Orthogonal Machine Learning: Power and Limitations

Lester Mackey*

Joint work with Vasilis Syrgkanis* and Ilias Zadik†

Microsoft Research New England*, Massachusetts Institute of Technology†

October 30, 2018

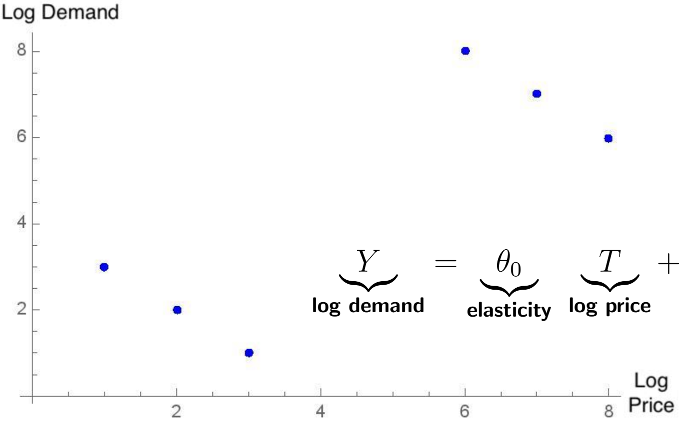
A Conversation with Vasilis

- **Vasilis:** Lester, I love Double Machine Learning!
- **Me:** What?
- **Vasilis:** It's a tool for accurately estimating treatment effects in the presence of many potential confounders.
- **Me:** I have no idea what you're talking about.
- **Vasilis:** Let me give you an example...

Example: Estimating Price Elasticity of Demand

Goal: Estimate *elasticity*, the effect a change in price has on demand

- Set prices of goods and services [Chernozhukov, Goldman, Semenova, and Taddy, 2017b]
- Predict impact of tobacco tax on smoking [Wilkins, Yurekli, and Hu, 2004]

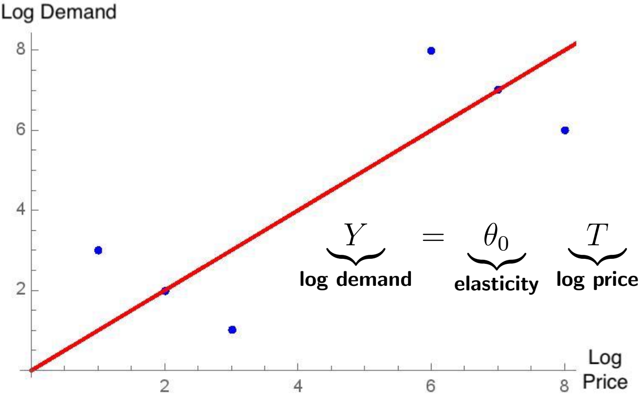


$$\underbrace{Y}_{\textbf{log demand}} = \underbrace{\theta_0}_{\textbf{elasticity}} \underbrace{T}_{\textbf{log price}} + \underbrace{\epsilon}_{\textbf{noise}}$$

Example: Estimating Price Elasticity of Demand

Goal: Estimate *elasticity*, the effect a change in price has on demand

- Set prices of goods and services [Chernozhukov, Goldman, Semenova, and Taddy, 2017b]
- Predict impact of tobacco tax on smoking [Wilkins, Yurekli, and Hu, 2004]



$$\underbrace{Y}_{\textbf{log demand}} = \underbrace{\theta_0}_{\textbf{elasticity}} \underbrace{T}_{\textbf{log price}} + \underbrace{\epsilon}_{\textbf{noise}}$$

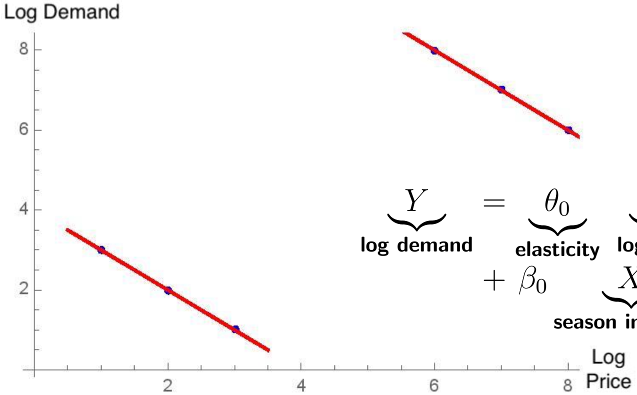
Conclusion: Increasing price increases demand!

