

4 Solutions

4.1 Divisor calculus

Problem 1. Some famous theorems.

- (a) Prove the radical axis theorem.
- (b) Prove the Miquel three circle theorem.
- (c) Prove the Miquel four circle theorem.
- (d) Prove the Miquel six circle theorem.
- (e) Prove Pappus's theorem
- (f) Prove Reim's theorem.
- (g) Prove Brokard's theorem.

Solution.

1. (Radical axis) Either

- (a) Start with $(ABCD), (CDEF), (EFAB)$. Pick a cubic passing through $A, B, C, D, E, F, J_1, J_2$, then

$$\begin{cases} A + B + C + D = -J_1 - J_2 \\ C + D + E + F = -J_1 - J_2 \\ E + F + A + B = -J_1 - J_2 \end{cases}$$

so in particular $A + B = C + D = E + F$, and the negative of this lies on AB, CD, EF .

- (b) (essentially Cayley-Bacharach) work with $(ABCD) \cup \overline{EFG}$, then if $(CDEF)$ and $\overline{ABG}$, we must have $A + B + C + D + E + F + G + J_1 + J_2 = 0$, so $(EFAB)$ is equivalent to $\overline{CDG}$.

2. (Miquel three circle) Start with $\overline{AFB}, \overline{BDC}, \overline{CEA}$ and we would like to know that

$$(AEF), (BFD), (CDE)$$

concur. Pick a cubic passing through $A, B, C, D, E, F, J_1, J_2$, then it suffices to show that

$$A + E + F + J_1 + J_2 = B + D + F + J_1 + J_2 = C + E + D + J_1 + J_2.$$

which is true because they are all equal to the cyclic expression $A + (B + C + D) + E + F + J_1 + J_2$. (C-B also works.)

3. (Miquel four circle) Start with $\overline{ABE}, \overline{CDE}, \overline{ADF}, \overline{BCF}$, and we would like to show that $(ABF), (CDF), (ADE), (BCE)$ concur. This is equivalent to

$$-M := A + B + F + J_1 + J_2 = C + D + F + J_1 + J_2 = A + D + E + J_1 + J_2 = B + C + E + J_1 + J_2.$$

which is true by checking that they are all equal to $A + B + (C + D + E) + F + J_1 + J_2$ under quadrilateral symmetry. (C-B also works.)

4. (Miquel six circle) We want to know that 5 of the following circles implies the 6th:

$$(ABCD), (AA'BB'), (BB'CC'), (CC'DD'), (DD'AA'), (A'B'C'D').$$

WLOG we're missing the last one. Pick a cubic passing through $A, B, C, D, A', B', C', J_1, J_2$. In particular, no D' because we don't have space for it, so we first show that $(AA'D)$ and $(A'B'C')$ reintersect on the circle, or that $A' + D = B' + C'$. We know

$$A + B + C + D = A + A' + B + B' = B + B' + C + C'$$

so taking the first term minus the third we get $A + D = B' + C'$. So D' is also on the circle, and finally $(A' + B') + (C' + D') = A + B + C + D = -J_1 - J_2$ so $(A'B'C'D')$ is a circle.

5. (Pappus) Basically equivalent to the fact that because

$$(A + B + C) + (D + E + F) + (G + H + I) = (A + D + G) + (B + E + H) + (C + F + I)$$

any 5 of them being zero implies the 6th is also zero.

6. (Reim) Start with $\overline{ACE}$, $\overline{BDF}$, $(ABCD)$ and $CD//EF$, and we would like to show $(ABEF)$. Pick $\omega \cup EF$ to be the cubic - it passes through $A, B, C, D, E, F, T := \infty_{CD} = \infty_{EF}, J_1, J_2$. It then suffices to show that $A + B + E + F = -J_1 - J_2$. This follows because

$$A + B + E + F = -C - D = T = -J_1 - J_2.$$

7. (Brokard) Let A, B, C, D lie on a circle ω with center O , and let

$$P = AC \cap BD, \quad Q = AB \cap CD, \quad R = AD \cap BC.$$

We will show that P is the pole of QR . Take $\omega \cup QR$ to be the cubic, then we get $A + B = C + D$ and $A + D = B + C$ (because Q and R are on the cubic). Thus, $2A = 2C$, so the tangents at A and C to ω meet on QR . So the by duality the pole of QR lies on AC . Similarly, it lies on BD , so it is precisely P . (It then also follows that O is the orthocenter of PQR .)

Problem 2. (USAMO 2014/3) Prove that there exists an infinite set of points

$$\dots, P_{-3}, P_{-2}, P_{-1}, P_0, P_1, P_2, P_3, \dots$$

in the plane with the following property: For any three distinct integers a, b , and c , points P_a, P_b , and P_c are collinear if and only if $a + b + c = 2014$.

Solution. Pick a nontorsion point Q on a elliptic curve (i.e. $nQ = 0$ is nonzero for $n \neq 0$) and let $P_n = (3n - 2014)Q$.

Note. The elementary version uses instead the graph of a cubic curve $y = x^3 - 2014x^2 + ax + b$, then for every three collinear points their x -coordinates must sum to 2014 by Vieta's relations.

Problem 3. (ISL 2004 G2) Let Γ be a circle with diameter AB , and let ℓ be a line perpendicular to AB , disjoint from Γ , closer to B than to A . Let C be another point on Γ and let AC intersect ℓ at D , and let E be the tangency point of the tangent from D to Γ chosen so that E and B lie on the same side of AC . Let BE intersect ℓ at F . Let G be the second intersection of AF with Γ .

Prove that the reflection of G across AB lies on CF .

