Heuristics for Inequalities

RI Math Core 2018

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1 Introduction

The goal of this handout is to outline some heuristics that sometimes work on inequalities. While most inequalities handouts out tell you what you need to know to solve inequalities (hence AM-GM, Cauchy, the list goes on), here we focus on the information you can obtain *without solving the inequality* (a.k.a. heuristics).

Knowledge of the standard tools is assumed and not specifically covered here. Proficiency with the standard tools is ideal but not required (and honestly a waste of time).

Disclaimer: these methods don't work on every problem, and it may even be counterproductive. For instance, IMO 2012 Q2 will lead you on a wild goose chase if you try to "motivate" it, but falls to a two-line weighted AM-GM.

1.1 Summary of Examples

Just in case you want to try them before you read the solutions.

1. Prove that for $x, y, z > 0$ with product 1,
$$\sum_{cyc} \frac{x^3}{(1+y)(1+z)} \geq \frac{3}{4}$$
2. (ISL 2016 A1/NTST 2017) Let a, b, c be positive real numbers such that $\min(ab, bc, ca) \geq 1$. Prove that
$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$
3. (CGMO 2011) Positive reals a, b, c, d satisfy $abcd = 1$. Prove that
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{9}{a+b+c+d} \geq \frac{25}{4}$$
4. (MEMO 2017) Determine the smallest possible real constant C such that the inequality
$$|x^3 + y^3 + z^3 + 1| \leq C|x^5 + y^5 + z^5 + 1|$$
holds for all real numbers x, y, z satisfying $x + y + z = -1$.
5. (ISL 2015 A8) Find the largest real constant a such that for all $n \geq 1$ and for all real numbers $x_0, x_1, ..., x_n$ satisfying $0 = x_0 < x_1 < x_2 < \cdots < x_n$ we have
$$\frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1} + \cdots + \frac{1}{x_n - x_{n-1}} \geq a\left(\frac{2}{x_1} + \frac{3}{x_2} + \cdots + \frac{n+1}{x_n}\right)$$

1.2 Quickfire General Advice

- Small cases. Build intuition.
- Easy stuff first.
- Extract as much information as possible from the problem (= apply heuristics).
- Take note of coincidences (like symmetries), most of them can be exploited somehow.
- Keep things neat. Symmetry is valuable, but feel free to force a WLOG ordering if that solves the problem
- Be aware of your "algebraic tolerance" if you have to get your hands dirty. Same advice also holds for coordinate geometry/complex.
- Don't be afraid to ditch your current approach. This is however much easier said than done.

2 Equality Cases

Why are equality cases important? Simply because the method of proof must encapsulate all equality cases.

Example 2.1. *('Vasc') Show that for all reals a, b, c,*

$$(a^2 + b^2 + c^2)^2 \geq 3(a^3b + b^3c + c^3a)$$

While this seems fairly innocuous at first glance ("trivial by Muirhead"?), it is an extremely difficult inequality due to the additional equality case at $(a : b : c) = \left(\sin^2\left(\frac{4\pi}{7}\right), \sin^2\left(\frac{2\pi}{7}\right), \sin^2\left(\frac{\pi}{7}\right)\right)$. Hence, we do not expect a fast proof by Cauchy or AM-GM, because there is almost no way to reproduce this strange equality case.

The inequality is in fact equivalent to

$$\sum_{cyc}(a^2 - 2ab + bc - c^2 + ca)^2 \geq 0$$

While we don't expect to pull such magical identities from mid-air, in general equality cases are the biggest hints as to which methods will work and which will not.

The equality case itself can suggest how exactly to use a particular technique. Take for instance:

Example 2.2. *Prove that for $x, y, z > 0$ with product 1,*

$$\sum_{cyc} \frac{x^3}{(1+y)(1+z)} \geq \frac{3}{4}$$

A naive person may come up with the following:

$$\frac{x^3}{(1+y)(1+z)} + (1+y) + (1+z) \geq 3x$$

However, it won't work:

$$\sum_{cyc} \frac{x^3}{(1+y)(1+z)} \geq x + y + z - 6 \not\geq \frac{3}{4}$$

If we consider the expected equality case at $(x, y, z) = (1, 1, 1)$, then we will know that this was doomed to begin with, since in the first step we applied AM-GM to $(\frac{1}{4}, 2, 2)$. Instead, a slight modification proves to be successful:

$$\frac{x^3}{(1+y)(1+z)} + \frac{1+y}{8} + \frac{1+z}{8} \geq \frac{3x}{4}$$

Now, summing cyclically,

$$\sum_{cyc} \frac{x^3}{(1+y)(1+z)} \geq \frac{x+y+z}{2} - \frac{3}{4} \geq \frac{3}{4}$$

where we used $x+y+z \geq 3\sqrt[3]{xyz} = 3$.