Problem: Demand increases in winter & price anticipates demand

Example: Estimating Price Elasticity of Demand

Goal: Estimate *elasticity*, the effect a change in price has on demand

- Set prices of goods and services [Chernozhukov, Goldman, Semenova, and Taddy, 2017b]
- Predict impact of tobacco tax on smoking [Wilkins, Yurekli, and Hu, 2004]



$$\underbrace{Y}_{\textbf{log demand}} = \underbrace{\theta_0}_{\textbf{elasticity}} \underbrace{T}_{\textbf{log price}} + \beta_0 \underbrace{X}_{\textbf{season indicator}} + \underbrace{\epsilon}_{\textbf{noise}}$$

Problem: What if there are 100s or 1000s of potential confounders?

Example: Estimating Price Elasticity of Demand

Goal: Estimate *elasticity*, the effect a change in price has on demand

Problem: What if there are 100s or 1000s of potential confounders?

- Time of day, day of week, month, purchase and browsing history, other product prices, demographics, the weather, ...

One option: Estimate effect of all potential confounders really well

$$\underbrace{Y}_{\text{log demand}} = \underbrace{\theta_0}_{\text{elasticity}} \underbrace{T}_{\text{log price}} + \underbrace{f_0(X)}_{\text{effect of potential confounders}} + \underbrace{\epsilon}_{\text{noise}}$$

- If nuisance function f_0 estimable at $O(n^{-1/2})$ rate then so is θ_0

Problem: Accurate nuisance estimates often unachievable when f_0 nonparametric or linear and high-dimensional

Example: Estimating Price Elasticity of Demand

Problem: What if there are 100s or 1000s of potential confounders?

Double Machine Learning [Chernozhukov, Chetverikov, Demirer, Duflo, Hansen, and Newey, 2017a]

$$\underbrace{Y}_{\text{log demand}} = \underbrace{\theta_0}_{\text{elasticity}} \underbrace{T}_{\text{log price}} + \underbrace{f_0(X)}_{\text{effect of potential confounders}} + \underbrace{\epsilon}_{\text{noise}}$$

- Estimate nuisance f_0 somewhat poorly: $o(n^{-1/4})$ suffices
- Employ *Neyman orthogonal* estimator of θ_0 robust to first-order errors in nuisance estimates; yields $\sqrt{n}$-consistent estimate of θ_0

Questions: Why $o(n^{-1/4})$? Can we relax this? When? How?

This talk:

- Framework for k-th order orthogonal estimation with $o(n^{-1/(2k+2)})$ nuisance consistency $\Rightarrow \sqrt{n}$-consistency for θ_0
- Existence characterization and explicit construction of 2nd-order orthogonality in a popular causal inference model

Estimation with Nuisance

Goal: Estimate target parameters $\theta_0 \in \Theta \subseteq \mathbb{R}^d$ (e.g., elasticities) in the presence of unknown nuisance functions $h_0 \in \mathcal{H}$

Given

- Independent replicates $(Z_t)_{t=1}^{2n}$ of a data vector $Z = (T, Y, X)$

Example (Partially Linear Regression (PLR))

- $T \in \mathbb{R}$ represents a treatment or policy applied (e.g., log price)
- $Y \in \mathbb{R}$ represents an outcome of interest (e.g., log demand)
- $X \in \mathbb{R}^p$ is a vector of associated covariates (e.g., seasonality)

These observations satisfy

$$Y = \theta_0 T + f_0(X) + \epsilon, \quad \mathbb{E}[\epsilon \mid X, T] = 0 \quad a.s.$$
$$T = g_0(X) + \eta, \quad \mathbb{E}[\eta \mid X] = 0 \quad a.s., \quad \mathrm{Var}(\eta) > 0$$

for noise η and ϵ, target parameter θ_0, and nuisance $h_0 = (f_0, g_0)$.

