STATS 300A: Theory of Statistics	Fall 2015

Lecture 19 — December 3

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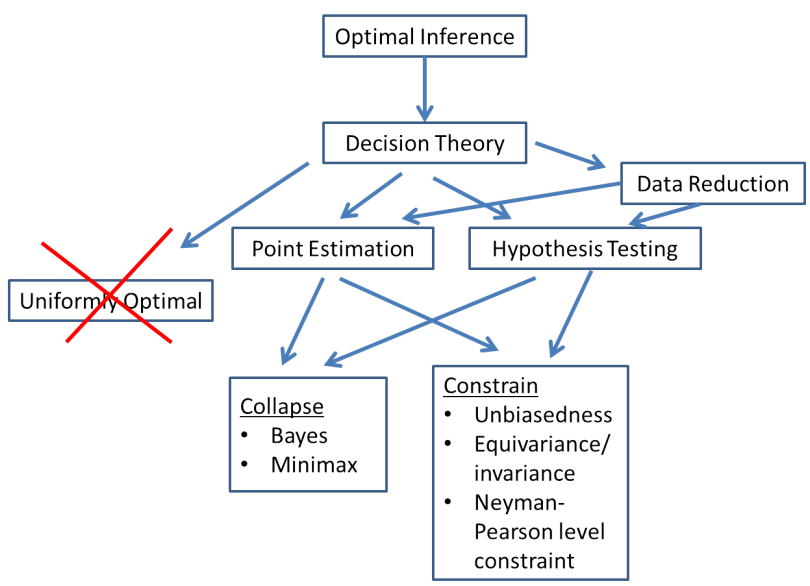
Warning: *These notes may contain factual and/or typographic errors.*

19.1 Recap

This quarter, we tackled the problem of drawing optimal inferences from data with a focus on the canonical tasks of point estimation and hypothesis testing. After adopting the framework of decision theory and ruling out the possibility of uniformly best decision procedures for most problems of interest, we explored optimal inference under constraints (unbiasedness, equivariance, invariance, and Neyman-Pearson level constraints) and with respect to scalar summaries of the risk function (average risk and worst case risk). Along the way, we found data reduction to be a useful tool in many decision problems.



19.2 Confidence Regions

A third canonical statistical task is that of deriving a region that, with high probability, contains a parameter of interest:

Definition 1 (Confidence region)**.** A set $S(X)$ satisfying $\mathbb{P}_\theta[\theta \in S(X)] \geq 1-\alpha$, for all $\theta \in \Omega$ is known as a $1-\alpha$ **confidence region**, with a $1-\alpha$ **confidence level**.

We will see in this abbreviated lecture that there is a direct correspondence between confidence regions and families of hypothesis tests and that all of our work on optimal testing translates directly into the design of optimal confidence regions. In other words, we can derive confidence regions directly from families of tests. We can already begin to see a connection in the definition of a confidence region: the constraint $\mathbb{P}_\theta[\theta \in S(X)] \geq 1-\alpha$ reminds us of a level constraint for a hypothesis test. In the next section, we show how a confidence region can be built up from a family of hypothesis tests.

19.2.1 From families of tests to confidence regions

Given a model $\mathcal{P} = \{\mathbb{P}_\theta : \theta \in \Omega\}$, we begin by defining an appropriate collection of tests. For each $\theta_0 \in \Omega$, let $\Omega_0(\theta_0)$ be a set containing θ_0, where $\Omega_0(\theta_0) \subset \Omega$. Next, define ϕ_{θ_0} to be a level α test for H_0: $\theta \in \Omega_0(\theta_0)$ vs. H_1: $\theta \notin \Omega_0(\theta_0)$, and define $A(\theta_0)$ as the acceptance region[1] of ϕ_{θ_0}. Because ϕ_{θ_0} is a level α test, and $\theta_0 \in \Omega_0(\theta_0)$, we have $P_\theta(X \in A(\theta)) \geq 1-\alpha$ for all $\theta \in \Omega$.

Now consider the region $S(X) = \{\theta \in \Omega : X \in A(\theta)\}$. Since $P_\theta(\theta \in S(X)) = P_\theta(X \in A(\theta)) \geq 1-\alpha$, $S(X)$ is a $1-\alpha$ confidence region! Different choices of null sets $\Omega_0(\theta_0)$ will lead to different forms of confidence regions. For example,

1. **One-sided tests** $\Omega_0(\theta_0) = \{\theta : \theta \leq \theta_0\}$ often yield **confidence bounds**

$$S(X) = \{\theta : u(X) \leq \theta\}$$

   where $u(X)$ denotes a data-dependent lower bound.

2. **Two-sided tests** $\Omega_0(\theta_0) = \{\theta_0\}$ often yield **confidence intervals**

$$S(X) = \{\theta : u(X) \leq \theta \leq v(X)\},$$

   where $v(X)$ denotes a data-dependent upper bound.

19.2.2 Optimality

Intuitively, we desire a $1-\alpha$ confidence region that is as narrow as possible, which we can achieve by minimizing the number of extraneous it contains. We make this notion precise in the following optimality property for confidence regions.

Definition 2 (UMA)**.** A $1-\alpha$ confidence region $S(X)$ is **uniformly most accurate** if, among all $1-\alpha$ regions, $\mathbb{P}_\theta(\theta' \in S(X)) = \mathbb{P}_\theta(X \in A(\theta'))$ is minimized for all (θ, θ') satisfying $\theta \notin \Omega_0(\theta')$.

A UMA confidence region has minimal probability of including undesirable points, those points θ' for which θ is in the alternative. This corresponds to finding a UMP test of $\Omega_0(\theta')$ for each θ', since the power $\beta(\theta) = 1 - \mathbb{P}_\theta(X \in A(\theta'))$. Hence, families of UMP tests give rise to UMA regions, and, similarly, **families of UMP constrained tests yield constrained UMA confidence regions**.

[1] We view the acceptance region as random when ϕ_{θ_0} is randomized.

19.3 Beyond 300A

- STATS 300A focused on **finite sample theory**. You can learn about **asymptotic theory** in STATS 300B.
- The STATS 300 sequence focuses on the theory of statistics. You can learn about **applied statistics** in STATS 306A/B and 315A/B.