Introduction

Let $f : \mathbb{R} \to \mathbb{C}$ be a function with period 1 (i.e. $f(x+1) = f(x)$ for all real x). We would like represent f as a sum of exponential functions:

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi inx}.$$

What should the coefficients be? Note that for any integer n,

$$\int_0^1 e^{2\pi inx}\,dx = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0, \end{cases}$$

so by working formally (and ignoring convergence issues), we can recover a single coefficient from f by integrating against a suitable exponential:

$$\begin{aligned}\int_0^1 f(x)e^{-2\pi inx}\,dx &= \int_0^1 \left(\sum_{m\in\mathbb{Z}} c_m e^{-2\pi i(m-n)x}\right) dx \\ &= \sum_{m\in\mathbb{Z}} \int_0^1 e^{-2\pi i(m-n)x}\,dx \\ &= c_n.\end{aligned}$$

However, it is possible that the sum of exponentials with coefficients $c_n(f) := \int_0^1 f(x)e^{-2\pi inx}\,dx$ does not converge to f. The main theorem we will prove here gives a criterion for pointwise convergence, as well as a criterion for term-by-term differentiation to be legitimate.

Theorem 1 (5.2.1(a),(b) in the course text)

Let $f : \mathbb{R} \to \mathbb{C}$ have period 1 and be continuous. Define the *Fourier coefficients* c_n of f by the formula

$$c_n = \int_0^1 f(x)e^{-2\pi inx}\,dx.$$

(a) We have $\sum_{n\in\mathbb{Z}} |c_n|^2 \leq \int_0^1 |f(x)|^2 dx$, so $c_n \to 0$, and if $\sum_{n\in\mathbb{Z}} |c_n|$ converges then

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi inx}$$

for all x and the partial sums converge uniformly to f.

(b) If for an integer $k \geq 2$, $c_n = O(1/n^k)$ as $n \to \infty$, then f is continuously differentiable $k-2$ times and the corresponding termwise derivatives of $\sum_{n\in\mathbb{Z}} c_n e^{2\pi inx}$ converges uniformly to the higher derivatives of f.

Overall approach.

For (a), the inequality

$$\sum_{n\in\mathbb{Z}} |c_n|^2 \leq \int_0^1 |f(x)|^2 dx$$

is best interpreted by placing f and $e^{2\pi inx}$ in the vector space of period 1 continuous functions. With an appropriate definition of inner product, the right side is the norm of f while c_n corresponds to the projection of f onto $e^{2\pi inx}$.

When $\sum_{n\in\mathbb{Z}} |c_n|$ converges, the series $g(x) = \sum_{n\in Z} c_n e^{2\pi inx}$ converges uniformly. It is easy to check that g has the same Fourier coefficients (the c_n's) as f, and because a continuous function is uniquely determined by its Fourier coefficients (Lemma 3) we conclude that $f = g$.

Part (b) follows by an inductive argument. The case $k = 2$ is given by part (a), while for $k > 2$, we simply note that the derivatives of the partial sums

$$f_N'(x) = \left(\sum_{n=-N}^{N} c_n e^{2\pi inx}\right)' = \sum_{n=-N}^{N} (2\pi i)nc_n e^{2\pi inx}$$

converges uniformly by part (a) (since $2\pi inc_n = O(1/n^{k-1}) = O(1/n^2)$), and using a lemma from calculus it follows that we can interchange the derivative and the limit (Lemma 5):

$$f'(x) = \lim_{N\to\infty} f_N'(x)$$

and from here we apply the inductive hypothesis for $k-1$ on f'.

Proofs

Proof of Theorem 1(a). Firstly, let V be the set of all continuous $\mathbb{C}$-valued functions on $\mathbb{R}$ with period 1. This is a $\mathbb{C}$-vector space by pointwise addition and scalar multiplication. In particular, if we also define $\omega_n(x) = e^{2\pi inx}$ for all integers n, then $\omega_n \in V$ for all integers n.