Two-stage Z-estimation with Sample Splitting

Goal: Estimate target parameters $\theta_0 \in \Theta \subseteq \mathbb{R}^d$ (e.g., elasticities) in the presence of unknown nuisance functions $h_0 \in \mathcal{H}$

Given

- Independent replicates $(Z_t)_{t=1}^{2n}$ of a data vector $Z = (T, Y, X)$
- Moment functions m that identify the target parameters θ_0:

$$\mathbb{E}[m(Z, \theta_0, h_0(X))|X] = 0 \;\; a.s. \text{ and } \mathbb{E}[m(Z, \theta, h_0(X))] \neq 0 \text{ if } \theta \neq \theta_0$$

- PLR model example: $m(Z, \theta, h_0(X)) = (Y - \theta T - f_0(X))T$

Two-stage Z-estimation with sample splitting

1. Fit estimate $\hat{h} \in \mathcal{H}$ of h_0 using $(Z_t)_{t=n+1}^{2n}$ (e.g., via nonparametric or high-dimensional regression)
2. $\hat{\theta}^{SS}$ solves $\frac{1}{n}\sum_{t=1}^{n} m(Z_t, \theta, \hat{h}(X_t)) = 0$

Con: Splitting statistically inefficient, possible detriment in first stage

Two-stage Z-estimation with Cross Fitting

Goal: Estimate target parameters $\theta_0 \in \Theta \subseteq \mathbb{R}^d$ (e.g., elasticities) in the presence of unknown nuisance functions $h_0 \in \mathcal{H}$

Given

- Independent replicates $(Z_t)_{t=1}^{2n}$ of a data vector $Z = (T, Y, X)$
- Moment functions m that identify the target parameters θ_0:

$$\mathbb{E}[m(Z, \theta_0, h_0(X))|X] = 0 \;\; a.s. \text{ and } \mathbb{E}[m(Z, \theta, h_0(X))] \neq 0 \text{ if } \theta \neq \theta_0$$

- PLR model example: $m(Z, \theta, h_0(X)) = (Y - \theta T - f_0(X))T$

Two-stage Z-estimation with cross fitting

[Chernozhukov, Chetverikov, Demirer, Duflo, Hansen, and Newey, 2017a]

0. Split data indices into K batches $I_1, \ldots, I_K$
1. For $k \in \{1, \ldots, K\}$, fit estimate $\hat{h}_k \in \mathcal{H}$ of h_0 excluding I_k
2. $\hat{\theta}^{CF}$ solves $\frac{1}{n} \sum_{k=1}^{K} \sum_{t \in I_k} m(Z_t, \theta, \hat{h}_k(X_t)) = 0$

Pro: Repairs sample splitting deficiencies

Goal: $\sqrt{n}$-Asymptotic Normality

Two-stage Z-estimators

- $\hat{\theta}^{SS}$ solves $\frac{1}{n}\sum_{t=1}^{n} m(Z_t, \theta, \hat{h}(X_t)) = 0$
- $\hat{\theta}^{CF}$ solves $\frac{1}{n}\sum_{k=1}^{K}\sum_{t\in I_k} m(Z_t, \theta, \hat{h}_k(X_t)) = 0$

Goal: Establish conditions under which $\hat{\theta}^{SS}$ and $\hat{\theta}^{CF}$ enjoy $\sqrt{n}$-asymptotic normality ($\sqrt{n}$-a.n.), that is

$$\sqrt{n}(\hat{\theta}^{SS} - \theta_0) \xrightarrow{d} N(0, \Sigma) \text{ and } \sqrt{2n}(\hat{\theta}^{CF} - \theta_0) \xrightarrow{d} N(0, \Sigma)$$

- Asymptotically valid confidence intervals for θ_0 based on Gaussian or Student's t quantiles
- Asymptotically valid association tests, like the Wald test

First-order Orthogonality

Definition (First-order Orthogonal Moments [Chernozhukov, Chetverikov, Demirer, Duflo, Hansen, and Newey, 2017a])

Moments m are *first-order orthogonal* w.r.t. the nuisance $h_0(X)$ if

$$\mathbb{E}\left[\nabla_\gamma m(Z, \theta_0, \gamma)|_{\gamma=h_0(X)} \,|\, X\right] = 0.$$

- Principle dates back to early work of [Neyman, 1979]
- Grants first-order insensitivity to errors in nuisance estimates
  - Annihilates first-order term in Taylor expansion around nuisance
  - Recall: m is 0-th order orthogonal, $\mathbb{E}[m(Z, \theta_0, h_0(X)) \,|\, X] = 0$
- Not satisfied by $m(Z, \theta, h(X)) = (Y - \theta T - f(X))T$
- Satisfied by $m(Z, \theta, h(X)) = (Y - \theta T - f(X))(T - g(X))$

