Constellation Design in an Energy-based Noncoherent Massive SIMO System

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Abstract

An uplink system with a single antenna transmitter and a single receiver with a large number of antennas is considered. We propose an average energy-detection-based single-shot noncoherent communication scheme which does not use the instantaneous channel state information, but uses only the knowledge of the channel distribution. The suggested system uses a transmitter that modulates information on the power of the symbols, and a receiver which exploits only the average energy across the antennas to decode the transmitted symbols. We present three different scenarios with the channel knowledge known to varying degrees of certainty. Specifically, we consider constellation designs for the cases when the transmitter and receiver have knowledge of (1) the channel fading distribution, (2) the first, second and fourth moments of the channel fading distribution, and (3) the moments of the channel distribution with some bounded uncertainty. We present numerical results on how these designs perform in typical scenarios, and show specific examples where each design should be employed. Our analysis shows that an optimized constellation for a specific channel distribution makes it very sensitive to uncertainties in the channel statistics. Furthermore, overestimating, rather than underestimating, the channel conditions could lead to significant performance loss.

Index Terms

Massive MIMO, Noncoherent Communications, Energy Receiver, Constellation Design

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I. INTRODUCTION

As the demand for mobile data in wireless broadband communications increases dramatically every year, there is an interest in an improved cellular PHY and MAC layer from both academia and industry. Large antenna MIMO arrays, while not new in astronomy or radar applications, have generated a lot of recent interest in cellular for this exact reason[1], [2]. The potential gains from massive antenna arrays are many. While beamforming and directivity gains have been traditionally associated with such systems, recent work also show significant savings on the baseband processing involved with massive antenna arrays. Not only that, sophisticated manufacturing techniques and high carrier frequencies make it increasingly feasible to pack in a larger number of antennas within a fixed form factor. These attractive gains of massive MIMO are, however, usually based on crucial but optimistic assumptions about channel state information (CSI) at the transmitter and receiver [3] and ideal hardware.

Regarding the former, even in today’s multiantenna systems (such as LTE-A), channel estimation and pilot overhead occupies a significant amount of time and frequency slots (≈ 15% in [4]). In a massive MIMO cellular scenario as proposed in [5], the base station has many more antennas than the number of users. For such a system, accurately estimating the channel is a real challenge. As mentioned in [5], operating in the time division duplex (TDD) mode may help address some of the channel estimation issues. The fact that the number of pilot signals grows linearly with the number of antennas in the FDD mode makes the channel reciprocity in the TDD mode more attractive. For example, in a SIMO system, the transmitter needs to transmit just one pilot sequence for the receiver to estimate all the channels and use for subsequent transmissions. However, this relies on channel reciprocity which may not hold due to different transceiver circuitries in the transmit and receive path [6]. Furthermore, initial investigations on the rate loss incurred by even a small training overhead in a massive SIMO system shows that in several scenarios of low SNR, or high mobility, or a large line of sight (LOS) component, a noncoherent system achieves a better probability of bit error than a coherent system for the same effective rate [7]. Under those circumstances, a noncoherent system seems an attractive alternative.

Another important challenge in designing a coherent massive MIMO system is the increased complexity of both the transmitter and receiver hardware [8], [9]. While the number of RF chains
goes up with an increasing number of antennas thereby causing increased complexity and energy consumption, hardware impairments such as phase noise and I/Q imbalances also become more severe at both the transmitter and the receiver. Proposing architectures which require simple and energy efficient analog circuit designs is thus an important research direction for realizing much of the benefits from Massive MIMO. Spatial Multiplexing (SM) [10], [11] is one example of a promising system which has only one RF chain. However, even there, we have several important challenges, such as fast antenna switching, small directional beamforming gain and the need for accurate CSI at the receiver.