Solution. Work in the cubic $\Gamma \cup \ell$. Since $\ell \perp AB$, both GH and the tangent at B passes through ∞_ℓ . Thus,

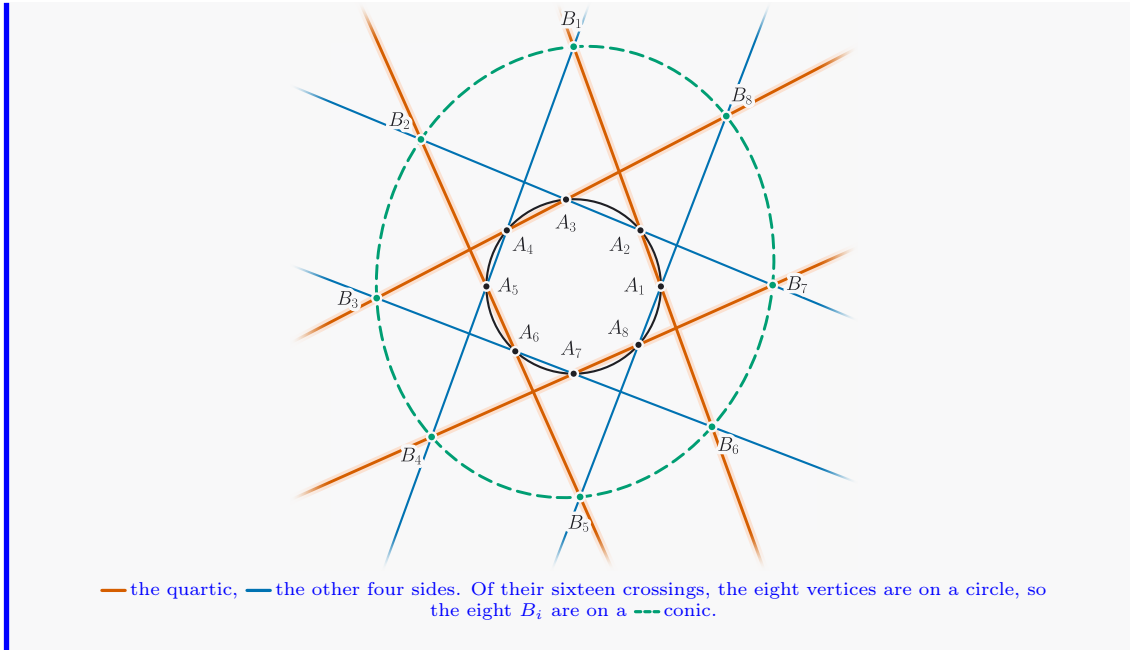
$$C + H = C + A + G + H - (A + G) = 2E + 2B - A + G = B + E = -F.$$

Problem 4. (Octagrammum Mysticum; also ELMO SL 2012/A10) Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be a cyclic octagon. Let B_i be the intersection of A_iA_{i+1} and $A_{i+3}A_{i+4}$. (Take $A_9 = A_1, A_{10} = A_2$, etc.) Prove that $B_1, B_2, \dots, B_8$ lie on a conic.

Solution. Work in the quartic $A_1A_2 \cup A_3A_4 \cup A_5A_6 \cup A_7A_8$. On the remaining sides of the octagon, we have that

$$\begin{cases} A_2 + A_3 + B_2 + B_7 = 0, \\ A_4 + A_5 + B_4 + B_1 = 0, \\ A_6 + A_7 + B_6 + B_3 = 0, \\ A_8 + A_1 + B_8 + B_5 = 0. \end{cases}$$

Adding them gives that the sum of all the A_i 's and B_i 's is zero. But the sum of just the A_i 's is zero (because they are cyclic), so the rest lie on a conic.



Problem 5. (USA TSTST 2021/1) Let $ABCD$ be a quadrilateral inscribed in a circle with center O . Points X and Y lie on sides AB and CD , respectively. Suppose the circumcircles of ADX and BCY meet line XY again at P and Q , respectively. Show that $OP = OQ$.

Solution. Let (ADX) meet AB again at Z , (BCY) meet CD again at W , and let WZ meet (ADX) and (BCY) again at R and S respectively. Work on the quartic $(ABCD) \cup XY \cup WZ$. Then, we have

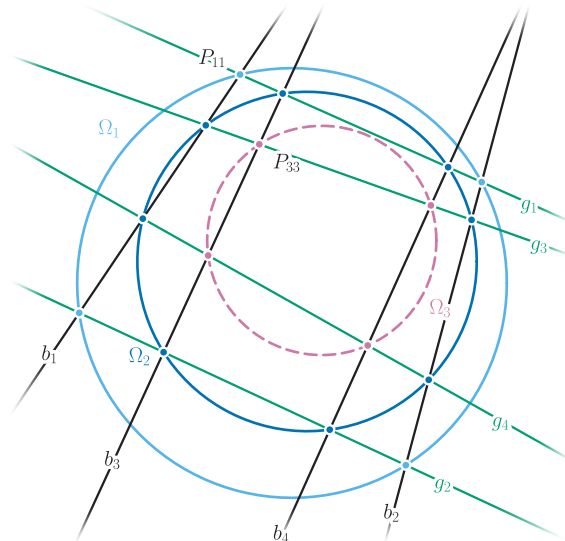
$$\begin{cases} A + D + X + P + Z + R + J_1 + J_2 = 0 \\ B + C + Y + Q + W + S + J_1 + J_2 = 0 \\ A + B + X + W = 0, & C + D + Y + Z = 0 \end{cases}$$

Adding the first two and cancelling out the lines gives

$$P + Q + R + S + 2(J_1 + J_2) = 0.$$

In other words, $(PQRS)$ is doubly tangent to $(ABCD)$ (i.e. they are concentric, so $OP = OQ$).

Problem 6. (Daniel Pham Nguyen) Four green lines g_i and four black lines b_j meet in sixteen points $P_{ij} = g_i \cap b_j$. If the four outer points lie on a circle Ω_1 and the eight middle points lie on a circle Ω_2 , prove that the four inner points lie on a circle Ω_3 .



Redrawn from <https://danielphamnguyen.com/2025/01/30/generalized-cayley-bacharach-theorems/>

Solution. Let me restate the problem. We have lines g_1, g_2, g_3, g_4 and b_1, b_2, b_3, b_4 , with $P_{ij} = g_i \cap b_j$. Order the lines so that the four outer points have $i, j \in \{1, 2\}$, the eight middle points form the two off-diagonal blocks, and the four inner points have $i, j \in \{3, 4\}$.