Main result of Chernozhukov et al. [2017a]: under 1st-order orthogonality, $\hat{\theta}^{SS}, \hat{\theta}^{CF}$ $\sqrt{n}$-a.n. when $\|\hat{h}_i - h_{0,i}\| = o_p(n^{-1/4}), \forall i$

Higher-order Orthogonality

Definition (k-Orthogonal Moments)

Moments m are k-*orthogonal*, if for **all** $\alpha \in \mathbb{N}^\ell$ with $\|\alpha\|_1 \le k$:

$$\mathbb{E}\left[D^\alpha m(Z, \theta_0, \gamma)|_{\gamma=h_0(X)} \middle| X\right] = 0.$$

where

$$D^\alpha m(Z, \theta, \gamma) = \nabla_{\gamma_1}^{\alpha_1} \nabla_{\gamma_2}^{\alpha_2} \ldots \nabla_{\gamma_\ell}^{\alpha_\ell} m(Z, \theta, \gamma)$$

and the γ_i's are the coordinates of the ℓ nuisance functions

- Grants k-th-order insensitivity to errors in nuisance estimates
  - Annihilates terms with order $\le k$ in Taylor expansion around nuisance

Asymptotic Normality from k-Orthogonality

Theorem ([Mackey, Syrgkanis, and Zadik, 2018])

Under k-orthogonality and standard identifiability and regularity assumptions, $\|\hat{h}_i - h_{0,i}\| = o_p(n^{-1/(2k+2)})$ for all i suffices for $\sqrt{n}$-a.n. of $\hat{\theta}^{SS}$ and $\hat{\theta}^{CF}$ with $\Sigma = J^{-1} V J^{-1}$ for $J = \mathbb{E}[\nabla_\theta m(Z, \theta_0, h_0(X))]$ and $V = \mathtt{Cov}(m(Z, \theta_0, h_0(X)))$.

- Actually suffices to have **product** of nuisance function errors decay $(n^{1/2} \cdot \sqrt{\mathbb{E}[\prod_{i=1}^{\ell} |\hat{h}_i(X) - h_{0,i}(X)|^{2\alpha_i} \mid \hat{h}]} \xrightarrow{p} 0$ for $\|\alpha\|_1 = k+1)$: if one is more accurately estimated, another can be estimated more crudely
- We prove similar results for non-uniform orthogonality
- $o_p(n^{-1/(2k+2)})$ rate holds the promise of coping with more complex or higher-dimensional nuisance functions

Question: How do we construct k-orthogonal moments in practice?

Second-order Orthogonality for PLR: Limitations

Question: Can we construct k-orthogonal moments in practice?

$$Y = \theta_0 T + f_0(X) + \epsilon, \quad \mathbb{E}[\epsilon \mid X, T] = 0 \quad a.s.$$
$$T = g_0(X) + \eta, \quad \mathbb{E}[\eta \mid X] = 0 \quad a.s., \quad \mathrm{Var}(\eta) > 0$$

Theorem ([Mackey, Syrgkanis, and Zadik, 2018])

Suppose the conditional distribution of η given X is a.s. Gaussian. Then no 2-orthogonal twice differentiable m yields $\sqrt{n}$-consistency.

- We use Stein's lemma ($\mathbb{E}[q'(Z)] = \mathbb{E}[Zq(Z)]$ for $Z \sim N(0, 1)$) to show 2-orthogonality implies $\mathbb{E}[\nabla_\theta m(Z, \theta_0, h_0(X))] = 0$ and hence infinite asymptotic variance for the Z-estimator
- Sad, but non-Gaussian residuals are common in pricing where T = log price, and η is a random log percentage discount (25% off now through Sunday!) over the log baseline price $g_0(X)$

Second-order Orthogonality for PLR: Power

Question: How do we construct k-orthogonal moments in practice?

$$Y = \theta_0 T + f_0(X) + \epsilon, \quad \mathbb{E}[\epsilon \mid X, T] = 0 \quad a.s.$$

$$T = g_0(X) + \eta, \quad \mathbb{E}[\eta \mid X] = 0 \quad a.s., \quad \mathrm{Var}(\eta) > 0$$

Exploit non-Gaussianity: η conditionally Gaussian given $X \Leftrightarrow$ $\mathbb{E}[\eta^{r+1}|X] = r\mathbb{E}[\eta^2|X]\mathbb{E}[\eta^{r-1}|X]$ for all $r \in \mathbb{N}$