The difficulty of channel state acquisition has inspired a lot of prior work, especially in noncoherent communication. The earliest incarnations of noncoherent systems were actually motivated not by the complications in CSI acquisition but by the simplicity of the receiver circuitry. The use of envelope detectors can be traced back to the well-studied quadrature, or square law receiver [12], [13] employed in the noncoherent detection of several well-known modulation schemes, such as Frequency-Shift Keying (FSK), Amplitude Shift Keying (ASK) [14] and Pulse Position Modulation (PPM) [15]. Since their spectral efficiency is generally worse than that of coherent counterparts, systems started implementing phase acquisition circuitry at the receivers. This was helped in no small measure by the sophistication of device manufacturing. However, as we moved to higher and higher frequencies carrier frequencies, and faster varying channels, coherent phase acquisition and baseband processing become more and more difficult, thereby leading to a renewed interest in noncoherent communication systems. A fundamental contribution towards the understanding of noncoherent communication is the notion of unitarily invariant codes [16], [17], [18], which perform space-time coding over the Grassman manifold associated with the channel matrix. On a similar note, [19] focuses on the noncoherent ML decoder and proposes signal constellation designs using a metric motivated by a union bound on the probability of error for a high SNR analysis. Similar metrics, motivated again by a high SNR analysis, are also presented in [20] where the worst-case chordal distance is employed to place the codewords as far apart as possible. A related research direction can be found in [21] which proposes a noncoherent communication system that uses the Generalized Likelihood Ratio Test (GLRT) to jointly recover the channel and the transmitted symbols, whenever one wants to avoid the estimation of the large-scale statistics of the channel. In this, the authors propose a minimum distance criterion for code design by characterizing the performance of the GLRT in
the AWGN channel at high SNR. Interestingly, even though the GLRT decoder has in general worse performance than the ML decoder, it is identical to the latter for unitary signaling and i.i.d. fading [12]. Even though joint channel and transmitted symbol estimation is an interesting research problem, in a typical practical deployment, we expect the transmitter and receiver will at least try to estimate the long-scale statistics of the channel, and thus the ML decoder should be preferred. Note that the idea of using the long-term channel information to simplify the design for massive MIMO systems can be found in [22] where authors compare the instantaneous versus long-term transmit beamforming, an idea initially presented in [23].

Last but not least, note that noncoherent communication is in general less spectrally-efficient, which traditionally has been a crucial disadvantage against the pilot-based coherent schemes. However, with a trend towards higher and higher carrier frequencies, the issue of simple circuit designs, inexpensive hardware components and energy efficiency becomes as crucial to system design as spectral efficiency [1].

A. Contributions

In this work, motivated by the difficulty of CSI acquisition in channel conditions with low-coherence time, and the need for low hardware complexity and low energy consumption, we consider a noncoherent energy-based SIMO system operating in a flat narrowband channel with independent and identically distributed (i.i.d.) channel realizations across the antennas, such that only the large-scale channel and noise statistics are known. These quantities can be estimated on time scales larger than those needed for estimating instantaneous phase for which we need resource-consuming training sequences. Furthermore, to alleviate the need for precise phase knowledge, we consider schemes which encode information only in the power of the transmitted symbols. The receiver decodes by computing the average received energy across all the antennas. This leads to a simple and energy efficient hardware implementation as there is no need for oscillators or phase synchronization [24], [25].

Surprisingly even with all these simplifications, the achievable rates for the above scheme are no different from coherent schemes in a scaling law sense with an increasing number of antennas (i.e., schemes with perfect CSIT and CSIR) [26]. In fact the energy based decoder is the noncoherent maximum likelihood (ML) decoding in a Rayleigh fading channel. However the analysis in [26] is an asymptotic analysis; to achieve reasonable BERs according to the
achievable scheme one would need on the order of 1000 antennas. Our goal in this work is to bring this number down.

In particular, we consider the problem of optimizing the transmit constellation points and investigate whether (and by how much) the number of receive antennas required for a certain performance can be brought down. We present practical constellation designs for any given SNR under different assumptions on the availability of CSI. We propose robust single user constellation designs for an average energy based receiver based on the error exponent with the number of the antennas. Through analysis and simulations, we find that the suggested schemes can outperform several existing noncoherent (and learning based) schemes. Our designs are applicable even in cases where the long term statistics are not known precisely. To the best of our knowledge, this line of work is the first to consider an average energy-based encoding and decoding procedure for a noncoherent large antenna system.

The rest of the paper is organized as follows. We present the system model in Section II, and summarize our previous work on its asymptotic characterization in Section III. Then, Section IV-A presents the constellation design problem and Sections IV-B describes the solution to this problem when the channel distribution is perfectly known. Section IV-C shows how the constellation design problem can be simplified when only the first, second and fourth moments of the fading distribution are known, and Section IV-D addressed the case when the latter are imperfectly known. Finally, in Section V, we present plots showing the numerical performance of the suggested schemes with representative statistics. Section VI, summarizes this work.