For $f, g \in V$, define

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx.$$

It is easy to verify that $\langle\cdot,\cdot\rangle$ is a Hermitian product on V (i.e. it is conjugate symmetric and linear in the first component). Note that the Fourier coefficients can be expressed simply as

$$c_n = \int_0^1 f(x)e^{-2\pi inx}\,dx = \langle f, \omega_n\rangle$$

and furthermore,

$$\langle\omega_m, \omega_n\rangle = \int_0^1 e^{2\pi i(m-n)x}\,dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

so the set $\{\omega_n\}_{n\in\mathbb{Z}}$ is an orthonormal set in V.

Applying Lemma 2 below for any $f \in V$ and the orthonormal set $\{\omega_{-N}, \omega_{-N+1}, ..., \omega_N\}$ we conclude that

$$\sum_{n=-N}^{N} |c_j|^2 \le \|f\| = \int_0^1 |f(x)|^2\,dx$$

so taking the limit $N \to \infty$ we conclude that $\sum_{n\in\mathbb{Z}} |c_j|^2 \le \int_0^1 |f(x)|^2\,dx$. In particular, $|c_n|^2 \to 0$ as $n \to \infty$, so $c_n \to 0$ as $n \to \infty$.

With the additional assumption that $\sum_{n\in\mathbb{Z}} |c_n|$ converges, we can apply Lemma 4 below to obtain uniform convergence for

$$g(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi inx}.$$

Finally, we must show that $f = g$. First we verify that g has the same Fourier coefficients as f. Because the sum in g uniformly converges, we may swap the order of integration and summation:

$$\begin{aligned}\int_0^1 g(x)e^{-2\pi inx}\,dx &= \int_0^1 \left(\sum_{m\in\mathbb{Z}} c_m e^{2\pi i(m-n)x}\right)dx \\ &= \sum_{m\in\mathbb{Z}} \int_0^1 c_m e^{2\pi i(m-n)}\,dx \\ &= c_m.\end{aligned}$$

Now we apply Lemma 3 below to conclude that $f = g$. ☐

Proof of Theorem 1(b). We proceed by induction on k.

The base case $k = 2$ is trivial, since by assumption f is continuous.

Suppose the statement is true for $k = m \geq 2$. When $k = m + 1 \geq 3$, we aim to show that f is differentiable, the derivative f' is continuous, and the derivative has Fourier coefficients that are $O(1/n^{k-1})$, allowing us to invoke the inductive hypothesis.

Define the partial sums

$$f_{M,N}(x) = \sum_{n=-M}^{N} c_n e^{2\pi i n x}$$

with derivatives

$$f'_{M,N}(x) = \sum_{n=-M}^{N} (2\pi i n) c_n e^{2\pi i n x}.$$

By assumption, $(2\pi i n)c_n = O(1/n^m) = O(1/n^2)$, so $\sum_{n\in\mathbb{Z}} |(2\pi i n)c_n|$ converges. Applying Lemma 4, we obtain a continuous, period 1 function

$$g(x) = \sum_{n\in\mathbb{Z}} (2\pi i n) c_n e^{2\pi i n x}$$

where the convergence $f'_N \to g$ is uniform over x. The preservation of continuity under uniform limits gives that g is continuous. By Lemma 5 on $\{f_N\}$, we get $f' = g$, so f is continuously differentiable.

However, the Fourier coefficients of g are $(2\pi i n)c_n = O(1/n^m)$, so by applying the case $k = m$ on f', we see that f' is continuously differentiable $m - 2$ times and the corresponding termwise ℓ-th derivatives converges uniformly to $f^{(\ell)}$ for $0 \leq \ell \leq m - 2$. Hence, f' must be continuously differentiable $m - 1$ times with the corresponding termwise ℓ-th derivatives converging uniformly to $f^{(\ell)}$ for $0 \leq \ell \leq m - 1$.