Theorem ([Mackey, Syrgkanis, and Zadik, 2018])

Suppose that, for some $r \in \mathbb{N}$, $\mathbb{E}[\eta^{r+1}] \neq r\mathbb{E}[\mathbb{E}[\eta^2|X]\mathbb{E}[\eta^{r-1}|X]]$. *If we know* $\mathbb{E}[\eta^r|X]$, *then the 2-orthogonal moments*

$$\begin{aligned} & m(Z, \theta, q(X), g(X), \mu_{r-1}(X)) \\ \triangleq & (Y - q(X) - \theta(T - g(X))) \\ & \quad \times ((T - g(X))^r - \mathbb{E}[\eta^r|X] - r(T - g(X))\mu_{r-1}(X)) \end{aligned}$$

satisfy our standard identifiability and regularity conditions.

- $o(n^{-1/6})$ nuisance estimation error suffices for $\sqrt{n}$-a.n.

Second-order Orthogonality for PLR: Power

Question: How do we construct k-orthogonal moments in practice?

$$Y = \theta_0 T + f_0(X) + \epsilon, \quad \mathbb{E}[\epsilon \mid X, T] = 0 \quad a.s.$$

$$T = g_0(X) + \eta, \quad \mathbb{E}[\eta \mid X] = 0 \quad a.s., \quad \mathrm{Var}(\eta) > 0$$

Exploit non-Gaussianity: η conditionally Gaussian given $X \Leftrightarrow$ $\mathbb{E}[\eta^{r+1}|X] = r\mathbb{E}[\eta^2|X]\mathbb{E}[\eta^{r-1}|X]$ for all $r \in \mathbb{N}$

Theorem ([Mackey, Syrgkanis, and Zadik, 2018])

Suppose that, for some $r \in \mathbb{N}$, $\mathbb{E}[\eta^{r+1}] \neq r\mathbb{E}[\mathbb{E}[\eta^2|X]\mathbb{E}[\eta^{r-1}|X]]$. Then, except for the $(q(X), \mu_r(X))$ and $(g(X), \mu_r(X))$ pairings,

$$\begin{aligned} & m(Z, \theta, q(X), g(X), \mu_{r-1}(X), \mu_r(X)) \\ \triangleq & (Y - q(X) - \theta(T - g(X))) \\ & \quad \times ((T - g(X))^r - \mu_r(X) - r(T - g(X))\mu_{r-1}(X)) \end{aligned}$$

is 2-orthogonal and satisifes our standard conditions.

- $o(n^{-1/3})$ error for $\mu_r(X)$ and $o(n^{-1/6})$ for rest suffice for $\sqrt{n}$-a.n.

PLR with High-dimensional Linear Nuisance

High-dimensional Linear Nuisance Setting

$$Y = \theta_0 T + \langle X, \beta_0 \rangle + \epsilon, \quad \mathbb{E}[\epsilon \mid X, T] = 0 \quad a.s.$$
$$T = \langle X, \gamma_0 \rangle + \eta, \quad \mathbb{E}[\eta \mid X] = 0 \quad a.s., \quad \mathrm{Var}(\eta) > 0$$

- $\beta_0, \gamma_0 \in \mathbb{R}^p$ are s-sparse, (η, ϵ, X) independent, $q_0 = \theta_0 \beta_0 + \gamma_0$

How many relevant confounders (non-zeros) can we tolerate?

- Lasso can estimate β_0, γ_0 with $O(\sqrt{s \log p / n})$ error
- Zeroth-order orthogonality rate $O(n^{-1/2})$: $s = O(1/\log p)$
  - $m = (Y - \theta T - \langle X, \beta \rangle) T$
- First-order orthogonality rate $o(n^{-1/4})$: $s = o(n^{1/2}/\log p)$
  [Chernozhukov, Chetverikov, Demirer, Duflo, Hansen, and Newey, 2017a]
  - $m = (Y - \theta T - \langle X, \beta \rangle)(T - \langle X, \gamma \rangle)$
  - $m = (Y - \langle X, q \rangle - \theta(T - \langle X, \gamma \rangle))(T - \langle X, \gamma \rangle)$