The outer points lie on a circle Ω_1 , and the eight middle points lie on a circle Ω_2 . We want the inner points to be cyclic.

For the final step, we will need P_{33} to be off the radical axis of Ω_1, Ω_2 . We can arrange this by relabeling the inner lines: the four inner points are not collinear, so they cannot all lie on that axis.

Choose a conic $\mathcal{C}$ through the four outer points and P_{33} . Work throughout on the fixed quartic

$$\mathcal{K} = \Omega_2 \cup \mathcal{C}.$$

Let d_o be the sum of the four outer points, d_1 the sum of the four middle points on $g_1 \cup g_2$, and d_2 the sum of the four middle points on $b_1 \cup b_2$. The outer green and black line pairs give, respectively,

$$d_o + d_1 = 0, \quad d_o + d_2 = 0,$$

so $d_1 = d_2$.

Now the *inner* green lines $g_3 \cup g_4$ meet $\mathcal{C}$ at P_{33} and three further points, whose sum we call t_1 . Similarly, the inner black lines $b_3 \cup b_4$ give a residual triple t_2 . Their sections on $\mathcal{K}$ are

$$d_2 + P_{33} + t_1 = 0, \quad d_1 + P_{33} + t_2 = 0.$$

Hence $t_1 = t_2$. These triples lie on the nonsingular conic $\mathcal{C}$, so they are not collinear. The quartic three-point rule therefore identifies them as actual triples of points. They lie on both inner line pairs, so they must be P_{34}, P_{43}, P_{44} . Thus all four inner points lie on $\mathcal{C}$.

Write $t_3 = P_{34} + P_{43} + P_{44}$. Comparing the outer black and inner green line sections also gives

$$d_o = -d_2 = P_{33} + t_3.$$

If $\mathcal{C}$ is a circle, we are done (in fact, it must be Ω_1). Otherwise, pick the circle Ω_3 through the three points in t_3 , and let R be its fourth intersection with $\mathcal{C}$. Write

$$\Omega_1 \cdot \Omega_2 = J_1 + J_2 + U + V, \quad \Omega_3 \cdot \Omega_2 = J_1 + J_2 + U' + V'.$$

The two circle sections on our fixed quartic give

$$\begin{aligned} d_o + J_1 + J_2 + U + V &= 0, \\ t_3 + R + J_1 + J_2 + U' + V' &= 0. \end{aligned}$$

Substituting $d_o = P_{33} + t_3$ and canceling yields

$$R + U' + V' = P_{33} + U + V.$$

By our choice of P_{33} , the right-hand triple is not collinear: UV is the radical axis of Ω_1, Ω_2 . The quartic rule again identifies the actual triples. Since $P_{33} \notin \Omega_2$ but $U', V' \in \Omega_2$, we get $R = P_{33}$. Therefore Ω_3 contains all four inner points.

Remark. We also get $\{U', V'\} = \{U, V\}$, so the three circles are coaxal. These residual intersections may be complex; for concentric circles, the radical axis is the line at infinity.

Problem 7. (EGMO 2022/6) Let $ABCD$ be a cyclic quadrilateral with circumcenter O . Let the internal angle bisectors at A and B meet at X , the internal angle bisectors at B and C meet at Y , the internal angle bisectors at C and D meet at Z , and the internal angle bisectors at D and A meet at W . Further, let AC and BD meet at P . Suppose that the points X, Y, Z, W, O , and P are distinct.

Prove that O, X, Y, Z, W lie on the same circle if and only if P, X, Y, Z , and W lie on the same circle.

Solution. *Solution written by Opus 5.5, based on my notes.*

Let the internal angle bisectors of A , B , C , and D intersect the circle again at M_A , M_B , M_C , and M_D , respectively. Let S and T be the midpoints of AC and BD , respectively. Note that M_B and M_D are the midpoints of the two arcs AC , so they lie on the perpendicular bisector s of AC . Similarly M_A , M_C lie on the perpendicular bisector t of BD , and $s \cap t = O$.

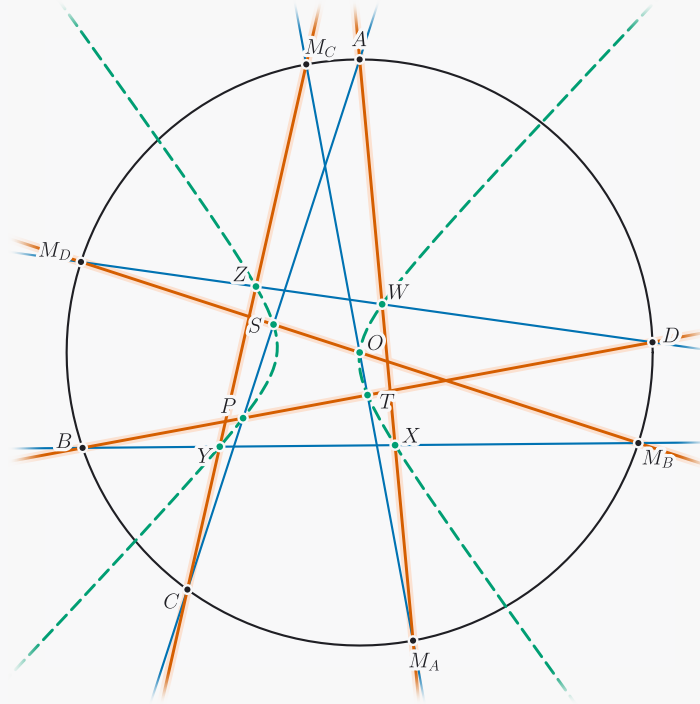
Work in the quartic made of the bisectors at A and C , the line s , and the line BD . The bisector at B , the bisector at D , the line t , and the line AC each meet it four times:

$$\begin{cases} X + Y + M_B + B = 0 \\ W + Z + M_D + D = 0 \\ M_A + M_C + O + T = 0 \\ A + C + S + P = 0 \end{cases}$$

The circle meets the quartic at A, M_A (bisector at A), C, M_C (bisector at C), M_B, M_D (on s) and B, D (on BD), so these eight points sum to zero. Adding the four equations and subtracting this, we get

$$X + Y + Z + W + O + P + S + T = 0,$$

so these eight points lie on a conic (two on each component, as they should be).