B. Notation

Notation: We use \([k]\) to denote the set \(\{1, 2, \cdots, k\}\) where \(k\) is an integer. \(\mathbb{C}^{n \times m}\) is the set of all complex-valued matrices of size \(n \times m\). For a matrix \(H \in \mathbb{C}^{n \times m}\), the \((i, j)\)-th element is denoted by \(H_{i,j}\) and for a vector \(h \in \mathbb{C}^{n \times 1}\), the \(i\)-th element is denoted as \(h_i\). \(\text{Re}(\cdot)\) and \(\text{Im}(\cdot)\) represent the real and imaginary terms, respectively. \(\mathcal{CN}(\mu, \mathbf{R})\) represents the distribution of circularly symmetric complex Gaussian (CSCG) random vectors with mean vector \(\mu\) and a covariance matrix \(\mathbf{R}\). The symbol \(\triangleq\) is used to denote a definition. The index \(i \in [n]\) is used to refer to a quantity related to the \(i\)-th antenna, \(\mathcal{P}\) refers to a set of power levels that the transmitter uses, \(k \in [L]\) to the \(k\)-th power level of \(\mathcal{P}\), and \(n\) is used to denote the number of receive antennas.
II. System Model

Consider one single antenna transmitter in a flat fading channel and a receiver with $n$ antennas, where $n$ is considered a large (but finite) number. The system is represented as

$$ y = hx + v, \quad (1) $$

with $y \in \mathbb{C}^{n \times 1}$, $x \in \mathbb{C}$, $v \in \mathbb{C}^{n \times 1}$, $h \in \mathbb{C}^{n \times 1}$ and each $v_i \sim \mathcal{CN}(0, \sigma^2)$, $h_i \sim f(h)$, such that $E[h_i] = \mu$, $E[|h_i - \mu|^2] = \sigma_h^2$, and $f(h)$ is the probability density function of the channel distribution. For normalization purposes and for notational simplicity, we also assume that $E[|h_i|^2] = 1$ and $E[|x|^2] = 1$ so that parameters such as large-scale shadowing, path-loss and antenna gain are incorporated in the $\sigma^2$. Then, the average SNR per antenna at the receiver for this model is

$$ \gamma \triangleq \frac{E[|h_i|^2]}{\sigma^2} = \frac{1}{\sigma^2}. $$

We further assume that the density function $f(h)$ is such that, for any fixed $x \in \mathbb{C}$, the moment generating function of $|y_i|^2$, i.e., $E[e^{\theta|y_i|^2}]$, exists and is twice differentiable in an interval around $\theta = 0$. Many fading distributions fall within this model, e.g., Rayleigh and Rician fading [27], in which case $h_i \sim \mathcal{CN}(\mu, \sigma^2)$. For notation simplicity, we refer to a $(K, \gamma)$ channel as a channel with Rician fading ($K$-factor in dB units and unit second moment) and additive Gaussian noise with power $\sigma^2 = -\gamma$ in dB.

An important aspect of this system model is the assumption that the channels are independent and identically distributed random variables. While this may appear artificial, several measurements performed to investigate how massive MIMO performs in real channels [?] [28], [29], [30], show that, despite the statistical difference between the measured channels and the i.i.d. channels, many of the observed practical gains can be predicted from theory.

This work focuses on symbol-by-symbol encoding schemes that use an energy-based transmitter and receiver design. This means that information is modulated on the power of the transmitted symbols, $|x|^2$, and the receiver estimates only the average power of the received signal, $\frac{|y|^2}{n}$. We describe this next.

A. Transmitter Architecture: Energy Encoder

The transmitter encodes information only in the power of the transmitted symbols, i.e., it transmits symbols with power levels from a codebook $\mathcal{P} = \{p_1, p_2, \cdots, p_L\}$, where $p_k \in \mathbb{R}_+$, subject to an average power constraint $\frac{1}{L} \sum_{k=1}^{L} p_k \leq 1$, assuming equiprobable signaling. Here $p_k \in \mathcal{P}$ is the power level of the $k^{th}$ symbol and $L$ is the cardinality of $\mathcal{P}$. In this point
we need to emphasize that the power of the transmitted symbols, and not the phase, carry information. Obviously, any set of transmitted symbols with powers that belong in codebook $\mathcal{P}$ are equivalent. Also, note that in this work constellation point refers to the power of the corresponding transmitted symbol. Contrary to the typical modulation techniques, which usually specify the amplitude and the phase of the transmitted symbols, we only describe how the powers of the transmitted symbols should be chosen.