PLR with High-dimensional Linear Nuisance

High-dimensional Linear Nuisance Setting

$$Y = \theta_0 T + \langle X, \beta_0 \rangle + \epsilon, \quad \mathbb{E}[\epsilon \mid X, T] = 0 \quad a.s.$$

$$T = \langle X, \gamma_0 \rangle + \eta, \quad \mathbb{E}[\eta \mid X] = 0 \quad a.s., \quad \mathrm{Var}(\eta) > 0$$

- $\beta_0, \gamma_0 \in \mathbb{R}^p$ are s-sparse, (η, ϵ, X) independent, $q_0 = \theta_0 \beta_0 + \gamma_0$

Theorem ([Mackey, Syrgkanis, and Zadik, 2018])

Suppose $\mathbb{E}[\eta^4] \neq 3\mathbb{E}[\eta^2]^2$, X *has i.i.d.* $N(0,1)$ *entries,* ϵ *and* η *are bounded by* C, *and* $\theta_0 \in [-M, M]$. *If* $s = o(n^{2/3}/\log p)$, *and we*

(a) *estimate* q_0, γ_0 *via Lasso with* $\lambda_n = 2CM\sqrt{3\log(p)/n}$ *and*

(b) *estimate* $\mathbb{E}[\eta^2]$ *and* $\mathbb{E}[\eta^3]$ *using* $\hat{\eta}_t \triangleq T'_t - \langle X'_t, \hat{\gamma} \rangle$,

$$\hat{\mu}_2 = \tfrac{1}{n}\textstyle\sum_{t=1}^n \hat{\eta}_t^2, \text{ and } \hat{\mu}_3 = \tfrac{1}{n}\textstyle\sum_{t=1}^n (\hat{\eta}_t^3 - 3\hat{\mu}_2\hat{\eta}_t),$$

for $(T'_t, X'_t)_{t=1}^n$ *an i.i.d. sample independent of* $\hat{\gamma}$,

then the moments $m = (Y - \langle X, q \rangle - \theta(T - \langle X, \gamma \rangle)) \times \left((T - \langle X, \gamma \rangle)^3 - \mu_3 - 3(T - \langle X, \gamma \rangle)\mu_2\right)$ *yield* $\sqrt{n}$*-a.n.*

High-dimensional PLR Experiments

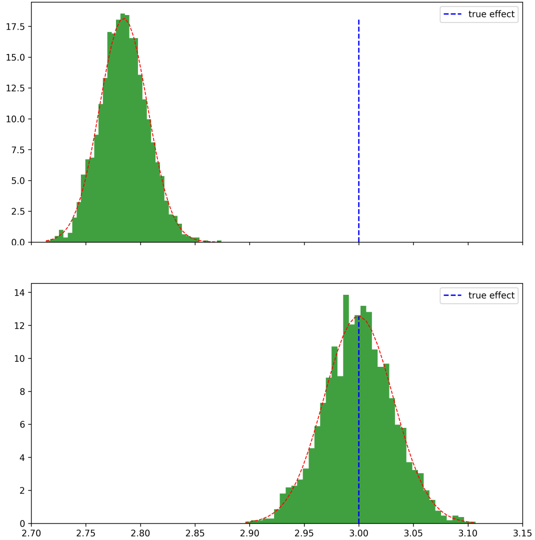
High-dimensional Linear Nuisance Setting

$$Y = \theta_0 T + \langle X, \beta_0 \rangle + \epsilon, \quad \mathbb{E}[\epsilon \mid X, T] = 0 \quad a.s.$$
$$T = \langle X, \gamma_0 \rangle + \eta, \quad \mathbb{E}[\eta \mid X] = 0 \quad a.s., \quad \mathrm{Var}(\eta) > 0$$

- $\beta_0, \gamma_0 \in \mathbb{R}^p$ are s-sparse, (η, ϵ, X) independent, $q_0 = \theta_0 \beta_0 + \gamma_0$
- Mimic price elasticity of demand setting: T represents log price and η drawn from discrete distribution representing random (log) discounts over baseline price