The same picture as in the Octagrammum: — the quartic and — the other four lines cross sixteen times, eight times on the circle, so the other eight crossings are on a — conic (here a hyperbola). The problem asks when that conic is a circle.

A conic is determined by five points. So if O, X, Y, Z, W are concyclic, this conic is their circle, and it passes through P as well. The converse is the same with O and P swapped.

Problem 8. (Poncelet's Coaxal Theorem; also CGMO 2017) Let $ABCD$ be a cyclic quadrilateral inscribed in ω . Let points P, Q be on arc AB . Let ω_1 touch AD, BC at S, T and pass through P, Q . Prove that if ST intersects AC, BD at U, V , then (1) U, V, P, Q concyclic and (2) $(UVPQ)$ is tangent to AC, BD .

Solution. Consider the quartic $\mathcal{Q} = \omega_1 \cup AC \cup BD$. We have the following:

$$\begin{cases} A + D + 2S = 0 \\ B + C + 2T = 0 \\ S + T + U + V = 0 \end{cases}$$

which gives $A + B + C + D = 2(U + V)$. The circle ω meets the circle component of $\mathcal{Q}$ at P, Q, J_1, J_2 , and the line components at A, C, B, D , so

$$A + B + C + D + P + Q + J_1 + J_2 = 0$$

Substituting the first into the second, we get

$$P + Q + J_1 + J_2 + 2(U + V) = 0$$

so these 8 points are on a conic, which we interpret as (1) P, Q, U, V being concyclic (since J_1, J_2 are on the conic) and (2) $(UVPQ)$ is tangent to AC at U and BD at V (due to the multiplicities).

4.2 Cubic curves

Solutions in this section written by Opus 5.5, based on my notes.

4.2.1 Exercises/Folklore

Problem 1. (folklore) Show that the circular points at infinity are isogonal conjugates with respect to any triangle.

Solution. At each vertex, isogonal conjugation reflects lines in the angle bisector. A reflection swaps J_1 and J_2 (up to a rotation it is $(x, y) \mapsto (x, -y)$, which sends $[1 : i : 0]$ to $[1 : -i : 0]$). So the reflection in the bisector at A swaps the lines AJ_1 and AJ_2 , and the same happens at B and C . The isogonals of AJ_1, BJ_1, CJ_1 are AJ_2, BJ_2, CJ_2 , which concur at J_2 .

In cross ratio language: if ℓ, ℓ' are the internal and external bisectors at A , the reflection in ℓ is the involution on lines through A that fixes ℓ and ℓ' , so it swaps AJ_1 and AJ_2 iff $(\ell, \ell'; AJ_1, AJ_2) = -1$, i.e. $(\infty_\ell, \infty_{\ell'}; J_1, J_2) = -1$. This holds because $\ell \perp \ell'$: the directions $[1 : m : 0]$ and $[1 : -1/m : 0]$ are always harmonic with respect to $[1 : \pm i : 0]$.

Problem 2. (Coharmonic hexagons) This is a generalization of the harmonic quadrilateral.

1. Suppose that $\angle A + \angle C + \angle E = 360^\circ$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

Show that there exists a point M such that

$$\triangle MAB \sim \triangle MED,$$

$$\triangle MBC \sim \triangle MFE,$$

$$\triangle MCD \sim \triangle MAF.$$

Solution. Use complex numbers, with the vertices labelled counterclockwise. The angle at A is $\arg \frac{f-a}{b-a}$, and similarly at C and E . So the angle condition says that

$$\frac{f-a}{b-a} \cdot \frac{b-c}{d-c} \cdot \frac{d-e}{f-e}$$

is a positive real number, and the length condition says its absolute value is 1: it equals 1. Rearranging,

$$\frac{a-b}{b-c} \cdot \frac{c-d}{d-e} \cdot \frac{e-f}{f-a} = -1.$$

This is the condition of Fact 15, now over $\mathbb{C}$, and the same proof works (send A, D to $0, \infty$). So there is a Möbius involution φ with $\varphi(a) = d$, $\varphi(b) = e$, $\varphi(c) = f$.

Let $M = \varphi(\infty)$, so $\varphi(M) = \infty$. With M as the origin, φ is $z \mapsto k/z$ for a constant k : an involution (invert, then reflect). Hence

$$ad = be = cf = k.$$

Now $\frac{a}{b} = \frac{e}{d}$ says $\triangle MAB \sim \triangle MED$, $\frac{b}{c} = \frac{f}{e}$ says $\triangle MBC \sim \triangle MFE$, and $\frac{c}{d} = \frac{a}{f}$ says $\triangle MCD \sim \triangle MAF$. (If φ fixes ∞ , the hexagon is centrally symmetric and M runs off to infinity.)

Problem 3. (Isogonal conjugates of a quadrilateral)

1. Show that in a quadrilateral $ABCD$, a point P has an isogonal conjugate (“ Q ”) if and only if $\angle(AP, PB) + \angle(CP, PD) = 0$ (with directed angles).
2. Show that the locus of such points is a cubic.
3. Show that $ABPCDQ$ is coharmonic.
4. Show that the midpoints of AC, BD, PQ are collinear.
5. Prove Fact 10.

Solution.

1. We use the pedal circle lemma at one vertex. Let V be a vertex with sides v_1, v_2 and let X, Y be two points. Then VX, VY are isogonal iff the feet of X, Y on v_1, v_2 are concyclic, and then the circle is centered at the midpoint of XY . (The feet X_i, Y_i on v_i satisfy $VX_i \cdot VY_i = VX \cdot VY \cos \angle(VX, v_i) \cos \angle(VY, v_i)$. The two products agree exactly when VX, VY are isogonal, and that is power of a point. The perpendicular bisectors of X_1Y_1 and X_2Y_2 pass through the midpoint of XY .)

