Solutions

Problem 1. (2024 Israel TST 6/2) Let n be a positive integer. Find all polynomials $Q(x)$ with integer coefficients so that the degree of $Q(x)$ is less than n and there exists an integer $m \geq 1$ for which

$$x^n - 1 \mid Q(x)^m - 1$$

Solution. Answer: $\pm x^k$ for $k \in \mathbb{Z}$ and $0 \leq k \leq n-1$. All work by letting $m = 2n$.

Let $Q(x) = \sum_{i=0}^{n-1} a_i x^i$. Let ω be a primitive n-th root of unity. It follows that $Q(\omega^i)$ is an m-th root of unity. Hence

$$n = \sum_{i=0}^{n-1} |Q(\omega^i)|^2 = \sum_{i=0}^{n-1} Q(\omega^i)Q(\omega^{-i}) = n \sum_{i=0}^{n-1} a_i^2.$$

Hence $\sum_{i=0}^{n-1} a_i^2 = 1$. As a_i are all integers, it follows that a_i are all zero, except one of them which is ± 1.

Problem 2. (USEMO 2021/5) Let $S(p)$ denote the sum of the squares of the coefficients of a polynomial $p(x)$. Prove that if $f(x)$, $g(x)$, and $h(x)$ are polynomials with real coefficients satisfying the indentity $f(x) \cdot g(x) = h(x)^2$, then

$$S(f) \cdot S(g) \geq S(h)^2$$

Solution. Indeed, $f(\omega^k)$ is just a rescaled version of $\hat{f}(k)$, so

$$S(f) = \frac{1}{n} \sum_{k=0}^{n-1} |\hat{f}(k)|^2$$

by Plancherel. Thus,

$$\begin{aligned} S(f) \cdot S(g) &= \frac{1}{n^2} \left(\sum_{k=0}^{n-1} |\hat{f}(k)|^2 \right) \left(\sum_{k=0}^{n-1} |\hat{g}(k)|^2 \right) \\ &\geq \frac{1}{n^2} \left(\sum_{k=0}^{n-1} |\hat{f}(k)\hat{g}(k)|^2 \right) \\ &= \frac{1}{n^2} \left(\sum_{k=0}^{n-1} |\hat{h}(k)|^2 \right)^2 \\ &= S(h)^2 \end{aligned}$$

where we used the Cauchy-Schwarz inequality.

Problem 3. (ISL 2018 A6) Let $m, n \geq 2$ be integers. Suppose that $f(x_1, \ldots, x_n)$ is a polynomial with real coefficients such that

$$f(x_1, \ldots, x_n) = \left\lfloor \frac{x_1 + \cdots + x_n}{m} \right\rfloor$$

for every $x_1, \ldots, x_n \in \{0, 1, \ldots, m-1\}$. Prove that the total degree of f is at least n.

Solution. Consider $g(x_1, \ldots, x_n) := x_1 + x_2 + \cdots + x_n - mf(x_1, \ldots, x_n)$. Then, we have

$$g(x_1, \ldots, x_n) = (x_1 + x_2 + \cdots + x_n \pmod{n})$$

where we take the value of g that is in $\{0, 1, \ldots, m-1\}$. Now, with ω denoting the n-th root of unity, consider the sum

$$\sum_{x_1, \ldots, x_n \in \{0,1,\ldots,m-1\}} \omega^{x_1 + x_2 + \ldots + x_n} g(x_1, \ldots, x_n)$$

If f has degree at most $n-1$ then g has degree at most $n-1$ as well, so the sum is 0 by the root of unity filter. However, the sum is also just

$$m^{n-1} \cdot (\omega + 2\omega^2 + \cdots + (m-1)\omega^{m-1})$$

and it is easy to check that this is nonzero by pairing up terms (e.g. for $m > 2$, $\omega + (m-1)\omega^{m-1}$ has a negative imaginary part).

Problem 4. (Ky Fan's inequality) Let $n \geq 3$ be an integer, and reals $a_1, \ldots, a_n$ sum to zero. Show that

$$\cos \frac{2\pi}{n} \sum_{k=1}^{n} a_k^2 \geq \sum_{k=1}^{n} a_k a_{k+1}.$$

Solution. We need the identity from the hint:

$$\langle f, \tau_a f \rangle = \sum_{k=1}^{n} e^{2\pi i a k} |\hat{f}(k)|^2$$

where $(\tau_a f)(x) = f(x+a)$. In particular, if the inner product is real then $e^{2\pi i a k}$ can be replaced with $\cos(2\pi a k)$.

Since $\widehat{f}(n) = \sum_{k=1}^{n} a_k = 0$, we have

$$\begin{aligned} \cos \frac{2\pi}{n} \sum_{k=1}^{n} a_k^2 - \sum_{k=1}^{n} a_k a_{k+1} &= \cos \frac{2\pi}{n} \langle f, f \rangle - \langle f, \tau_1 f \rangle \\ &= \sum_{k=1}^{n} \left(\cos \frac{2\pi}{n} - \cos \frac{2\pi c k}{n} \right) |f(k)|^2 \\ &\geq 0. \end{aligned}$$

Problem 5. (CTST 2020/1) Let ω be a n -th primitive root of unity. Given complex numbers $a_1, a_2, \cdots, a_n$, and p of them are non-zero. Let

$$b_k = \sum_{i=1}^{n} a_i \omega^{ki}$$

for $k = 1, 2, \cdots, n$. Prove that if $p > 0$, then at least $\frac{n}{p}$ numbers in $b_1, b_2, \cdots, b_n$ are non-zero.