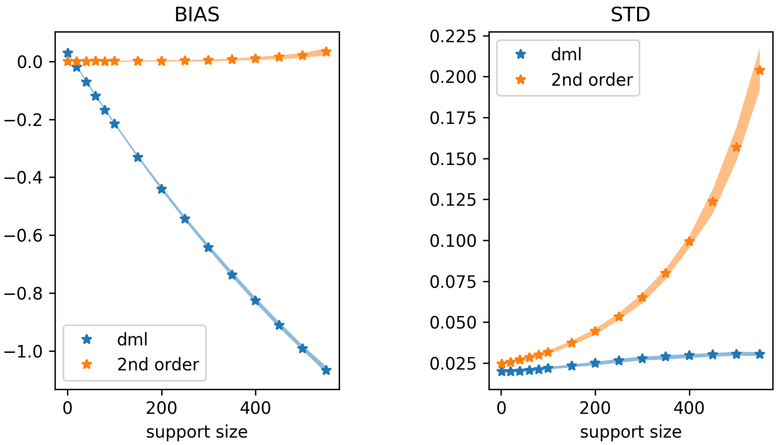
High-dimensional PLR: Fixed Sparsity

1st (top) vs. 2nd order, $s = 100$, $n = 5000$, $p = 1000$, $\theta_0 = 3$.



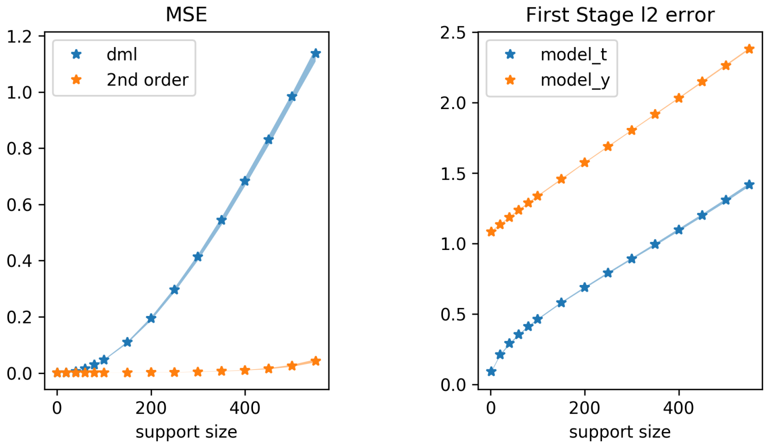
High-dimensional PLR: Varying Sparsity

1st vs. 2nd order, $n = 5000$, $p = 1000$, $\theta_0 = 3$.



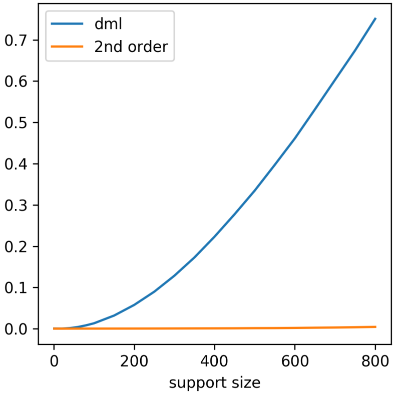
High-dimensional PLR: Varying Sparsity

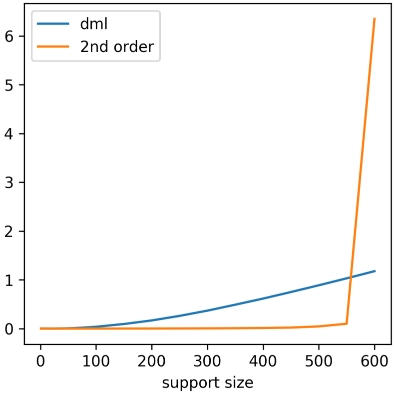
1st vs. 2nd order, $n = 5000$, $p = 1000$, $\theta_0 = 3$.



High-dimensional PLR: MSE for Varying n, p, s

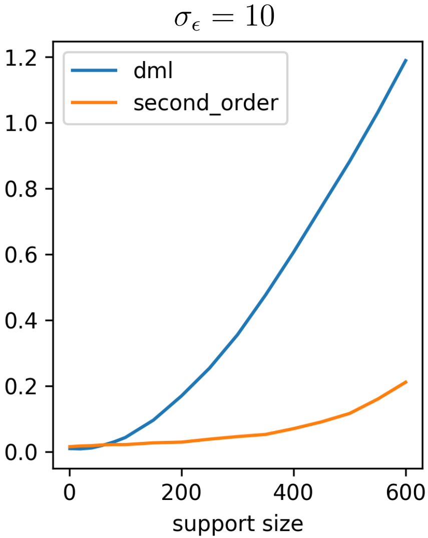
$n = 10000, p = 1000$ and $n = 5000, p = 2000$





High-dimensional PLR: Varying Noise Level

$$n = 5000, p = 1000$$



Recap

What have we accomplished?