If P has an isogonal conjugate Q , then at every vertex the four feet lie on a circle centered at the midpoint of PQ . Adjacent vertices share a side, so it’s the same circle every time, and the four feet of P are concyclic. Conversely, if the feet of P lie on a circle ω , let Q be the reflection of P in its center. The feet of Q are the second intersections of ω with the sides, so by the lemma Q is isogonal to P at every vertex.

So it remains to show that the condition holds iff the feet P_1, P_2, P_3, P_4 on AB, BC, CD, DA are concyclic. Using the circles with diameters PA, PB, PC, PD ,

$$\angle(P_1P_2, P_1P_4) = \angle(BC, BP) + \angle(AP, AD), \quad \angle(P_3P_2, P_3P_4) = \angle(CB, CP) + \angle(DP, DA).$$

These are equal iff $\angle(CP, BP) + \angle(AP, DP) = 0$, which rearranges to the condition.

2. The condition says that $\frac{(p-a)(p-c)}{(p-b)(p-d)}$ is real, i.e.

$$(p-a)(p-c)(\bar{p}-\bar{b})(\bar{p}-\bar{d}) = (\bar{p}-\bar{a})(\bar{p}-\bar{c})(p-b)(p-d).$$

The $p^2\bar{p}^2$ terms cancel, so this is a polynomial equation of degree 3 in $p, \bar{p}$, and hence in x, y . (Its degree 3 part is $p\bar{p} \cdot (\text{linear})$, which is why the cubic passes through J_1, J_2 .)

3. Let φ be the involution swapping $A \leftrightarrow C$ and $B \leftrightarrow D$. Its center is the Miquel point of $ABCD$. After a similarity, φ is $z \mapsto 1/z$, so $c = 1/a$ and $d = 1/b$. For each point write

$$\hat{z} = z + \frac{1}{z},$$

so $\hat{z}/2$ is the midpoint of z and $\varphi(z)$. Since $(z-a)(z-\frac{1}{a}) = z(\hat{z}-\hat{a})$ and $(z-a)(\frac{1}{z}-a) = a(\hat{a}-\hat{z})$, we get for every p :

$$\frac{(p-a)(p-c)}{(p-b)(p-d)} = \frac{\hat{p}-\hat{a}}{\hat{p}-\hat{b}}, \quad \frac{(p-a)(\frac{1}{p}-a)}{(b-a)(d-a)} = \frac{\hat{a}-\hat{p}}{\hat{a}-\hat{b}}.$$

The first identity says that P is on the cubic iff $\hat{p}, \hat{a}, \hat{b}$ are collinear. The second says that AP and $A\varphi(P)$ are isogonal at A (both sides are real) iff $\hat{p}, \hat{a}, \hat{b}$ are collinear. The same computation works at B, C, D ; at C , replacing a by $1/a$ doesn’t change $\hat{a}$. So for P on the cubic, its isogonal conjugate is $Q = \varphi(P)$: isogonal conjugation is just φ restricted to the cubic. Since φ swaps the opposite vertices $A \leftrightarrow C, B \leftrightarrow D, P \leftrightarrow Q$ of $ABPCDQ$, this hexagon is coharmonic (Exercise 2 in reverse).

4. The midpoints of AC, BD, PQ are $\hat{a}/2, \hat{b}/2, \hat{p}/2$, and we just saw that they are collinear. (Taking the conjugate pair $AB \cap CD, AD \cap BC$ shows that the line is the Newton–Gauss line.)

5. Let A', B' be on the cubic, with $C' = \varphi(A')$ and $D' = \varphi(B')$. Then φ is also the involution swapping $A' \leftrightarrow C', B' \leftrightarrow D'$, so the same coordinates work for $A'B'C'D'$. By part 3, the cubic of $A'B'C'D'$ is the set of p with $\hat{p}$ on the line $\hat{a}'\hat{b}'$, and its isogonal conjugation is φ . But $\hat{a}'$ and $\hat{b}'$ lie on the line $\hat{a}\hat{b}$, because A', B' are on the cubic of $ABCD$. So the two lines coincide: both quadrilaterals have the same cubic, and on it both conjugations are φ .

4.2.2 Problems

Problem 1. (IMO 2019/6) Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA , and AB at D, E , and F , respectively. The line through D perpendicular to EF meets ω at R . Line AR meets ω again at P . The circumcircles of triangle PCE and PBF meet again at Q .

Prove that lines DI and PQ meet on the line through A perpendicular to AI .

Solution. Let T be the intersection of DI and the line through A perpendicular to AI ; we want P, Q, T collinear. Since $EF \perp AI$, we have $AT \parallel EF$.

First, A, T, P, D are concyclic. Indeed, $RD \perp AT$ and $BC \perp DT$, and BC is the tangent to the incircle at D , so

$$\angle(PA, PD) = \angle(PR, PD) = \angle(DR, BC) = \angle(TA, TD).$$

Now pick a cubic passing through $B, C, D, E, F, I, T, J_1, J_2$ (this is the curve drawn in Section 1.1). The circles $(BDIF)$ and $(CDIE)$ (with diameters BI and CI) give

$$B + D + I + F + J_1 + J_2 = C + D + I + E + J_1 + J_2 = 0,$$

so $B + F = C + E$. Thus BF and CE meet on the cubic: A is on it, with $A = -(C + E)$. Then

$$A + T = -(C + E) - (D + I) = J_1 + J_2,$$

so the third point of AT is the third point of the line at infinity, i.e. ∞_{AT} . Since $EF \parallel AT$, this is also the third point of EF , so $A + T = E + F$.

