Survival Analysis: Martingale CLT

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If $M_1(\cdot)$ and $M_2(\cdot)$ are mean zero martingales defined on the same filtration, and that for every $t$, $E(M_j^2(t)) < \infty$ for $j = 1$ and $2$, then there exists a right-continuous predictable process $< M_1, M_2 > (\cdot)$ such that $M_1(\cdot)M_2(\cdot) - < M_1, M_2 > (\cdot)$ is a zero-mean martingale.

$< M_1, M_2 >$ tells us about the covariance function of $M_1(\cdot)$ and $M_2(\cdot)$ since

$$cov(M_1(t), M_2(s)) = E(< M_1, M_2 > (min(t, s)))$$
• if \( \langle M_1, M_2 \rangle (\cdot) = 0 \text{ a.s., then } M_1(\cdot)M_2(\cdot) \) is a martingale and \( M_1(\cdot) \) and \( M_2(\cdot) \) are called orthogonal.

• Suppose that \( N_i(\cdot), i = 1, \cdots, K \) are counting processes with continuous compensator \( A_i(\cdot), i = 1, \cdots, K \) respectively. Then if no two of the counting process can jump at the same time, \( \langle M_i, M_j \rangle (\cdot) = 0 \text{ a.s. for } i \neq j \), where \( M_i(\cdot) = N_i(\cdot) - A_i(\cdot) \).
Let \( N(\cdot) \) be a counting process with continuous compensator \( A(\cdot) \), such that \( M(\cdot) = N(\cdot) - A(\cdot) \) is a zero mean martingale. If \( H(\cdot) \) is a bounded, predictable process defined on the same filtration, the process \( Q(\cdot) \) defined by

\[
Q(t) = \int_0^t H(s) dM(s)
\]

is also a zero mean martingale.

Justifications:

\[
E \left\{ \int_s^t H(u) dM(u) | \mathcal{F}_s \right\} = E \left\{ E \left[ \int_s^t H(u) dM(u) | \mathcal{F}_u \right] | \mathcal{F}_s \right\} = E \left\{ \int_s^t H(u) E [dM(u) | \mathcal{F}_u] | \mathcal{F}_s \right\} = 0
\]
If $E(M(s)^2) < \infty$ and $N(\cdot)$ is bounded, then for all $t$:

$$< Q, Q > (t) = \int_0^t H^2(s)dA(s), a.s.$$ 

$$var[Q(t)] = E(< Q, Q > (t)) = E\left(\int_0^t H^2(s)dA(s)\right).$$
Martingale Integration

- Suppose that $N_i(\cdot), i = 1, 2, \cdots$ are bounded counting processes, $M_i(\cdot), i = 1, 2, \cdots$ are the corresponding zero-mean counting process martingales, each $M_i$ satisfies $E(M_i^2(t)) < \infty$ for any $t$ and that $H_i(\cdot), i = 1, 2, \cdots$ are bounded and predictable processes. Let

$$Q_i(t) = \int_0^t H_i(u) dM_i(u),$$

then $\langle Q_i, Q_j \rangle(t) = \int_0^t H_i(s) H_j(s) d < M_i, M_j > (s)$ a.s.

- If $M_i(\cdot), i = 1, 2, \cdots$ are orthogonal, then so are $Q_i(\cdot), i = 1, 2, \cdots$ and

$$Var\{\Sigma Q_i(t)\} = \sum_{i=1}^n E \left( \int_0^t H_i^2(s) dA_i(s) \right).$$
Martingale Central Limit Theorem

- $N_{in}(\cdot)$ is a counting process with continuous compensator $A_{in}(\cdot)$
- $H_{in}$ is locally bounded and predictable.
- No two of the counting processes can jump at the same time, so that the $n$ martingales $M_{in}(\cdot) = N_{in}(\cdot) - A_{in}(\cdot)$ are orthogonal.

Define

- $U_{in}(t) \overset{def}{=} \int_0^t H_{in}(s)dM_{in}(s)$
- $U_{in,\epsilon}(t) \overset{def}{=} \int_0^t H_{in}(s)1[|H_{in}(s)| \geq \epsilon]dM_{in}(s)$.
Martingale Central Limit Theorem

If

(a) \( < \sum_{i=1}^{n} U_{in}, \sum_{i=1}^{n} U_{in} > (t) \xrightarrow{p} \alpha(t) \)

(b) \( < \sum_{i=1}^{n} U_{in, \epsilon}, \sum_{i=1}^{n} U_{in, \epsilon} > (t) \xrightarrow{p} 0 \ \forall \epsilon > 0 \)

as \( n \to \infty \), then as \( n \to \infty \)

\[ \sum_{i=1}^{n} U_{in}(\cdot) \to U(\cdot) \]

weakly, where \( U(\cdot) \) is a zero-mean Gaussian process with independent increments and variance function \( \alpha(\cdot) \). The Gaussian process \( U(t) = \int_{0}^{t} f(s) dW(s) \), where \( \int_{0}^{t} f(s)^2 ds = \alpha(t) \) and \( W(\cdot) \) is a Weiner Process.
**Weiner Process**

- **Definition:** If $W(\cdot)$ is a Gaussian process satisfying $W(0) = 0$, $E(W(t)) = 0$ for all $t$, and $\text{Cov}(W(s), W(t)) = \min(s, t)$ for all $t, s$, then $W(\cdot)$ is a Wiener process.

1. $W(\cdot)$ is a zero-mean martingale.
2. The predictable quadratic variation process for $W(\cdot)$ satisfies
   \[
   \langle W, W \rangle(t) = t.
   \]
3. $Q(t) \overset{def}{=} \int_0^t f(s)dW(s)$ is a zero-mean Gaussian process with $Q(0) = 0$, independent increments, and variance function
   \[
   \text{var}(Q(t)) = \int_0^t f^2(s)ds.
   \]
Weak Convergence in Stochastic Process

- $X_n(\cdot)$ converges weakly to $X(\cdot)$ if for all bounded, continuous (with respect to the Skorohod topology), real value functions $f$, 

\[ E\{f(X_n)\} \rightarrow E\{f(X)\} \text{ as } n \to \infty. \]

- Example of $f(\cdot)$:
  1. $f(X_n(\cdot)) = \sup_{t \in [0, \tau]} |X_n(t)|$
  2. $f(X_n(\cdot)) = \int_0^\tau X_n(t)dt.$
  3. $f(X_n(\cdot)) = X_n(t_0)$
Weak Convergence in Stochastic Process

A sequence of random processes $X_n(\cdot) \in D[0, \tau]$ converges weakly to $X(\cdot)$ if

1. The finite dimensional distribution converge:
   
   $$(X_n(t_1), \cdots, X_n(t_K)) \rightarrow (X(t_1), \cdots, X(t_K))$$

   in distribution.

2. $X_n(\cdot)$ is tight which can be implied by the condition:

   $$\lim_{\delta \to 0} \lim_{n \to \infty} \sup P \left[ \sup_{|s-t|<\delta} |X_n(t) - X_n(s)| > \epsilon \right] = 0$$

The tightness condition is difficult to verify in general. In the MCLT, the tightness condition is guaranteed via condition (b).
Martingale CLT

- From MCLT

\[ \sum_{i=1}^{n} U_{in}(\cdot) \to U(\cdot) \quad \text{weakly.} \]

\[ \sup_{t \in [0, \tau]} |\sum_{i=1}^{n} U_{in}(t)| \to \sup_{t \in [0, \tau]} |U(t)| \]

in distribution.