STATS 300A Theory of Statistics
Stanford University, Fall 2015

Problem Set 7

Due: Thursday, November 19, 2015

Instructions:

- You may appeal to any result proved in class or proved in the course textbooks.
- Any request to "find" requires proof that all requested properties are satisfied.

Problem 1 (TSH 3.56)**.** Let the variables X_i $(i = 1, \ldots, s)$ be independently distributed with distribution Poi(λ_i). For testing the hypothesis $H_0 : \sum_{j=1}^{s} \lambda_j \leq a$ (for example, that the combined radioactivity of a number of pieces of radioactive material does not exceed a), show that there exists a UMP test, which rejects when $\sum_{j=1}^{s} X_j > C$.

Problem 2 (TSH 3.58)**.**

Problem 3. A $1-\alpha$ *confidence region* $S(X)$ for an estimand $g(\theta)$ is a set-valued function of the data satisfying $\mathbb{P}_\theta\left[g(\theta) \in S(X)\right] \geq 1 - \alpha$ for all $\theta \in \Omega$. For each of the following models, (i) find a uniformly most powerful unbiased level α test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ for θ_0 known, and (ii) find a $1 - \alpha$ confidence region for the parameter θ:

(a) $X = (X_1, \ldots, X_n)$ with independent coordinates $X_i \sim \mathcal{N}(t_i\theta, 1)$ for t_i known $\forall i$.

(b) For $\theta > 0$, X is a single scalar random angle with von Mises density

$$p_\theta(x) = \frac{\exp\left(\theta \cos x\right)}{2\pi I_0(\theta)} \mathbb{I}\left(x \in [0, 2\pi)\right)$$

where I_0 is a modified Bessel function with $I_0(0) = 1$. (This is a distribution on a circle, and is often used in the earth sciences to model the direction of wind, glaciers, water flows etc.) The following additional instructions apply to part (b):

- Simplify the form of your test as much as possible.
- You need not solve for rejection thresholds $C(\theta_0)$ explicitly.
- You should specify the equations that could be solved to determine any thresholds.
- Your confidence region may make use of any partially determined rejection thresholds $C(\theta_0)$.

Problem 4. Suppose that X is distributed according to a Cauchy location model with location parameter θ so that

$$p_\theta(x) = \frac{1}{\pi[1 + (x - \theta)^2]}, \quad \forall x \in \mathbb{R}.$$

For testing $H_0 : \theta = 0$ against $H_1 : \theta \neq 0$, derive a test ϕ with level $\alpha = 5\%$ that maximizes $\mathbb{E}_{\theta=1}[\phi(X)]$ under the constraint $\mathbb{E}_{\theta=1}[\phi(X)] = \mathbb{E}_{\theta=-1}[\phi(X)]$. Is this test uniformly most powerful unbiased?

Problem 5 (TSH 4.6 parts (i) and (ii)).