I really can't overstate the importance of the equality case. If anything, the next two methods both rely on knowledge of what the equality case is.

3 Smoothing

A proof by smoothing starts from the general case and attempts to "adjust" the variables in a way that makes the inequality tighter. As a (very simple) example:

Example 3.1. *(AM-GM) Show that for positive reals* $a_1, a_2, ..., a_n$*,*

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n}$$

For $x, y > 0$ with fixed sum, xy becomes larger as x and y are pushed closer together. We will apply this idea on some pairs of variables.

Denote $A = \frac{a_1+a_2+...+a_n}{n}$, then we select $a_i < A < a_j$ (if i or j does not exist then all $a_i = A$ and the conclusion is obvious). Take (a_i, a_j) and replace them with $(A, a_i + a_j - A)$. Clearly, the new pair has a larger product. So RHS increases while LHS stays the same, and thus the inequality has become tighter.

Repeating this step, we find that the number of i where $a_i = A$ strictly increases, so eventually $a_i = A$ for all i, where the inequality is true. Hence before this, the inequality must been true as well (since it only became tighter after each operation). In practice however, smoothing is never this clean. In the above example:

- For any fixed variable, there are only two separate terms containing it. This is rarely the case.
- $a_i a_j$ changes predictably, despite the fact that we parked a_i and a_j at very weird places. In practice this is much much worse, and usually we will be lucky to even map them both to $\frac{a_i+a_j}{2}$ or $\sqrt{a_i a_j}$.

  (In an n-variable setting, this step often comes from the $n = 2$ case.)

Depending on what smoothing step you come up with, you may be in for a rough bash or a smooth (heh) time.

3.1 Example - Mount Inequality erupts, NTST edition

Example 3.2. *(ISL 2016 A1/NTST 2017) Let a, b, c be positive real numbers such that* $\min(ab, bc, ca) \geq 1$*. Prove that*

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

Let's solve the two-variable case: for $x, y > 0, xy \geq 1$,

$$(x^2+1)(y^2+1) \leq \left(\left(\frac{x+y}{2}\right)^2 + 1\right)^2$$

This is fairly easy because we expect $(x-y)^2$ to be a factor. A fairly cute way is to note

$$(x^2+1)(y^2+1) \le (x-y)^2 + (xy+1)^2 \le \left(\left(\frac{x+y}{2}\right)^2 + 1\right)^2$$

or $(x-y)^2 \le \left(\frac{x-y}{2}\right)^2 \left(\left(\frac{x+y}{2}\right)^2 + xy + 2\right)$. This makes us want to do things like $(a,b) \to \left(\frac{a+b}{2}, \frac{a+b}{2}\right)$. We quickly check that the min condition still holds under this kind of operations. Now there are two options:

1) WLOG $a \ge b \ge c$. Then we smooth a and b together ($(a,b,c) \to (x,x,c)$ where $x = \frac{a+b}{2}$) and attempt to directly expand/factorize the resulting 2-variable expression. This is not too hard, since we expect $(x-c)^2$ to be a factor again.

2) Or we can brute force this and alternately apply $(a,b,c) \to \left(\frac{a+b}{2}, \frac{a+b}{2}, c\right)$ and $(a,b,c) \to \left(a, \frac{b+c}{2}, \frac{b+c}{2}\right)$. By using a monovariant (like $|a-b|+|b-c|+|c-a|$, which halves each time after the first step), we can see that eventually all three variables converge to $\frac{a+b+c}{3}$ (since the differences converge to 0 and their sum is invariant). Hence we just need to show that using a', b', c' that each differ from a,b,c by at most $\varepsilon \to 0$ respectively, the overall discrepancy should also $\to 0$.

The second approach is known as *Strong Variable Mixing*. While its more tricky to make rigorous but worth it if you just don't see any way to make the 3-variable case work (or if this was generalized to n-variables).