B. Receiver Architecture: Energy decoder

Assume the user transmits a symbol whose power is the $k^{th}$ constellation point from $\mathcal{P}$, i.e., $p_k$. In order for the receiver to detect $p_k$, it only computes the following statistic

$$\frac{\|y\|^2}{n} = \frac{\sum_{i=1}^{n} |y_i|^2}{n} \in \mathbb{R}^+, \quad (2)$$

i.e., it estimates only the average received power across all its antennas. Based on its knowledge of the statistics of the channel, the receiver divides the positive real line into non-intersecting intervals or decoding regions $\{I_k\}_{k=1}^L$, corresponding to each $p_k \in \mathcal{P}$, and returns

$$\hat{k} \in \left\{ \tilde{k} : \frac{\|y\|^2}{n} \in I_{\tilde{k}} \right\}. \quad (3)$$

Then, we refer to the constellation and decoding regions separately to prevent confusion? constellation $\mathcal{C}$ as the set that contains the codebook $\mathcal{P}$ and the corresponding decoding regions $\{I_k\}$, i.e., $\mathcal{C} = \{\mathcal{P}, I_1, \cdots, I_L\}$. The constellation $\mathcal{C}$ is decided by the system prior to the start of the communication based on the statistics on the channel.

The probability of error of the $k^{th}$ power level $p_k \in \mathcal{P}$ and the average Symbol Error Rate (SER) for any fixed constellation size $L$ is defined as

$$P_e(p_k) \triangleq P_r(\hat{k} \neq k), \quad P_s \triangleq \frac{1}{L} \sum_{k=1}^{L} P_e(p_k), \quad (4)$$

respectively, assuming equiprobable signaling.

C. Discussion

The use of energy detection based transmission and decoding is motivated by the fact [26] that such an encoding and decoding method is as good as a noncoherent maximum likelihood (ML) scheme in the Rayleigh fading channel. To see this, assume the transmitter sends a symbol
The noncoherent log likelihood function for Rician fading, i.e., $h_i \sim \mathcal{CN}(\mu, \sigma^2)$, is
\[
\log f_x^{NC}(y) = \frac{||y-\mu x^1||^2}{\sigma^2 + \sigma^2|x|^2} + n \log \left( \sqrt{\pi(\sigma^2 + \sigma^2|x|^2)} \right),
\]
and therefore, the noncoherent ML decoder is
\[
\hat{k} = \arg\max_{x:|x|^2=p_k, \forall k} \log f_x^{NC}(y).
\]
For $\mu = 0$, i.e., Rayleigh fading, the noncoherent ML decoder depends only on $||y||^2$, as is the case with the proposed energy decoder; for suitably chosen decoding regions $\{I_k\}$, it performs as well as the ML decoder. In general for $\mu \neq 0$, energy based detectors are not optimal. However, as shown in our numerical section for representative values for $\mu$, the gap to optimality may be small.

The suggested architecture requires a very simple one-dimensional statistic of the received signals, which allows for a very simple circuit design and a corresponding RF chain. Note that implementing this decoder only needs a set of analog envelope estimators and one A/D converter to quantize their average. A general noncoherent ML or coherent detector, on the other hand, requires much more complicated circuits.

### III. Error Exponent

I feel this can be rewritten In this section we justify the relevance of the error exponent as a metric for our designs.

#### A. SER Minimization

Consider the following problem of minimizing SER for any fixed constellation size and fixed $n$, i.e.,
\[
\begin{array}{c}
\text{minimize} \\
\{\mathcal{P}, I_1, \ldots, I_L\}
\end{array}
\log(P_s)
\]
subject to
\[
\frac{1}{L} \sum_{k=1}^{L} p_k \leq 1, \ 0 \leq p_k
\]
This is in general a difficult problem to solve. The scope of this work is to solve a specific relaxation of this problem motivated by the large $n$ asymptotics. Specifically, we consider maximizing the error exponent of SER, or a second-order approximation of it, with respect to $n$ which as we will show is analytically much more tractable. We now define the notion of the error exponent of SER with respect to $n$. 
B. Error Exponent Maximization

Fix any codebook $\mathcal{P}$. Define the receiver’s constellation points $r(p_k)$ to be the value of the average received energy when the transmitter sends the $k^{th}$ power level, i.e., $r(p_k) \triangleq p_k + \sigma^2$.