So the circles (ATD) and (DEF) meet the cubic again at the same point, which must be their other intersection P . Similarly (PCE) and (PBF) meet it again at the same point (as $C + E = B + F$), which must be Q . Finally,

$$P + Q = -(C + E + J_1 + J_2) = D + I = -T,$$

so P, Q, T are collinear.

Problem 2. (Romania TST 2017/3/4) Let $ABCD$ be a convex quadrilateral and let P and Q be variable points inside this quadrilateral so that $\angle APB = \angle CPD = \angle AQB = \angle CQD$. Prove that the lines PQ obtained in this way all pass through a fixed point, or they are all parallel.

Solution. With directed angles, $\angle APB = \angle CPD$ says $\angle(AP, BP) = \angle(CP, DP)$. So P lies on the isoptic cubic $\mathcal{K} = \mathcal{C}(ABDC)$, which is circular, and so does Q . Since $\angle APB = \angle AQB$ and P, Q are on the same side of AB , the points A, B, P, Q are concyclic, so on $\mathcal{K}$

$$A + B + P + Q + J_1 + J_2 = 0.$$

Hence the third point of PQ on $\mathcal{K}$ is $-(P + Q) = A + B + J_1 + J_2$, which doesn't depend on P and Q . So every line PQ passes through this fixed point of $\mathcal{K}$ (or they are all parallel, if it's at infinity).

Problem 3. (Sharygin 2024 II/9.3) Let (P, P') and (Q, Q') be two pairs of points isogonally conjugated with respect to a triangle ABC , and R be the common point of lines PQ and $P'Q'$. Prove that the pedal circles of points P, Q , and R are coaxial.

Solution. Write ω_X for the pedal circle of X . The useful cubics are the loci where ω_X passes through a fixed point.

Isogonal conjugates share a pedal circle, and ω_P has the same center as the circle on diameter PP' , namely the midpoint of PP' . So their power functions differ by a constant. Comparing them at the foot P_a of P on BC , where $(P_a - P) \cdot (P_a - P') = \delta(P) \delta(P')$, gives

$$\text{pow}(U, \omega_P) = (U - P) \cdot (U - P') - \delta(P) \delta(P'),$$

where δ is the signed distance to BC . The right side is linear in P and in P' separately, and P' has degree 2 in the coordinates of P . So, just like the line PP' for pivotal isocubics (Section 2.4), it has degree 3 in the coordinates of P , and for a fixed point U the locus $\mathcal{K}_U = \{X : U \in \omega_X\}$ is a cubic. It is invariant under isogonal conjugation, and it isn't pivotal: otherwise it would contain I, I_A, I_B, I_C , and U would lie on the incircle and all three excircles. So conjugation acts on $\mathcal{K}_U$ as $X \mapsto X + d$.

Now let ω_P and ω_Q meet at U and V (possibly imaginary). Then $P, P', Q, Q' \in \mathcal{K}_U$, and on $\mathcal{K}_U$

$$-(P' + Q') = -(P + Q) - 2d = -(P + Q).$$

So PQ and $P'Q'$ meet $\mathcal{K}_U$ again at the same point, which must be R . Hence $U \in \omega_R$. Likewise $V \in \omega_R$, so $\omega_P, \omega_Q, \omega_R$ are coaxial.

Remark. A cubic-free proof: let R_a be the foot of R on BC . Projecting the lines RPQ and $RP'Q'$ onto BC gives

$$\frac{\text{pow}(R_a, \omega_P)}{\text{pow}(R_a, \omega_Q)} = \frac{R_a P_a \cdot R_a P'_a}{R_a Q_a \cdot R_a Q'_a} = \frac{RP \cdot RP'}{RQ \cdot RQ'},$$

and the feet on the other two sides give the same ratio. So the pedal triangle of R lies on a circle of the pencil of ω_P and ω_Q , and that circle is ω_R .

Problem 4. (USA TST 2016/3) Let ABC be an acute scalene triangle and let P be a point in its interior. Let A_1, B_1, C_1 be projections of P onto triangle sides BC, CA, AB , respectively. Find the locus of points P such that AA_1, BB_1, CC_1 are concurrent and $\angle PAB + \angle PBC + \angle PCA = 90^\circ$.

Solution. The answer is $P \in \{I, O, H\}$. Let P' be the isogonal conjugate of P .

The first condition says that the pedal triangle of P is a cevian triangle, so P is on the Darboux cubic, the pivotal isocubic with pivot the de Longchamps point L .

The second condition puts P on the McCay cubic, the pivotal isocubic with pivot O . To see this, put the circumcircle on the unit circle. The condition $\angle PAB + \angle PBC + \angle PCA \equiv 90^\circ$ says that $\frac{(p-a)(p-b)(p-c)}{(b-a)(c-b)(a-c)}$ is purely imaginary, i.e.

$$(p-a)(p-b)(p-c) = (abc)^2(\bar{p}-\bar{a})(\bar{p}-\bar{b})(\bar{p}-\bar{c}).$$

On the other hand, the isogonal conjugate is

$$p' = \frac{a+b+c-p-(ab+bc+ca)\bar{p}+abc\bar{p}^2}{1-p\bar{p}},$$

and expanding $\bar{p}p' = p\bar{p}'$ (that is, O, P, P' collinear) gives exactly the same equation. For P inside the triangle the angle sum is between 0° and 180° , so “ $\equiv 90^\circ$ ” is the same as “ $= 90^\circ$ ”.

Now count. (The picture in Section 2.5 shows both cubics: — Darboux and — McCay.) Both cubics are pivotal isocubics, so both pass through $A, B, C, I, I_A, I_B, I_C$. Both also pass through O and H : they're isogonal conjugates, and the Euler line OH passes through both pivots O and L . That's nine common points. Two cubics without a common component meet only nine times (for a scalene triangle both are irreducible), so these are all the common points. The ones inside the triangle are I, O and H . (As a sanity check, their pedal triangles are the contact, medial and orthic triangles, and their angle sums are $\frac{A}{2} + \frac{B}{2} + \frac{C}{2}$, $(90^\circ - C) + (90^\circ - A) + (90^\circ - B)$ and $(90^\circ - B) + (90^\circ - C) + (90^\circ - A)$.)