3.2 Normalization

A polynomial P is *homogeneous* if $P(ka,kb,kc) = k^{\deg P} P(a,b,c)$. Intuitively, this means that each term in P should have the same total degree (i.e. sum of degree in each variable). Similarly, we call an inequality *homogeneous* if $LHS - RHS$ is homogeneous.

This gives us the seemingly useless smoothing step $(a,b,c) \to (ka,kb,kc)$, but what we can do now is to set $k = \frac{1}{a+b+c}$, which means that we can assume $a+b+c=1$ and preserve full generality. This is sometimes useful:

Example 3.3. *(Cauchy-Schwarz) Show that for non-negative reals* $a_1, a_2, ..., a_n; b_1, b_2, ..., b_n$,

$$(a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2) \ge (a_1b_1 + a_2b_2 + ... + a_nb_n)^2$$

Not only is the above inequality homogeneous, it is homogeneous if we just consider $\{a_i\}$ alone (or $\{b_i\}$ alone). So we can assume $a_1^2 + a_2^2 + ... + a_n^2 = 1$ and $b_1^2 + b_2^2 + ... + b_n^2 = 1$, then the conclusion immediately follows from:

$$\sum a_i b_i \le \sum \frac{1}{2}(a_i^2 + b_i^2) = 1$$

However, usually conditions explicitly given in the problem ("Given positive reals a,b,c with sum 1...") are superfluous and so should be promptly thrown away by substituting copies of it into the equation until it is homogeneous. For instance, if a,b,c have sum 1, then expressions like a^2+1 should be re-written as $a^2 + (a+b+c)^2$.

Disclaimer: normalization is very rare in practice, but it's a good demonstration of the underlying principles. Homogenization doesn't work all the time, you will have to judge whether it's helpful or not. Also, if the condition itself is not homogeneous you're screwed anyway.

3.3 Convexity

Sums of convex/concave functions can be smoothed:

Fact 3.1. *Let $f(x)$ be convex on the interval I. Suppose $a < b$ are both in I, and suppose epsilon > 0 is a real number for which $a + \epsilon \leq b - \epsilon$. Then $f(a) + f(b) \geq f(a + \epsilon) + f(b - \epsilon)$.*

This means that

- For **convex** functions f, we can decrease the sum $f(a)+f(b)$ by "smoothing" a and b together, and increase the sum by "unsmoothing" a and b apart.
- For **concave** functions f, we can increase the sum $f(a) + f(b)$ by "smoothing" a and b together, and decrease the sum by "unsmoothing" a and b apart.

This quickly implies things like Jensen's inequality (or even Karamata's if you're feeling brave), but much more is possible. In general, if a function f is made up of concave and convex parts, then to maximize sums of f we simply smooth along the concave parts and unsmooth along the convex parts, so we can WLOG conclude things like there is at most one variable within a convex part and all variables within a concave part are equal.

This is very messy to execute in practice, but usually the following special case is (typically) more than sufficient:

Fact 3.2. *($n-1$ EV) Let $a_1, a_2, ..., a_n$ be reals with fixed sum. Let f be a function with exactly one inflection point (i.e. $f''(x) = 0$ has exactly one root). Then if $f(a_1) + f(a_2) + ... + f(a_n)$ achieves maximal or minimal value, then (at least) $n - 1$ of them are equal to each other.*

Note: if the product is fixed instead of a sum, you should do a variable substitution ($y_i = \ln x_i$) before proceeding as all of the above only work with constant sum.

If there are no extra conditions like fixed sum/product, then we can still think about single-variable smoothing. If the equality case lies at the endpoints of an interval, sometimes this is the reason why:

- If $f(x)$ is **convex** on the interval $a \leq x \leq b$, then $f(x)$ attains a maximum, and that value is either $f(a)$ or $f(b)$.
- If $f(x)$ is concave on the interval $a \leq x \leq b$, then $f(x)$ attains a minimum, and that value is either $f(a)$ or $f(b)$.

3.4 Other ways to smooth

Traditionally, smoothing has only referred to pushing variables together, but it can be a lot more generic. It all depends on what kind smoothing steps you come up with.