To see this, note that

$$\frac{||y||^2}{n} = \left| h \sqrt{p_k} + v \right|^2 = \frac{||h||^2}{n} p_k + \frac{||v||^2}{n} + 2 \frac{\text{Re}(h^*v)}{n} \sqrt{p_k},$$

so, in the limit of large $n$, due to the law of large numbers and the independence of $h$ and $v$, it follows that

$$\lim_{n \to \infty} \frac{||y||^2}{n} = r(p_k).$$

In practice though, the system has finite $n$, and thus we need to analyze how the statistic $\frac{||y||^2}{n}$ varies around the value $r(p_k)$. To do so, it is helpful to denote

$$u_{k,i} = |h_i \sqrt{p_k} + \nu_i|^2 - E \left[ |h_i \sqrt{p_k} + \nu_i|^2 \right] = |h_i \sqrt{p_k} + \nu_i|^2 - r(p_k)$$

as the random variation of the received energy at the $i^{th}$ antenna around the expected value. Note that $\{u_{k,i}\}_{i=1}^n$ are independent realizations of the same zero-mean random variable $U_k \sim g_k(u)$ whose m.g.f.

$$M_k(\theta) \triangleq E[e^{\theta U_k}],$$

which depends on the statistics of the channel and the noise, and the power level $p_k$.

In [26], starting from (4), and using a union upper bound approach, we relaxed the objective in (6) by upper bounding it as follows

$$P_s \leq \frac{1}{L} \sum_{k=1}^L \left( e^{-nI_{R,k}(d_{R,k})} + e^{-nI_{L,k}(d_{L,k})} \right),$$

(9)

where

$$I_{L,k}(d) \triangleq \sup_{\theta > 0} (\theta d - \log(M_k(-\theta))), \quad I_{R,k}(d) \triangleq \sup_{\theta > 0} (\theta d - \log(M_k(\theta))),$$

(10)

are denoted as the left and right rate functions of $p_k$, $\sup_{x \in \mathcal{A}} f(x) \triangleq y_0$ such that $y_0 \leq y$ for all $y > f(x)$ and $x \in \mathcal{A}$, is the least upper bound of $f(x)$ in $\mathcal{A}$. $d_{L,k}, d_{R,k}$ specify the maximum distance to the left and right respectively of the received statistic $\frac{||y||^2}{n}$ from $r(p_k) = p_k + \sigma^2$ in order to decide that the value $p_k$ was transmitted. This means the decoding regions are chosen as $\mathcal{I}_k = (r(p_k) - d_{L,k}, r(p_k) + d_{R,k})$. Define as

$$I_k \triangleq \min \left( I_{L,k}(d_{L,k}), I_{R,k}(d_{R,k}) \right)$$
the rate function of the constellation point \( p_k \). Then, it was shown in [26] that

\[
I_e \triangleq \lim_{n \to \infty} -\frac{\log(P_s)}{n} = \min_{k \in [L]} I_k, \tag{11}
\]
i.e., the error exponent of SER, denoted as \( I_e \), is the same as the worst rate function of the constellation points. In other words, for a finite \( n \) large enough, the probability of error performance is dominated by the constellation point with the worst rate function. Therefore, the constellation points \( \{p_k\} \) and the corresponding decoding regions \( \mathcal{I}_k \) could be chosen in such a way as to maximize the error exponent of SER, i.e.,

\[
\max_{\{P, \mathcal{I}_1, \ldots, \mathcal{I}_L\}} I_e
\]
subject to

\[
\frac{1}{L} \sum_{k=1}^{L} p_k \leq 1, 0 \leq p_k. \tag{12}
\]

This problem is interesting for three main reasons. First, for large but finite \( n \), the suggested design guarantees that it achieves the best decay with increasing \( n \), even if it does not explicitly solve (6). Secondly, an interesting aspect of this approach is that it characterizes explicitly the impact of \( n \) on the SER, for \( n \) large, by separating it from the impact of the channel distribution, i.e., it can explicitly provide what are the expected gains on the SER by increasing or decreasing \( n \). Third, for \( n \) going to infinity, this design is asymptotically optimal with respect to (6).