By induction, the statement is thus true for all integers $k \geq 2$. ☐

Lemmata

Lemma 2

In a $\mathbb{C}$-vector space V with an Hermitian inner product $\langle \cdot, \cdot \rangle$, suppose that $\{e_1, e_2, ..., e_m\}$ is a set of orthonormal vectors in V. Then for any $v \in V$, the following inequality holds:

$$\sum_{j=1}^{m} |\langle v, e_j \rangle|^2 \leq \|v\|^2.$$

Proof. This is essentially an analogue of Pythagoras' Theorem. Define

$$v_j = \langle v, e_j \rangle e_j; \qquad w = v - \sum_{j=1}^{m} v_j.$$

Then v is the sum of orthogonal vectors $v_1, v_2, ..., v_m, w$, and we claim that the sum of squares of their norms is $\|v\|^2$.

We can check this algebraically by expanding out $\|w\|^2$:

$$\begin{aligned}\|w\|^2 &= \|v\|^2 - \sum_{j=1}^{n} (\langle v, v_j\rangle + \langle v_j, v\rangle) + \sum_{j,k=1}^{n} \langle v_i, v_j\rangle \\ &= \|v\|^2 - \sum_{j=1}^{n} \left(\overline{\langle v, e_j\rangle}\langle v, e_j\rangle + \langle v, e_j\rangle\langle e_j, v\rangle\right) + \sum_{j=1}^{n} \langle v_j, v_j\rangle \\ &= \|v\|^2 - \sum_{j=1}^{n} |\langle v, e_j\rangle|^2.\end{aligned}$$

□

Lemma 3

Let $f, g : \mathbb{R} \to \mathbb{C}$ be continuous, period 1 functions. Suppose f and g have the same Fourier coefficients, i.e.

$$\int_0^1 f(x)e^{-2\pi inx}\,dx = \int_0^1 g(x)e^{-2\pi inx}\,dx$$

for all integers $n \in \mathbb{Z}$. Then $f = g$.

Proof. First we reduce to the case where $g = 0$. Knowing that case, upon rewriting the condition as

$$\int_0^1 (f - g)(x)e^{-2\pi inx}\,dx = 0$$

for all $n \in \mathbb{Z}$ we may immediately conclude that $f - g = 0$, as desired. Hence hereafter, we will assume that $g = 0$.

Next, we reduce to the case where f is real-valued. Define two real-valued functions

$$u(x) = \frac{f(x) + \overline{f(x)}}{2}, \quad v(x) = \frac{f(x) - \overline{f(x)}}{2i}.$$

Because the Fourier coefficients of $\overline{f}$ are all zero, so must be the Fourier coefficients of u and v. Since we can recover f from u and v, it suffices to prove the statement for u and v, thus we may assume f is real-valued.

Finally, for any real t,

$$\int_0^1 f(x+t)e^{-2\pi inx}\,dx = \int_0^1 f(x)e^{-2\pi in(x-t)}\,dx = e^{2\pi int}\int_0^1 f(x)e^{-2\pi inx}\,dx = 0$$

so without loss of generality, if $f \neq 0$ then we may assume (by a suitable translation) that $f(0) \neq 0$. We may further assume that $f(0) > 0$ (by considering $\frac{1}{f(0)}f$ instead of f).

Define a *trigonometric polynomial* $p : \mathbb{R} \to \mathbb{C}$ to be any function expressible as

$$p(x) = \sum_{n=-N}^{N} c_n e^{2\pi inx}$$

for some integer $N > 0$ and complex coefficients $c_{-N}, c_{-N+1}, ..., c_N$. It follows that

$$\begin{aligned}\int_0^1 f(x)p(x)\,dx &= \int_0^1 f(x)\sum_{n=-N}^{N} c_n e^{2\pi inx}\,dx \\ &= \sum_{n=-N}^{N} c_n \int_0^1 f(x)e^{2\pi inx}\,dx = 0.\end{aligned}$$

The approach now is to construct a real-valued trigonometric function p which is concentrated heavily around integer values (i.e. large near $x \in \mathbb{Z}$, approximately zero elsewhere). Trigonometric polynomials are closed under pointwise addition and multiplication, so because cos is a trigonometric polynomial, one possible candidate is $p_N(x) = (1+\cos 2\pi x)^N$ for large N due to the exponential-like growth near 0.