Philosophically, you should think of smoothing as a kind of WLOG: you are reducing all your cases down to a smaller subset (which should be easier to deal with). Here's an inexhaustive list of some ways to smooth:

- If only $(a-b), (b-c), (c-a)$ appear on one side, you can consider $(a, b, c) \to (a-\epsilon, b-\epsilon, c-\epsilon)$.
- You can rewrite the variables as elementary symmetric polynomials (i.e. $(a + b + c, ab + bc + ca, abc) = (p, q, r)$) and smooth those instead. The only edge cases are boundary conditions and if two of a, b, c are equal.
- In general, anything that changes a small part of the inequality predictably qualifies as a smoothing step. In this way, smoothing in inequalities is a little like smoothing in combinatorics. The more synergistic your smoothing steps are, the less algebra you have to do.

3.5 Example - What does a smoothing bash look like?

Quick note: this seems to be a "classic example" on how to smooth, and is essentially what happens if you don't have a good smoothing step. But there's no other way to do this problem, so I won't complain.

Example 3.4. *(CGMO 2011) Positive reals a, b, c, d satisfy $abcd = 1$. Prove that*

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{9}{a+b+c+d} \geq \frac{25}{4}$$

.

The usual tools fail spectacularly here, especially since nothing can be done about $abcd = 1$ without the inequality melting into an intractable mess.

Let's try to do something like $(a, b, c, d) \to (\sqrt{ab}, \sqrt{ab}, c, d)$. For this to work for any a, b, c, d would be too much to hope for, but let's just try simplifying the statement anyway to see what we need to assume. We want:

$$\frac{2}{\sqrt{ab}}+\frac{9}{2\sqrt{ab}+c+d} \leq \frac{1}{a}+\frac{1}{b}+\frac{9}{a+b+c+d}$$

It's actually not too bad if we group $\frac{1}{a}+\frac{1}{b}$ with $\frac{2}{\sqrt{ab}}$. For cleanliness, we do $(a, b, c, d) = (w^2, x^2, y^2, z^2)$, then:

$$\begin{aligned}
&\frac{2}{wx}+\frac{9}{2wx+y^2+z^2} \leq \frac{1}{w^2}+\frac{1}{x^2}+\frac{9}{w^2+x^2+y^2+z^2} \\
\Leftrightarrow\ &\frac{9}{2wx+y^2+z^2}-\frac{9}{w^2+x^2+y^2+z^2} \leq \frac{(w-x)^2}{w^2x^2} \\
\Leftrightarrow\ &\frac{9(w-x)^2}{(2wx+y^2+z^2)(w^2+x^2+y^2+z^2)} \leq \frac{(w-x)^2}{w^2x^2} \\
\Leftrightarrow\ &\frac{9}{(2wx+y^2+z^2)(w^2+x^2+y^2+z^2)} \leq \frac{1}{w^2x^2} \\
\Leftrightarrow\ &(2wx+y^2+z^2)(w^2+x^2+y^2+z^2) \geq 9w^2x^2
\end{aligned}$$

Of course, the last bit works as long as $y^2 + z^2 \geq wx$. This pretty much means that we can always smooth a and b together if c or d is the largest among a, b, c, d. So if $a \leq b \leq c \leq d$, we can perform this on a, b and a, c repeatedly which will converge to $a = b = c$.

At this point, we can just bash out the last bit: the original inequality is equivalent to

$$\frac{(a-1)^2(12a^6+24a^5+36a^4-27a^3-14a^2-a+12)}{4a(3a^4+1)} \geq 0$$

(where naturally we predicted that $(a-1)^2$ was a factor). It remains to show that $f(a) = 12a^6 + 24a^5 + 36a^4 - 27a^3 - 14a^2 - a + 12 \geq 0$. A reasonable way to do this is to split $f(a)$ as $g(a) - h(a)$ where g is increasing and h is decreasing, then do a piecewise bash by cutting our desired interval into smaller intervals of the form $[p, q]$ such that $g(p) - h(q) \geq 0$. For instance:

$a \geq 1$ (technically this is not required, but anyway):

$$12a^6 + 24a^5 + 36a^4 \geq 60a^3 \geq 27a^3 + 14a^2 + a$$

$\frac{1}{\sqrt{2}} \leq a \leq 1$: $24a^5 + 36a^4 \geq 27a^3$, $12a^6 \geq 3a^2$ and $12 \geq a + 11a^2$.

