STATS 300A: Theory of Statistics **Fall 2014**

Lecture 18 — December 2

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Warning: *These notes may contain factual and/or typographic errors.*

18.1 Computing a Maximal Invariant in Stages

Last time, we discussed the role of maximal invariants in deriving UMPI tests and found maximal invariants for some common classes of transformations. In this abbreviated lecture, we learn how to compute a maximal invariant in stages when a group decomposes into generating subgroups.

Example 1 (Elementwise linear transformations)**.** Suppose $X = (X_1, X_2, \ldots, X_n)$, and consider the group of scale and shift transformations $\mathcal{G}$ with elements g satisfying $gX = (aX_1 + b, aX_2 + b \ldots aX_n + b)$ for some $a \neq 0$ and $b \in \mathbb{R}$.

One can check that $\mathcal{G}$ is **generated** by the shift subgroup $\mathcal{D}$ with elements d satisfying $dX = (X_1 + b, \ldots, X_n + b)$ for $b \in \mathbb{R}$ and the scale subgroup $\mathcal{E}$ with elements e satisfying $eX = (aX_1, \ldots, aX_n)$ for $a \neq 0$. That is, $\mathcal{G}$ is the smallest group containing $\mathcal{D}$ and $\mathcal{E}$.

We will derive a maximal invariant for $\mathcal{G}$ in stages by (1) deriving an MI $Y = S(X)$ under one subgroup, $\mathcal{D}$, (2) finding the group $\mathcal{E}'$ induced by $\mathcal{E}$ on Y-space, and (3) deriving an MI $T(Y)$ under $\mathcal{E}'$:

1. We have seen previously that, under the shift subgroup $\mathcal{D}$, an MI is the vector of differences $S(X) = (Y_1, Y_2, \ldots, Y_{n-1})$, where $Y_i = X_i - X_n$.

2. For each e in the scale group $\mathcal{E}$, $S(eX) = (eY_1, \ldots, eY_{n-1}) = eS(X) = eY$. Hence, $\mathcal{E}$ induces an isomorphic group $\mathcal{E}' \cong \mathcal{E}$ on Y-space (note that $\mathcal{E}$ is a group of transformations on $\mathbb{R}^n$ whereas $\mathcal{E}'$ is a group of transformations on $\mathbb{R}^{n-1}$).

3. As seen in an earlier lecture, an MI under $\mathcal{E}'$ is the vector of differences

$$T(Y) = \left(\frac{Y_1}{Y_{n-1}}, \frac{Y_2}{Y_{n-1}}, \ldots, \frac{Y_{n-2}}{Y_{n-1}}\right) = \left(\frac{X_1 - X_n}{X_{n-1} - X_n}, \frac{X_2 - X_n}{X_{n-1} - X_n}, \ldots, \frac{X_{n-2} - X_n}{X_{n-1} - X_n}\right).$$

The good news is that any statistic $T(S(X))$ derived in this stagewise manner is a maximal invariant for the original group $\mathcal{G}$:

Theorem 1 (TSH 6.2.2)**.** Suppose that $\mathcal{D}$ and $\mathcal{E}$ are subgroups that generate a group of transformations $\mathcal{G}$. Suppose that $Y = S(X)$ is MI for $\mathcal{D}$ and that for any $e \in \mathcal{E}$, $S(X_1) = S(X_2) \Rightarrow S(eX_1) = S(eX_2)$. Define the induced group $\mathcal{E}'$ with elements e' on $Y = S(X)$: $e'(Y) \triangleq S(eX)$. If $T(Y)$ is MI for $\mathcal{E}'$, then $T(S(X))$ is MI for the full group $\mathcal{G}$.

The bad news is that order matters. Indeed, suppose that in the example above, we had reversed the subgroup order and first derived an MI under the scale subgroup $\mathcal{E}$. We would get $S(X) = (X_1/X_n, \ldots, X_{n-1}/X_n) = Z$. To find the induced group of transformations $\mathcal{D}$ in Step 2, note that

$$S(dX) = \left(\frac{X_1 + a}{X_n + a}, \ldots, \frac{X_{n-1} + a}{X_n + a} \right). \tag{18.1}$$

Unfortunately, there is no transformation d' for which the RHS above is equal to $d'(Z)$. Hence, there is no induced subgroup $\mathcal{D}'$, and so Step 2 fails.

In summary, Steps 1, 2, and 3 do not necessarily work for any choice of an ordered pair $(\mathcal{D}, \mathcal{E})$ of generating subgroups. However, anytime we can carry out these three steps, we obtain a maximal invariant for $\mathcal{G}$.