In [26] we showed the following about the left and right rate functions for any \( p_k \):

**Lemma 1.** The right and left rate functions \( I_{R,k}(d), I_{L,k}(d) \), respectively, of the power level \( p_k \) enjoy the following properties:

- They satisfy

\[
\lim_{d \to 0} \frac{I_{R,k}(d)}{d^2} = \lim_{d \to 0} \frac{I_{L,k}(d)}{d^2} = \frac{1}{2} \text{E}[U_k^2], \quad \text{where} \quad U_k = |h\sqrt{p_k} + v|^2 - p_k - \sigma^2,
\]

with \( h \sim f(h) \) and \( v \sim \mathcal{CN}(0, \sigma^2) \).

- They are non-negative, convex and monotonically increasing for positive \( d \) for a fixed non-negative \( p_k \), and monotonically decreasing for non-negative \( p_k \) for a fixed positive \( d \).

- It holds that \( I_{L,k}(0) = I_{R,k}(0) = 0 \) for any non-negative \( p_k \).

The above lemma provides important insights on the dependence of the rate functions from the system’s parameters. Specifically, for small \( d \), which practically means large constellations, increasing \( p_k \) leads to smaller rate functions, i.e., worse SER performance. This shows that the
constellation points that correspond to high power levels have smaller rate functions than those with low power levels. Actually, this exact behavior of the rate functions is exploited in our constellation designs: space the power levels onto the positive real line in such a way such that all the constellation points experience the same rate function. Then, we can guarantee that the proposed design has a positive error exponent with a large but finite $n$, and explicitly characterize the dependence of the achieved probability of error performance as a function of $n$.

IV. CONSTELLATION DESIGNS

A. Overview

In this section we consider three cases, each corresponding to a different assumption on the availability of statistical information. We start from (12).

- Subsection IV-B presents a design which assumes that the encoder knows perfectly the channel distribution. This constellation is denoted as $C^{(1)}_{K,\gamma}$.
- Subsection IV-C presents a design in which only the first, second and fourth moments of the channel distribution, are perfectly known. This constellation is denoted as $C^{(2)}_{K,\gamma}$.
- Subsection IV-D presents a design in which even the latter are imperfectly known, denoted as $C^{(2,a)}_{K,\gamma}$, where $a$ is the uncertainty in dB around the nominal values $K$ and SNR.

We also denote as $C^{(\text{min})}$ a minimum distance constellation design that was proposed in [31], [26]. This is asymptotically optimal only for $\sigma^2 \to \infty$. The new approach presented in this work generalizes the minimum distance design criterion to very general scenarios without constraints on the SNR. Furthermore, as a byproduct of the above designs, it is possible to propose a constellation in which the family of the channel distribution is known, but the distribution’s parameters are imperfectly known.

B. Perfect knowledge of channel distribution

We first discuss the constellation design with perfect knowledge of the channel distribution at the receiver which solves (12). Since the exact channel distribution is known, $M_k(\theta)$ is also
known at the receiver and transmitter for any chosen $p_k$. Then, (12) is written as

$$\text{maximize} \quad \min_{k \in [L]} \left( I_{L,k}(d_{L,k}), I_{R,k}(d_{R,k}) \right)$$

subject to

$$0 \leq p_1 < p_2 < \cdots < p_L, d_{L,k} \geq 0, d_{R,k} \geq 0$$

(13)

assuming decoding regions of the form $\mathcal{I}_k = (r(p_k) - d_{L,k} : r(p_k) + d_{R,k}]$, where for simplicity we assume that $d_{L,1} = d_{R,L} = \infty$. Algorithm 2 describes in detail how to get the solution of the optimization problem (13) and a detailed proof is shown in Appendix A. To exemplify the procedure and provide an intuitive argument of the validity of the suggested construction we consider the case with $L = 4$ as shown in Fig. 1. The design is based on the following two properties that result from Lemma 1:

1) Both $I_{L,k}(d)$ and $I_{R,k}(d)$ are non-negative and monotonically increasing functions of $d$ for a fixed $p_k$. This means that increasing the size of the decoding regions always helps to increase the resulting rate functions, and therefore increase the minimum amongst them.