$\frac{1}{2} \leq a \leq \frac{1}{\sqrt{2}}$: $24a^5 + 36a^4 \geq 24a^3$, and $12 \geq 3a^3 + 14a^2 + a$.

$0 \leq a \leq \frac{1}{2}$: $12 > 27a^3 + 14a^2 + a$. Done!

4 Double Roots

4.1 Some motivation

Inequalities (with equality cases) are closely related to double roots. Think about this:

Example 4.1. *Let P be a polynomial with real coefficients. If $P(x) \geq 0$ for all x and $P(\alpha) = 0$, then $(x - \alpha)^2$ divides $P(x)$.*

Geometrically, this is obvious as we must have $P'(\alpha) = 0$, otherwise P will dip below 0 either right before or right after α.

Algebraically, we can set $y = x - \alpha$ and expand $P(x) = a_0 + a_1 y + O(y^2)$ where $x \approx \alpha$. Then it's clear that when x moves in the range $[\alpha - \varepsilon, \alpha + \varepsilon]$ we can ignore the $O(y^2)$ part and treat f as linear (which means it has to be identically 0).

In fact, we only need $P(x) \geq 0$ around the neighbourhood of α, so conditions like $x > 0$ are generally not a problem.

Perhaps more interestingly, does this work in more variables?

Example 4.2. *Let $P(x, y)$ be a bivariate polynomial. Suppose that $P(x, y) \geq 0$ for all x, y and $P(x, x) = 0$ for all x. Does $(x - y)^2$ divide $P(x, y)$?*

Solution: actually we expect something very similar to work. Treating $\mathbb{R}[x, y]$ as $\mathbb{R}[y][x - y]$ (read: polynomials in $x - y$ with coefficients which are polynomials in y), we write:

$$P(x, y) = P_0(y) + (x - y)P_1(y) + O((x - y)^2)$$

But for a fixed value of y, $(x-y)^2$ divides $P(x, y)$. Taking large x, we get that $P_0(y)+(x-y)P_1(y) \equiv 0$ for fixed y. So for all y, $P_1(y) \equiv 0$ and $P_0(y) - yP_1(y) \equiv 0$, and hence the fact is proven.

This is an *extremely* useful heuristic to have! Broadly speaking, this essentially means that whenever we have an equality case, we should expect a double root.

Unfortunately there's nothing similar for x, y, z, since we expect expressions like $\sum k_a(b - c)^2$.

4.2 Example - Absolute value, absolute nightmare?

Example 4.3. *(MEMO 2017) Determine the smallest possible real constant C such that the inequality*

$$|x^3 + y^3 + z^3 + 1| \leq C|x^5 + y^5 + z^5 + 1|$$

holds for all real numbers x, y, z satisfying $x + y + z = -1$.

Since the inequality varies over reals (instead of non-negative reals), we can throw away most of the standard stockpile and instead try to use our "bare hands". It's very likely this ends up as a factorization/sum of squares-type deal.

The -1 clearly isn't going to help, so let's homogenize:

$$|(x + y + z)^2 \cdot (x^3 + y^3 + z^3 - (x + y + z)^3)| \leq C|x^5 + y^5 + z^5 - (x + y + z)^5|$$

But, now note that $RHS^2 - LHS^2$ does have an equality case at $x = -y$, so we expect $(x + y)^2$ to divide it, or at least $(x + y)|RHS - LHS$. In addition, $LHS = 0$ whenever $x = -y$. Hence it's also reasonable to think that $(x + y)$ divides it (and by symmetry, so must $(y + z)$ and $(z + x)$). Indeed, it's not hard to verify that

$$(x + y + z)^3 - (x^3 + y^3 + z^3) = 3(x + y)(y + z)(z + x)$$

Similarly, we should expect

$$(x + y + z)^5 - (x^5 + y^5 + z^5) = (A(x^2 + y^2 + z^2) + B(xy + yz + zx))(x + y)(y + z)(z + x)$$

for some constants A, B which are easily determined by checking two specific sets of (x, y, z). I'll save you the trouble and tell you $A = B = 5$.

Anyway, this means that after throwing away $(x + y)(y + z)(z + x)$, the remaining terms are non-negative (so we can toss the absolute values). Moreover, the remaining degree is 2. Easy peasy.