1. Introduced a notion of k-orthogonality for two-stage Z-estimation with nuisance, generalizing Neyman orthogonality
2. Showed that $o(n^{-\frac{1}{2k+2}})$ nuisance estimate error suffices for $\sqrt{n}$-asymptotic normality of target parameters
3. Established that non-normality of $\eta|X$ necessary for the existence of useful 2-orthogonal moments in PLR model
4. Derived explicit 2-orthogonal moments for PLR given knowledge of non-normality
5. Used 2-orthogonal moments to tolerate $o(\frac{n^{\frac{2}{3}}}{\log p})$ sparsity in high-dimensional PLR
6. Showed benefits over standard $o(\frac{n^{\frac{1}{2}}}{\log p})$ first-order orthogonal moments in synthetic demand estimation experiments

Future Directions

Many opportunities for future development

1. Second-order orthogonality
   - How to select optimal / improved double orthogonal moments
   - How to construct moments for other causal inference models
2. k-th order orthogonality for $k > 2$
   - When are k-th order orthogonal moments available and useful?
   - How do we construct them explicitly?
3. Lower bounds: (non-Gaussian) examples where first-order orthogonality provably worse than second-order orthogonality
4. Implications for Lasso debiasing [Zhang and Zhang, van de Geer, Buhlmann, Ritov, and Dezeure, 2014, Javanmard and Montanari, 2015]?
5. Applications to problems with non-Gaussian treatment residuals

References I

V. Chernozhukov, D. Chetverikov, M. Demirer, E. Duflo, C. Hansen, and W. Newey. Double/debiased/neyman machine learning of treatment effects. *American Economic Review*, 107(5):261–65, May 2017a.

V. Chernozhukov, M. Goldman, V. Semenova, and M. Taddy. Orthogonal machine learning for demand estimation: High dimensional causal inference in dynamic panels. *arXiv preprint arXiv:1712.09988*, 2017b.

A. Javanmard and A. Montanari. De-biasing the Lasso: Optimal Sample Size for Gaussian Designs. *ArXiv e-prints*, Aug. 2015.

L. Mackey, V. Syrgkanis, and I. Zadik. Orthogonal machine learning: Power and limitations. In J. Dy and A. Krause, editors, *Proceedings of the 35th International Conference on Machine Learning*, volume 80 of *Proceedings of Machine Learning Research*, pages 3375–3383, Stockholmsmssan, Stockholm Sweden, 10–15 Jul 2018. PMLR.

J. Neyman. C() tests and their use. *Sankhy: The Indian Journal of Statistics, Series A (1961-2002)*, 41(1/2):1–21, 1979. ISSN 0581572X.

S. van de Geer, P. Buhlmann, Y. Ritov, and R. Dezeure. On asymptotically optimal confidence regions and tests for high-dimensional models. *Ann. Statist.*, 42(3):1166–1202, 06 2014. doi: 10.1214/14-AOS1221.

N. Wilkins, A. Yurekli, and T.-w. Hu. Economic analysis of tobacco demand. *Economics of Tobacco Toolkit*, 80576, 2004.

C. H. Zhang and S. Zhang. Confidence intervals for low dimensional parameters in high dimensional linear models. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 76(1):217–242. doi: 10.1111/rssb.12026.

Experiment Specification

- η is drawn from a discrete distribution with values $\{0.5, 0, -1.5, -3.5\}$ taken with probabilities $(.65, .2, .1, .05)$.
- ϵ is drawn independently from a uniform $U(-\sigma_\epsilon, \sigma_\epsilon)$ distribution.
- Importantly, the coordinates of the s non-zero entries of the coefficient β_0 are the same as the coordinates of the s non-zero entries of γ_0.
- Each non-zero coefficient was generated independently from a uniform $U(0, 5)$ distribution.
- The regularization parameter λ_n of each Lasso was $\sqrt{\log(p)/n}$.
- For each instance of the problem, i.e., each random realization of the coefficients, we generated 2000 independent datasets to estimate the bias and standard deviation of each estimator. We repeated this process over 100 randomly generated problem instances, each time with a different draw of the coefficients γ_0 and β_0, to evaluate variability across different realizations of the nuisance functions.