Problem 5. (USAMO 2023/6) Let ABC be a triangle with incenter I and excenters I_a, I_b , and I_c opposite A, B , and C , respectively. Let D be an arbitrary point on the circumcircle of $\triangle ABC$ that does not lie on any of the lines II_a, I_bI_c , or BC . Suppose the circumcircles of $\triangle DII_a$ and $\triangle DI_bI_c$ intersect at two distinct points D and F . If E is the intersection of lines DF and BC ,

prove that $\angle BAD = \angle EAC$

Solution. Redefine E as the point on BC with $\angle BAD = \angle EAC$. Then we just need to show that D, E, F are collinear.

It's natural to want D on the cubic as well as the circular points. So take the pivotal isocubic $\mathcal{K}$ of ABC whose pivot is the isogonal conjugate D' of D . Since D is on the circumcircle, D' is at infinity, and since AD, AE are isogonal, $D' = \infty_{AE}$. So $\mathcal{K}$ passes through $A, B, C, I, I_a, I_b, I_c, D, D'$, and also through J_1, J_2 , because they are conjugates and $J_1 J_2$ passes through the pivot. The point E is on $\mathcal{K}$ too: by the pivotal relation $B + C = A + D'$, the lines BC and $AD' = AE$ have the same third point, which is $E = -(B + C) = -(A + D')$.

Now $I + I_a = -A = I_b + I_c$ (the lines $AI I_a$ and $AI_b I_c$), so the circles (DII_a) and $(DI_b I_c)$ meet $\mathcal{K}$ again at the same point. That point must be their second intersection F . Using $J_1 + J_2 = -D'$ (the line at infinity),

$$F = -(D + I + I_a + J_1 + J_2) = -(D - A - D') = -(D + E).$$

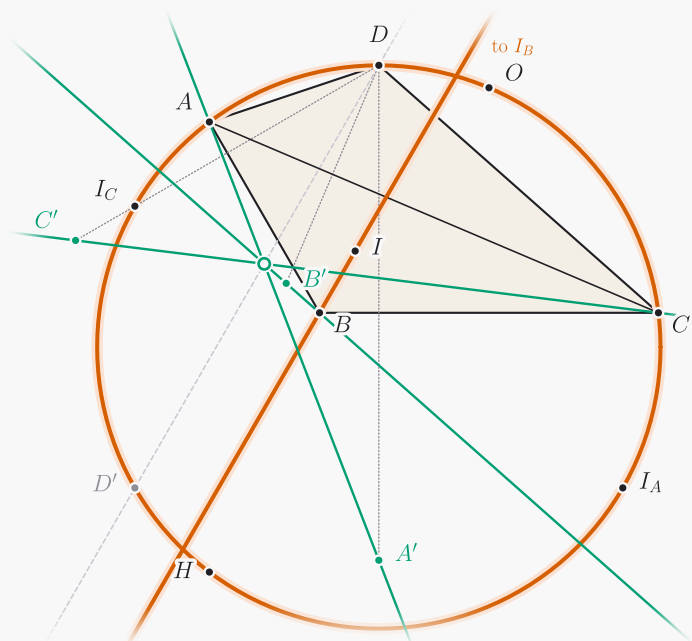
So D, E, F are collinear.

Problem 6. (239 Olympiad 2010/9.7) In a convex quadrilateral $ABCD$, We have $\angle B = \angle D = 120^\circ$. Points A', B' and C' are symmetric to D relative to BC, CA and AB , respectively. Prove that lines AA', BB' and CC' are concurrent.

Solution. A', B', C' are the reflections of D in the sides of ABC , so we need D to be on the Neuberg cubic $\mathcal{N}$ of ABC .

Let Σ be the circle with diameter $I_A I_C$. It passes through A and C , since the internal and external bisectors at A are perpendicular, and likewise at C . Also $\angle AI_A C = \frac{1}{2} \angle B = 60^\circ$, and I_A is on the same side of AC as B . So the arc of Σ on the other side of AC is exactly the locus of points X with $\angle AXC = 120^\circ$, and $D \in \Sigma$. So is the circumcenter O : since $\angle B = 120^\circ$, the point O is on the other side of AC , and $\angle AOC = 360^\circ - 2\angle B = 120^\circ$.

So Σ and $\mathcal{N}$ share the seven points $A, C, I_A, I_C, O, J_1, J_2$. A conic and a cubic without common components meet only six times, so Σ is a component of $\mathcal{N}$. In particular $D \in \mathcal{N}$, so AA', BB', CC' concur.



When $\angle B = 120^\circ$, the Neuberg cubic of ABC is the circle on diameter $I_A I_C$ together with the bisector of angle B (through I and I_B). Every D with $\angle ADC = 120^\circ$ is on the circle, and AA', BB', CC' meet on DD' .

Problem 7. (ISL 2024 G5) Let ABC be a triangle with incentre I , and let Ω be the circumcircle

of triangle BIC . Let K be a point in the interior of segment BC such that $\angle BAK < \angle KAC$. The angle bisector of $\angle BKA$ intersects Ω at points W and X such that A and W lie on the same side of BC , and the angle bisector of $\angle CK A$ intersects Ω at points Y and Z such that A and Y lie on the same side of BC .

Prove that $\angle WAY = \angle ZAX$.

Solution. The point is that A lies on a “product of distances” locus, and that locus is an isoptic cubic. This is the generalisation from the shortlist comments.

Lemma. Let B, C, R, S lie on a circle Ω with center O_Ω and radius ρ , and let $K = BC \cap RS$. Let the two bisectors of the angles between BC and RS meet Ω at W, X and at Y, Z . Then

$$\mathcal{M} = \{P : PB \cdot PC = PR \cdot PS\}$$

is the isoptic cubic $\mathcal{K} = \mathcal{C}(WYXZ)$, the locus of P with $\angle(WP, YP) = \angle(ZP, XP)$.

