STATS 300A Theory of Statistics
Stanford University, Fall 2015

Problem Set 3

Due: Thursday, October 15, 2015

Instructions:

- You may appeal to any result proved in class or proved in the course textbooks.
- Any request to find a statistic with certain properties requires proof that those properties are satisfied.

Problem 1. (Nonparametric Unbiased Risk Estimation)

In this question, we will estimate the risk (and hence the quality) of an estimator under minimal assumptions on the data-generating distribution. Let $X = (X_1, \ldots, X_{n+1})$ be an i.i.d. sample from a model $\mathcal{P}$, under which $\mathbb{E}_{\mathbb{P}}[|X_1|] < \infty$ for each $\mathbb{P} \in \mathcal{P}$, and $T(X) = (X_{(1)}, \ldots, X_{(n+1)})$ is complete. Suppose that $\tilde{\delta}_n(X)$ is an estimate of the mean $g(\mathbb{P}) = \mathbb{E}_{\mathbb{P}}[X_1]$ that only makes use of the first n coordinates. That is,

$$\tilde{\delta}_n(x_1, \ldots, x_n, x_{n+1}) = \delta_n(x_1, \ldots, x_n)$$

for some δ_n and every valid input vector $(x_1, \ldots, x_{n+1})$. Derive the UMVUE of the mean squared error,

$$\mathbb{E}_{\mathbb{P}}\left[(\tilde{\delta}_n(X) - g(\mathbb{P}))^2\right] = \mathbb{E}_{\mathbb{P}}\left[(\delta_n(X_1, \ldots, X_n) - g(\mathbb{P}))^2\right]$$

provided that this risk is finite for all $\mathbb{P}$. (The UMVUE would have a similar form if your goal was to estimate $g(\mathbb{P}) = \mathbb{E}_{\mathbb{P}}[\delta_u(X_1)]$ for any statistic δ_u with $\mathbb{E}_{\mathbb{P}}[|\delta_u(X_1)|] < \infty$ for all $\mathbb{P}$.)

Problem 2 (Two-sample Location Model)**.** Suppose $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ have joint density $f(x_1 - \xi, \ldots, x_m - \xi; y_1 - \eta, \ldots, y_n - \eta)$, and consider the problem of estimating the location parameter difference $\Delta = \eta - \xi$. Prove the equivalents of TPE Chapter 3 Theorems 1.4, 1.8, 1.10, and 1.17 and Corollaries 1.11, 1.12, and 1.14 for estimators satisfying the following equivariance property:

$$\delta(\mathbf{x} + a, \mathbf{y} + b) = \delta(\mathbf{x}, \mathbf{y}) + (b - a).$$

(In your equivalent of Theorem 1.17, you should assume that $\mathbf{X}$ and $\mathbf{Y}$ are independent.)

Problem 3 (Two-sample MRE)**.** Suppose $X_1, \ldots, X_m \overset{\text{iid}}{\sim} \text{Exp}(\xi, 1)$ and, independently, $Y_1, \ldots, Y_n \overset{\text{iid}}{\sim} \text{Exp}(\eta, t)$ for t fixed and known. Using the notion of equivariance from the previous problem, determine the minimum risk equivariant estimator of $\Delta = \eta - \xi$ under squared error loss when $m \neq n$.

Problem 4 (Another MRE Characterization)**.** Suppose that δ_0 is location equivariant and unbiased, and let $\mathcal{U}$ be the class of all location invariant functions u for which $u(X)$ is an unbiased estimator of zero. Show that δ_0 is MRE with respect to squared error loss iff $\operatorname{Cov}[\delta_0(X), u(X)] = 0$ for all $u \in \mathcal{U}$.

Problem 5 (Feedback)**.** (This "problem" is not graded.)

(a) How much time did you spend on this problem set?

(b) Which problems did you find valuable?