4.3 Generalized double roots = partial derivatives?

Of course, not all inequalities look like polynomials. In general, the best that we have is $f'(\alpha) = 0$, or for multivariate f:

$$\left.\frac{\partial f}{\partial x_i}\right|_{x_i=\alpha_i} = 0$$

where $(x_i) = (\alpha_i)$ is the equality case.

If you're analyzing the behaviour of each variable, you can try to solve for where the minima happens, but it's not strictly necessary.

Note: derivatives are good as a heuristic but bad as a proof technique. Even ignoring boundary cases, we only know that (minimum at $\alpha \Rightarrow f'(\alpha) = 0$) but not ($f'(\alpha) = 0 \Rightarrow$ minimum at α).

4.4 Undetermined coefficients

Some methods attempt to bound each term in the inequality by a term of some predetermined form, like the tangent line trick or isolated fudging. In either case, the variable coefficients must be determined, and assuming a "double root" can help us make an educated guess.

Example 4.4. *(Nesbitt) Show that for $a, b, c > 0$,*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

Perhaps we predict that for some r:

$$\frac{a}{b+c} \geq \frac{3a^r}{2(a^r + b^r + c^r)}$$

Let's cross-multiply and do a check at $b = c = 1$:

$$f(a) = 2a(a^r + b^r + c^r) - 3a^r(b+c) = 2a^{r+1} + 4a - 6a^r \geq 0$$

We expect $f'(1) = 0$, so $f'(1) = 2(r+1) + 4 - 6r = 0$. Hence $r = \frac{3}{2}$. Indeed:

$$a^{\frac{5}{2}} + ab^{\frac{3}{2}} + ab^{\frac{3}{2}} \geq 3a(ab^2)^{\frac{1}{2}} = 3a^{\frac{3}{2}}b$$

and a similar inequality when c is replaced with b. Adding them together, we are done!

While this may seem like a lot of work just for Nesbitt (why not do a one-line Cauchy?), this does have a comparative advantage sometimes.

4.5 Example - OH MY GOD IT'S AN A8

By the way, I couldn't solve this, but I got surprisingly far using some of the ideas discussed.

Example 4.5. *(ISL 2015 A8) Find the largest real constant a such that for all $n \geq 1$ and for all real numbers $x_0, x_1, ..., x_n$ satisfying $0 = x_0 < x_1 < x_2 < \cdots < x_n$ we have*

$$\frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1} + \cdots + \frac{1}{x_n - x_{n-1}} \geq a\left(\frac{2}{x_1} + \frac{3}{x_2} + \cdots + \frac{n+1}{x_n}\right)$$

Maybe the first thing we do is to try some small cases: $n = 1$ is trivial and gives us $a \leq \frac{1}{2}$. $n = 2$ is as follows:

$$\frac{1}{x_1} + \frac{1}{x_2 - x_1} \geq a\left(\frac{2}{x_1} + \frac{3}{x_2}\right)$$

Simplifying slightly:

$$\frac{1-2a}{x_1} + \frac{1}{x_2 - x_1} \geq \frac{3a}{x_2}$$

Cauchy tells us that minimum on the left hand side is in fact $\frac{(\sqrt{1-2a}+1)^2}{x_2}$, so solving we have that $a \leq \frac{12}{25}$. a seems like it's dropping (as n increases) but *very* slowly.

$n = 3$ is a little too complicated for us to get a nice looking bound for a, so we'll stop there with small cases.

Looking at $n = 2$, it is reasonable to imagine an approach that looks like summing inequalities of the form

$$\frac{?}{x_1} + \frac{?}{x_2 - x_1} + ... + \frac{?}{x_k - x_{k-1}} \geq \frac{a(k+1)}{x_k}$$

...except we have no idea what the coefficients should be. In fact, if this inequality is "tight", we need the equality case of Cauchy to hold (i.e. "proportional" sequences), and so we need to know the equality case for x_i first.