2) Both $I_{L,k}(d)$ and $I_{R,k}(d)$ are monotonically decreasing functions of $p_k$ for a fixed $d$. This means that transmitting with low power levels should be always preferred.

Based on these two properties, we have the following sequential construction: Assume there exist a constellation with error exponent $t^*$ that satisfies the power constraint. This means that the left and right rate functions of all the constellation points at the receiver are at least $t^*$. To find this constellation choose first the minimum possible value for $p_1$. Then, choose the boundary
of the decoding region to the right of \( r(p_1) = p_1 + \sigma^2 \), i.e., \( c_1 \) as show in Fig. 1, such that the right rate function of \( r(p_1) \), i.e., \( I_{R,1}(c_1 - r(p_1)) \), is at least \( t^* \) on the boundary. Then, choose the smallest \( p_2 \) such that \( r(p_2) > c_1 \) and the left right rate function of \( r(p_2) \), \( I_{L,2}(r(p_2) - c_1) \), is at least \( t^* \). Note that choosing a higher \( p_2 \) is always an option but this will lead to a design that uses more power than necessary. We perform this procedure sequentially until we find \( p_L \).

Then, we check if the average power constraint is satisfied. If that is the case, the assumption that there exists a constellation with error exponent at least \( t^* \) that satisfies the power constraint was correct. If not, we should discard this constellation, decrease \( t^* \) and repeat the procedure.

Proof of the validity of this procedure is presented in Appendix A.

For \( L = 4 \), Figs 2-(a) and 2-(b) show the normalized empirical histogram of the received statistic \( \frac{||y||^2}{n} \) in Rayleigh fading channel of the suggest constellation \( \mathcal{C}^{(1)}_{K,\gamma} \) and a \( \mathcal{C}^{(min)} \) (power levels are equally spaced on the positive real line). The circles and diamonds on the x-axis show the \( r(p_k) \) and the \( c_k \) (boundaries of the decoding regions) respectively. Observe that for the \( \mathcal{C}^{(min)} \) constellation there is a significant overlap in the histogram and therefore the receiver cannot decode reliably the information. Also, observe that as \( p_k \) increases, the variation around \( r(p_k) \) also increases, due to the special nature of the energy detector at the receiver.
Number of antennas

\[ \log_{10}(P_s) \]

\[ ML_K = -\infty \text{ dB} \]
\[ Energy_K = -\infty \text{ dB} \]
\[ ML_K = -10 \text{ dB} \]
\[ Energy_K = -10 \text{ dB} \]
\[ ML_K = 0 \text{ dB} \]
\[ Energy_K = 0 \text{ dB} \]

\( K = 0 \text{ dB}, \gamma = 5 \text{ dB} \)

(a) Comparison of Energy and ML decoder in Rician fading with \( \gamma = 10 \text{ dB} \)

(b) \( K = 0 \text{ dB}, \gamma = 5 \text{ dB} \)

Fig. 3: (a) Comparison of the energy decoder with the noncoherent ML decoder in Rician fading, (b) Comparison of \( C_{K,\gamma}^{(1)}, C_{K,\gamma}^{(2)} \) as a function of the constellation size \( L \) in Rician fading

Furthermore, Fig. 3 shows a SER numerical comparison of two systems that transmit using the same codebook \( \mathcal{P} \) in Rician fading, but a different decoder: The first system uses the suggested decoder (3) and the decoding regions resulting from the above procedure, and the second system uses an ML noncoherent decoder (5). First, observe that there is not an evident difference in the performance of the two systems in Rayleigh fading. This means that focusing on the error exponents as a surrogate of the likelihood function works very well in simplifying the decoder and separating the impact of \( n \) from the SER performance. Secondly, in Rician fading, the difference in performance is small, even for cases with relatively strong LOS component.

