for 1.5 pages, stream-of-consciousness style.

Maybe the first thing we do is to try some small cases: $n = 1$ is trivial and gives us $a \leq \frac{1}{2}$. $n = 2$ is as follows:

$$\frac{1}{x_1} + \frac{1}{x_2 - x_1} \geq a\left(\frac{2}{x_1} + \frac{3}{x_2}\right)$$

Simplifying slightly:

$$\frac{1-2a}{x_1} + \frac{1}{x_2 - x_1} \geq \frac{3a}{x_2}$$

Cauchy tells us that minimum on the left hand side is in fact $\frac{(\sqrt{1-2a}+1)^2}{x_2}$, so solving we have that $a \leq \frac{12}{25}$. a seems like it's dropping (as n increases) but *very* slowly.

$n = 3$ is a little too complicated for us to get a nice looking bound for a, so we'll stop there with small cases.

Looking at $n = 2$, it is reasonable to imagine an approach that looks like summing inequalities of the form

$$\frac{*}{x_1} + \frac{*}{x_2 - x_1} + ... + \frac{*}{x_k - x_{k-1}} \geq \frac{a(k+1)}{x_k}$$

...except we have no idea what the coefficients should be. In fact, if this inequality is "tight", we need the equality case of Cauchy to hold (i.e. "proportional" sequences), and so we need to know the equality case for x_i first.

Here's where the heuristics begin. Now let's think about adjusting just one variable x_k so that the inequality is optimally tight. We isolate all terms where x_k appear and treat it as a function:

$$f(x) = \frac{1}{(x - x_{k-1})} + \frac{1}{(x_{k+1} - x)} - \frac{a(k+1)}{x}$$

If x_k was minimal, we should expect either x_k to be at the endpoints (which makes f shoot off to infinity) or $f'(x_k) = 0$. We should thus pay attention to

$$f'(x) = -\frac{1}{(x - x_{k-1})^2} + \frac{1}{(x_{k+1} - x)^2} + \frac{a(k+1)}{x^2}$$

Sometimes, here's where we try and solve the equation $f'(x) = 0$ but I assure you it doesn't end well here. Instead, we'll just keep the equation as it is and see what we can do.

To minimize future confusion, we are going to replace all the x's with y's. Specifically, we will keep $\{x_i\}$ as variables while $\{y_i\}$ will denote the specific value that keeps $LHS - RHS$ minimal.

Since $f'(y_i) = 0$, we have:

$$\frac{a(k+1)}{y_k^2} = \frac{1}{(y_k - y_{k-1})^2} - \frac{1}{(y_{k+1} - y_k)^2}$$

(Quick sanity check: multiply both sides by y_k and sum over k. Realise things somewhat work out. Also RHS looks like it telescopes.)

This doesn't seem very useful at the moment, but let's try to go back to the initial Cauchy and make it work. With y_i's instead of x_i's, we have

$$\frac{*}{y_1} + \frac{*}{y_2 - y_1} + \cdots + \frac{*}{y_k - y_{k-1}} \overset{\text{-ish}}{=} \frac{a(k+1)}{y_k}$$

or

$$\left(\frac{*}{y_1} + \frac{*}{y_2 - y_1} + \cdots + \frac{*}{y_k - y_{k-1}}\right)(y_1 + (y_2 - y_1) + \cdots + (y_k - y_{k-1})) \overset{\text{-ish}}{=} a(k+1)$$

What's the equality case? It's when the coefficients above are proportional to $(y_k - y_{k-1})^2$, and the right scaling is $a(k+1)/y_n^2$, so we get

$$\sum_{j \le k} \frac{a(k+1)(y_j - y_{j+1})^2}{y_k^2} \cdot \frac{1}{y_j - y_{j+1}} = \frac{a(k+1)}{y_k}$$

with equality holds all the time. But let's sum over k and see what happens:

$$\sum_j \left((y_j - y_{j+1})^2 \cdot \frac{1}{y_j - y_{j+1}} \sum_{k \ge j} \frac{a(k+1)}{y_k^2} \right) \cdot = \sum_k \frac{a(k+1)}{y_k}$$

But the sum $\sum_{k \ge j} \frac{a(k+1)}{y_k^2}$ telescopes to $\frac{1}{(y_j - y_{j-1})}$ so we have precisely the original inequality!