Proof. In $|P - B|^2|P - C|^2 = |P - R|^2|P - S|^2$ the $|P|^4$ terms cancel, and the cubic terms are $-2|P|^2\langle P, B + C - R - S \rangle$. So $\mathcal{M}$ is a circular cubic, and its real point at infinity is perpendicular to the line $M_{BC}M_{RS}$ through the midpoints of BC and RS . It shares these points with $\mathcal{K}$:

- W, X, Y, Z . For $P \in \Omega$, computing $2[PBC]$ two ways gives $PB \cdot PC = 2\rho d(P, BC)$, and points on the bisectors are equidistant from BC and RS .
- K . It is on $\mathcal{M}$ by power of a point, and on $\mathcal{K}$ because $WX \perp YZ$.
- O_Ω . All four distances are ρ . On $\mathcal{K}$ it is the conjugate of K : since $WX \perp YZ$, at each vertex of $WYXZ$ the diagonal is the altitude of the triangle formed by that vertex and its two neighbours, and altitudes are isogonal to circumradii. So $O_\Omega = K + d$, where $d = W - X = Y - Z$.
- J_1, J_2 and the real point at infinity ∞ . Far out, $\frac{(p-y)(p-z)}{(p-w)(p-x)} = 1 + \frac{w+x-y-z}{p} + O(p^{-2})$, so the real asymptote of $\mathcal{K}$ is parallel to $M_{WX}M_{YZ}$. The four midpoints lie on the circle with diameter $O_\Omega K$. On it, M_{WX} and M_{YZ} are the midpoints of the two arcs $M_{BC}M_{RS}$, because KW and KY bisect the angles between BC and RS . So $M_{WX}M_{YZ} \perp M_{BC}M_{RS}$.

On $\mathcal{K}$, $W + X = Y + Z = -K$, so $J_1 + J_2 = -(W + X + Y + Z) = 2K$ and $\infty = -(J_1 + J_2) = -2K$. So the nine common points sum to

$$(W + X + Y + Z + J_1 + J_2) + K + O_\Omega + \infty = 0 + K + (K + d) - 2K = d \neq 0.$$

If $\mathcal{M}$ were a different cubic, these nine points would be all of $\mathcal{M} \cap \mathcal{K}$, which sums to 0. So $\mathcal{M} = \mathcal{K}$. ■

Now the problem. Let AK meet Ω at R, S . Then $K = BC \cap RS$, and W, X, Y, Z are the points of the lemma. The center of Ω is on AI , so the reflection F of B in AI lies on Ω and on AC , with $AF = AB$. By power of a point,

$$AB \cdot AC = AF \cdot AC = AR \cdot AS.$$

So $A \in \mathcal{M} = \mathcal{K}$, which says $\angle(WA, YA) = \angle(ZA, XA)$. With the given configuration this is $\angle WAY = \angle ZAX$.

Remark. The lemma also has a short coordinate proof. Take Ω to be the unit circle, $f(p) = (p - b)(p - c)$ and $g(p) = (p - r)(p - s)$. A chord uv passes through K iff $u + v = k + uv\bar{k}$, i.e. iff $(p - u)(p - v) = (p^2 - kp) + uv(1 - \bar{k}p)$. So $f, g, (p - w)(p - x)$ and $(p - y)(p - z)$ all lie in a 2-dimensional space, and

$$\frac{(p - w)(p - x)}{(p - y)(p - z)} = c \cdot \frac{u - \lambda}{u - \mu}, \quad u = \frac{f(p)}{g(p)}, \quad \lambda = \frac{f(w)}{g(w)}, \quad \mu = \frac{f(y)}{g(y)}.$$

Since $W, Y \in \mathcal{M}$, $|\lambda| = |\mu| = 1$, so this Möbius map sends the circle $|u| = 1$ to a line through 0. At $p = 0$ we have $|u| = |bc/rs| = 1$, and the left side is $wx/yz = -1$ because $WX \perp YZ$. So that line is $\mathbb{R}$, and $|f(p)| = |g(p)|$ iff $\frac{(p - w)(p - x)}{(p - y)(p - z)} \in \mathbb{R}$, which is the equation of $\mathcal{K}$.

Problem 8. (Taiwan TST 2018/2/3) Given six points A, B, C, D, E, F such that

$$\triangle BCD \stackrel{+}{\sim} \triangle ECA \stackrel{+}{\sim} \triangle BFA$$

and let I be the incenter of $\triangle ABC$. Prove that the circumcenter of $\triangle AID, \triangle BIE, \triangle CIF$ are collinear.

Solution. AD, BE, CF concur, for synthetic reasons. From $\triangle BCD \sim \triangle ECA$, the spiral similarity at C taking $B \mapsto D$ also takes $E \mapsto A$. So BE and DA meet at a point of (CBD) and of (CEA) . Similarly, from $\triangle BCD \sim \triangle BFA$, the spiral similarity at B taking $C \mapsto D$ takes $F \mapsto A$, so CF and DA meet at a point of (BCD) and of (BFA) . Both points are the second intersection of AD with (BCD) , so AD, BE, CF concur at a point X , and

$$X \in (BCD) \cap (CEA) \cap (AFB).$$

Let $\mathcal{K}$ be the circular pivotal isocubic of ABC through X . Its pivot is at infinity, in the direction of XX' , and it passes through I, J_1, J_2 . Then D is on $\mathcal{K}$. Indeed, the line AX and the circle (BXC) meet $\mathcal{K}$ again at $-(A+X)$ and $-(B+X+C+J_1+J_2)$. These are equal, because $A = B+C+J_1+J_2$ on a circular pivotal isocubic. The common point is on both AX and (BXC) , so it is D . Similarly E, F are on $\mathcal{K}$, and

$$A + D = B + E = C + F = -X.$$

So the circles $(AID), (BIE), (CIF)$ meet $\mathcal{K}$ again at the same point $-(A + D + I + J_1 + J_2)$. All three circles pass through I and this point, so they are coaxial, and their centers are collinear.