Here's where the heuristics begin. Now let's think about adjusting just one variable x_k so that the inequality is optimally tight. We isolate all terms where x_k appear and treat it as a function:

$$f(x) = \frac{1}{(x - x_{k-1})} + \frac{1}{(x_{k+1} - x)} - \frac{a(k+1)}{x}$$

If x_k was minimal, we should expect either x_k to be at the endpoints (which makes f shoot off to infinity) or $f'(x_k) = 0$. We should thus pay attention to

$$f'(x) = -\frac{1}{(x - x_{k-1})^2} + \frac{1}{(x_{k+1} - x)^2} + \frac{a(k+1)}{x^2}$$

Sometimes, here's where we try and solve the equation $f'(x) = 0$ but I assure you it doesn't end well here. Instead, we'll just keep the equation as it is and see what we can do.

To minimize future confusion, we are going to replace all the x's with y's. Specifically, we will keep $\{x_i\}$ as variables while $\{y_i\}$ will denote the specific value that keeps $LHS - RHS$ minimal.

Since $f'(y_i) = 0$, we have:

$$\frac{a(k+1)}{y_k^2} = \frac{1}{(y_k - y_{k-1})^2} - \frac{1}{(y_{k+1} - y_k)^2}$$

(Quick sanity check: multiply both sides by y_k and sum over k. Realise things somewhat work out. Also RHS looks like it telescopes.)

This doesn't seem very useful at the moment, but let's try to go back to the initial Cauchy and make it work.

$$\sum_{k \leq n} \frac{a(n+1)(y_k - y_{k+1})^2}{y_n^2} \cdot \frac{1}{y_k - y_{k+1}} \geq \frac{a(n+1)}{y_n}$$

In fact, this is "dummy Cauchy" since equality holds all the time. But let's sum over n and see what happens:

$$\sum_k \left((y_k - y_{k+1})^2 \cdot \frac{1}{y_k - y_{k+1}} \sum_{n \geq k} \frac{a(n+1)}{y_n^2} \right) \cdot \geq \sum_n \frac{a(n+1)}{y_n}$$

But $\sum_{n \geq k} \frac{a(n+1)}{y_n^2} \leq \frac{1}{(y_k - y_{k-1})^2}$ by the telescope, and this is tight if $y_n - y_{n-1} \to \infty$ when $n \to \infty$!

That's way too big of a coincidence to pass up (even though this is still a "dummy equation").

Let's see whether we can turn this into something concrete: for one, let's go back to the Cauchy statement with x_i but keeping y_i for the coefficients:

$$\sum_{k=1}^{n} \frac{a(n+1)(y_k - y_{k+1})^2}{y_n^2} \cdot \frac{1}{x_k - x_{k+1}} \geq \frac{a(n+1)}{x_n}$$

Summing once again (over n):

$$\sum_{k=1}^{N} \frac{1}{x_k - x_{k+1}} > \sum_{k=1}^{N} \left((y_k - y_{k+1})^2 \cdot \sum_{n=k}^{N} \frac{a(n+1)}{y_n^2} \right) \cdot \frac{1}{x_k - x_{k+1}} \geq \sum_{n=1}^{N} \frac{a(n+1)}{x_n}$$

Holy crap. As long as we have a working a and y_i, we are done! Unfortunately I could not construct them, so I failed here. Booooo.

For completeness: $a = \frac{4}{9}$ and $y_i = \binom{i+2}{3}$ magically works. If you have a good way of figuring this out please tell me.

5 More Problems

These aren't specifically solvable with the above methods, but I just think they look ISL-like:

1. Show that for reals $a_1, a_2, ..., a_n$ and any subset $S \subseteq \{1, 2, ..., n\}$,

$$\left(\sum_{i \in S} a_i \right)^2 \leq \sum_{1 \leq i \leq j \leq n} (a_i + a_{i+1} + ... + a_j)^2$$

2. Given complex numbers $z_1, z_2, ..., z_n$ such that $|z_1| + |z_2| + ... + |z_n| = 1$. Show that there exists a subset $S \subseteq \{1, 2, ..., n\}$ such that

$$\left| \sum_{i \in S} z_i \right| \geq \frac{1}{4}$$

3. Determine, for each positive integer n, the maximal k such that for all reals $x_1, x_2, ..., x_n$, the following inequality holds:

$$\sqrt{x_1^2 + x_2^2 + ... + x_n^2} \geq k \min\{|x_1 - x_2|, |x_2 - x_3|, ..., |x_n - x_1|\}$$

4. Find the greatest real number k such that, for any positive a, b, c with $a^2 > bc$,

$$(a^2 - bc)^2 > k(b^2 - ca)(c^2 - ab)$$

.