Solution. Instead of b_k, equivalently consider $\hat{a}_k$, and note the triangle inequality

$$|\hat{a}_k| \leq \frac{1}{n} \sum_{i=1}^{n} |a_i|.$$

Now, use Plancherel to get

$$\sum_{k=1}^{n} |\hat{a}_k|^2 = \sum_{i=1}^{n} |a_i|^2 \geq \frac{(\sum |a_i|)^2}{p} \geq \frac{n}{p} \cdot |\hat{a}_k|^2$$

and by picking the largest $|\hat{a}_k|$ we get the desired result.

Comments:

- here's an alternative approach: as a function of k, $\{b_k\}$ is the sum of p exponentials, and thus there is a linear recurrence relation of degree at most p. Thus, if less than n/p of the b_k are nonzero, then there are p consecutive terms that are zero, so all b_k are zero. Now, use the inverse Fourier transform to get that all a_i are zero.
- This fact can be written as $|\operatorname{supp} a| \cdot |\operatorname{supp} \hat{a}| \geq n$ and so can be considered some kind of "uncertainty principle". See Tao 2004.

Problem 6. (HMIC 2017/5, rephrased) Suppose that a function $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ satisfies $f(s)+f(t) = f(s+t)$ for $(1-\epsilon)$ of all pairs $s, t \in \mathbb{F}_2^n$. Show that there exists a linear map $f' : \mathbb{F}_2^n \to \mathbb{F}_2$ that matches f on at least $(1-10\epsilon)$ of the inputs.

Solution. Let $g_v(x) = (-1)^{v \cdot x}$ for $v \in \mathbb{F}_2^n$. We see that the number of values where g_v and f match is captured by

$$\hat{f}(v) := \sum_s f(s) g_v(s) = 2A - N.$$

where $N = 2^n$ and A is exactly the number of values where f and g_v match. (This DFT is $\sqrt{N}$-larger than the usual DFT for convenience.) Thus:

$$\sum_{v \in \mathbb{F}_2^n} \hat{f}(v)^2 = N \sum_s f(s)^2 = N^2 \qquad \text{(Plancherel)}$$

$$\begin{aligned}
\sum_{v \in \mathbb{F}_2^n} \hat{f}(v)^3 &= \sum_{s^{(1)}, s^{(2)}, s^{(3)}, v} f(s^{(1)}) f(s^{(2)}) f(s^{(3)}) g_v(s^{(1)} + s^{(2)} + s^{(3)}) \\
&= N \cdot \sum_{s^{(1)} + s^{(2)} + s^{(3)} = 0} f(s^{(1)}) f(s^{(2)}) f(s^{(3)}) \\
&\geq (1 - 2\epsilon) N^3.
\end{aligned}$$

Subtracting, we have

$$\sum_v \hat{f}(v)^2 \cdot (\hat{f}(v) - (1 - 2\epsilon)N) \geq 0.$$

In particular, for at least one such v we must have $\hat{f}(v) \geq (1-2\epsilon)N$, or equivalently that $A \geq (1-\epsilon)N$.

Problem 7. (CTST 2025/15) Let X be a finite set of real numbers, d be a real number, and $\lambda_1, \lambda_2, \cdots, \lambda_{2025}$ be 2025 non-zero real numbers. Define

- A to be the set of 2025-tuples $(x_1, x_2, \cdots, x_{2025}) \in X^{2025}$ such that $\sum_{i=1}^{2025} \lambda_i x_i = d$
- B to be the set of 2024-tuples $(x_1, x_2, \cdots, x_{2024}) \in X^{2024}$ such that $\sum_{i=1}^{2024} (-1)^i x_i = 0$
- C to be the set of 2026-tuples $(x_1, x_2, \cdots, x_{2026}) \in X^{2026}$ such that $\sum_{i=1}^{2026} (-1)^i x_i = 0$

Show that $|A|^2 \leq |B| \cdot |C|$.

Solution. By Fact 7,

$$\begin{aligned}
A &= \lim_{T \to \infty} \frac{1}{T} \int_0^T \sum_{x \in X^{2025}} e^{it \sum_{j=1}^{2025} \lambda_j x_j} \, dt \\
&= \lim_{T \to \infty} \frac{1}{T} \int_0^T \prod_{j=1}^{2025} \sum_{x_j \in X} e^{it \lambda_j x_j} \, dt \\
&= \lim_{T \to \infty} \frac{1}{T} \int_0^T \prod_{j=1}^{2025} \Phi(\lambda_j t) \, dt
\end{aligned}$$

where $\Phi(t) := \sum_{x \in X} e^{itx}$. In a similar vein,

$$\begin{aligned}
|B| &= \lim_{T \to \infty} \frac{1}{T} \int_0^T |\Phi(t)|^{2024} \, dt \\
|C| &= \lim_{T \to \infty} \frac{1}{T} \int_0^T |\Phi(t)|^{2026} \, dt
\end{aligned}$$

First note that by Cauchy-Schwarz,

$$|B| \cdot |C| \geq \lim_{T \to \infty} \left(\frac{1}{T} \int_0^T |\Phi(t)|^{2025} \, dt \right)^2.$$

Now, the key is to take 2025 independent copies of this integral, each aptly rescaled:

$$\begin{aligned}
&\frac{1}{2025}\sum_{j=1}^{2025}\left(\lim_{T\to\infty}\frac{1}{\lambda_j T}\int_0^{\lambda_j T}|\Phi(t_j)|^{2025}\,dt_j\right)\\
&=\lim_{T\to\infty}\frac{1}{T}\int_0^{T}\frac{1}{2025}\sum_{j=1}^{2025}|\Phi(\lambda_j t_j)|^{2025}\,dt_j\\
&\geq\lim_{T\to\infty}\frac{1}{T}\int_0^{T}\prod_{j=1}^{2025}|\Phi(\lambda_j t_j)|\,dt_j\\
&=|A|.
\end{aligned}$$