Specifically, by continuity of f around 0, we may fix $0 < \varepsilon < 1/2$ such that f is bounded below by a positive constant on $[-\varepsilon, \varepsilon]$. Define the constants

$$\alpha = 1 + \cos \pi\varepsilon$$
$$\beta = 1 + \cos 2\pi\varepsilon$$

which play a role in the lower and upper bounds of p_N:

$$p_N(x) \begin{cases} \geq \alpha^N & \text{if } x \in [-\varepsilon/2, \varepsilon/2], \\ \leq \beta^N & \text{if } x \in [\varepsilon, 1-\varepsilon], \\ \geq 0 & \text{otherwise.} \end{cases}$$

By assumption, $0 = \int_0^1 f(x)p_N(x)\,dx = \int_{-\varepsilon}^{1-\varepsilon} f(x)p_N(x)\,dx = 0$, so

$$\int_{-\varepsilon}^{\varepsilon} f(x)p_N(x)\,dx = -\int_{\varepsilon}^{1-\varepsilon} f(x)p_N(x)\,dx.$$

However,

$$\int_{-\varepsilon}^{\varepsilon} f(x)p_N(x) \geq \int_{-\varepsilon/2}^{\varepsilon/2} f(x)p_N(x) \geq \varepsilon \inf_{x\in[-\varepsilon,\varepsilon]} f(x) \cdot \alpha^N$$

while

$$\left|\int_{\varepsilon}^{1-\varepsilon} f(x)p_N(x)\,dx\right| \leq \int_{\varepsilon}^{1-\varepsilon} |f(x)| \cdot p_N(x)\,dx \leq \sup_{x\in[0,1]} |f(x)| \cdot \beta^N.$$

Since $\alpha > \beta > 0$, this is a contradiction as $N \to \infty$. □

Lemma 4

Let $c_1, c_2, \ldots$ be a sequence of real numbers where $\sum |c_n|$ converges. Then the function

$$g(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}$$

converges uniformly over all x.

Proof. The function g converges absolutely pointwise, so it remains to show that this convergence is uniform over x. To do this, we compare g with its partial sums $g_{M,N}(x) = \sum_{n=-M}^{N} c_n e^{2\pi i n x}$:

$$\begin{aligned} |g(x) - g_{M,N}(x)| &= \left|\sum_{n\in\mathbb{Z}-[-M,N]} c_n e^{2\pi i n x}\right| \\ &\leq \sum_{n\in\mathbb{Z}-[-M,N]} |c_n| \to 0 \qquad \text{as } M, N \to \infty \end{aligned}$$

where the above bound is uniform over x.

□

Lemma 5

Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ with each f_n' continuous, such that $\{f_n\}$ converges pointwise to a function f. If $\{f_n'\}$ converges uniformly on $[a.b]$ to a function g then f is differentiable and

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

Proof. Since $\{f_n'\}$ is a uniformly convergent sequence of continuous functions with pointwise limit g, the function g is continuous and we can pass this limit through definite integration. More specifically, for all $x \in [a, b]$ we have

$$\int_a^x g(t)\,dt = \lim_{n\to\infty} \int_a^x f_n'(t)\,dt = \lim_{n\to\infty} (f_n(x) - f_n(a)) = f(x) - f(a).$$

This says $f(x) = f(a) + \int_a^x g(t)\,dt$ with g continuous, so by the Fundamental Theorem of Calculus f is differentiable with $f' = g$; i.e., f' coincides with the pointwise limit of the functions f_n'. $\square$