That's way too big of a coincidence to pass up (even though this is still a "dummy equation"). Let's see whether we can turn this into something concrete. One idea is just to reuse the coefficients we found for $\{y_i\}$ back into the statement for $\{x_i\}$:

$$\sum_{j=1}^{k} \frac{a(k+1)(y_j - y_{j+1})^2}{y_k^2} \cdot \frac{1}{x_j - x_{j+1}} \ge \frac{a(k+1)}{x_k}$$

Summing once again (over k):

$$\sum_{k=1}^{n} \frac{1}{x_k - x_{k+1}} > \sum_{k=1}^{n} \left((y_k - y_{k+1})^2 \cdot \sum_{k=j}^{n} \frac{a(k+1)}{y_k^2} \right) \cdot \frac{1}{x_k - x_{k+1}} \ge \sum_{k=1}^{n} \frac{a(k+1)}{x_k}$$

Holy crap, wow. We've figured out how to solve the problem without solving the problem. Now, as long as we have a working a and y_i, we are done!

Unfortunately I could not construct them, so I failed here. Booooo. For completeness, $a = \frac{4}{9}$ and $y_i = \binom{i+2}{3}$ magically works. If you have a good way of figuring this out please tell me. ■

5 Problems

1. (St Petersburg 1991) Given two continuous functions $f, g : [0,1] \to [0,1]$ with f increasing, prove that
$$\int_0^1 f(g(x))\,dx \leq \int_0^1 f(x)dx + \int_0^1 g(x)\,dx.$$

2. (Alibaba Contest Finals 2020/18) Let $a_1, a_2, ..., a_n$ be positive reals whose squares sum to 1. Show that there exists a choice of $\varepsilon_i \in \{-1, 1\}$ such that
$$\left|\sum_{i=1}^{n} \varepsilon_i a_i\right| \leq \frac{1}{a_1 + a_2 + ... + a_n}.$$

3. (SMO Senior 2012) Show that for $a, b, c, d \geq 0$, and $a + b = c + d = 2$,
$$(a^2 + c^2)(a^2 + d^2)(b^2 + c^2)(b^2 + d^2) \leq 25$$

4. (239 Olympiad 2019 11.7) Let $a_1, \ldots, a_n; b_1, \ldots, b_n; c_1, \ldots, c_n$ be positive real numbers. Let m_k be the maximum product $a_i b_j c_\ell$ across all triples (i, j, ℓ) satisfying $\max(i, j, \ell) = k$. Prove that
$$(a_1 + \cdots + a_n)(b_1 + \cdots + b_n)(c_1 + \cdots + c_n) \leqslant n^2(m_1 + \cdots + m_n).$$

5. (CHKMO 2014 Q3, modified) Find all pairs (a, b) of integers a and b satisfying
$$(b^2 + (a - b))^2 = a^3 b$$

A Single Variable Polynomial bounding

In one of the examples, we had this polynomial:

$$f(a) = 12a^6 + 24a^5 + 36a^4 - 27a^3 - 14a^2 - a + 12$$

and the claim that somehow $f(a) \geq 0$ for all nonnegative a. (Actually, we're hoping for $f(a) > 0$, since otherwise we should be factoring this more.)

The general approach starts by splitting f as $g - h$ where g is increasing and h is decreasing. A convenient choice is

$$\begin{aligned} g(a) &= 12a^6 + 24a^5 + 36a^4 + 12 \\ h(a) &= 27a^3 + 14a^2 + a \end{aligned}$$

Then, along an interval $[p, q]$, for $p \leq a \leq q$ we have $f(a) \geq g(p) - h(q)$, so if we cut our intervals just right this will be nonnegative.

For instance:

$$\begin{aligned} a \geq 1: &\quad 12a^6 + 24a^5 + 36a^4 \geq 60a^3 \geq 27a^3 + 14a^2 + a \\ \frac{1}{\sqrt{2}} \leq a \leq 1: &\quad 24a^5 + 36a^4 \geq 27a^3, 12a^6 \geq 3a^2, 12 \geq a + 11a^2 \\ \frac{1}{2} \leq a \leq \frac{1}{\sqrt{2}}: &\quad 24a^5 + 36a^4 \geq 24a^3, 12 \geq 3a^3 + 14a^2 + a \\ 0 \leq a \leq \frac{1}{2}: &\quad 12 > 27a^3 + 14a^2 + a \end{aligned}$$

So our polynomial was positive, and that's done (phew)!