C. Perfect knowledge of the first, second, fourth moments

The constellation design presented above assumes that the receiver knows exactly the m.g.f. of the channel distribution, which may not be realistic in a practical scenario. In this section, we relax this assumption and consider the case that the encoder and decoders only know the first, second and fourth moments of the channel distribution. To this end, we are going to use Lemma 1 as it provides an approximation of the rate functions for small \( d \). To see this, denote \( h = h_{re} + jh_{im} \), where \( h_{re}, h_{im} \in \mathbb{R} \). Using Lemma 1 leads to

\[
I_{R,k}(d_{R,k}) \approx \tilde{I}_{R,k}(d_{R,k}) \triangleq \frac{d_{R,k}^2}{2s(p_k)}, \quad I_{L,k}(d_{L,k}) \approx \tilde{I}_{L,k}(d_{L,k}) \triangleq \frac{d_{L,k}^2}{2s(p_k)},
\]

for small \( d_{R,k} \) and \( d_{L,k} \), with \( s(p_k) \triangleq \mathbb{E}[U_k^2] = \alpha_1 p_k^2 + \alpha_2 p_k + \alpha_3 \), where

\[
\alpha_1 \triangleq \mathbb{E}[h_{re}^4] + \mathbb{E}[h_{im}^4] + 2 \mathbb{E}[h_{re}^2] \mathbb{E}[h_{im}^2] - 1, \quad \alpha_2 \triangleq 2\sigma^2, \quad \alpha_3 \triangleq \sigma^4,
\]
Algorithm 2: Constellation design: Perfect channel distribution knowledge

\[ [t^*, C^*] = \text{Bisection}( ); \]

function: \( C = \text{ConstellationDesign}( t ) \)

\( p_1 = 0; \quad d_{L,1} = \infty; \quad d_{R,L} = \infty; \)

for \( k = 1, 2, 3, \ldots, L \) do

if \( k = 1 \) then

\( p_1 = 0 \)

else

Find the smallest \( p_k > p_{k-1} + d_{R,k-1} \) such that \( I_{L,k}(p_k - p_{k-1} - d_{R,k-1}) = t \)

\( d_{L,k} = p_k - p_{k-1} - d_{R,k-1} \)

end if

if \( k \neq L \) then

Find the smallest \( d_{R,k} > 0 \) such that \( I_{R,k}(d_{R,k}) = t \)

end if

\( \mathcal{I}_k = [p_k + \sigma^2 - d_{L,k}, p_k + \sigma^2 + d_{R,k}] \)

end for

\( \mathcal{P} = [p_1, p_2, \ldots, p_L], S_t = \frac{1}{T} \sum_{k=1}^{L} p_k \)

return \( C = [\mathcal{P}, \mathcal{I}_1, \ldots, \mathcal{I}_L], S_t \)

end ConstellationDesign

due to the gaussianity of the noise and the fact that the noise and the channel are independent random variables. Observe that this approximation depends only on the first, second and fourth moment of the channel distribution. For example, in the case of Rician fading with K-factor equal to \( K \) and unit second moment it can be shown that \( E[U_k^2] = \sigma^4 + 2p_k \sigma^2 + \frac{(1+2K)}{(1+K)^2} p_k^2 \), which means that \( \alpha_1 = \frac{1+2K}{(1+K)^2} \). That is, in Rician fading, both sides need to know just the power of the LOS component and SNR, and still use the approach that we describe in this section.

To start with, substituting the objective function of (13) using (14) leads to the following
optimization problem

\[
\text{maximize } \min_{k \in [L]} \left( \tilde{I}_{L,k}(d_{L,k}), \tilde{I}_{R,k}(d_{R,k}) \right)
\]

\[
0 \leq p_1 < p_2 < \cdots < p_L, d_{L,k} \geq 0, d_{R,k} \geq 0
\]

(15)

Note that the objective of problem (13) has been substituted in (20) with an expression which is still non-negative and non-decreasing in \(d\) for a fixed \(p_k\) and non-increasing in \(p_k\) for a fixed \(d\); i.e., all the properties and arguments that led to Algorithm 2 are still valid. Thus, the approach of solving this problem is similar to the one presented in Section IV-B, with the only difference that \(\tilde{I}_{L,k}(d_{L,k}), \tilde{I}_{R,k}(d_{R,k})\) exhibit an easily-interpretable dependance on \(p_k\) and \(d_{L,k}, d_{R,k}\) respectively. Algorithm (3) contains the simplified algorithm and Appendix B shows the detailed proof.

\[\text{Algorithm 3: Constellation design: Perfect knowledge of the first, second, fourth moments}\]

\[\{t^*, C^*\} = \text{Bisection}(\ );\]

\[\text{function: } C = \text{ConstellationDesign}( t )\]

\[p_1 = 0; \quad d_{L,1} = \infty; \quad d_{R,L} = \infty;\]

\[d_{R,k} = \sqrt{2ts(0)};\]

\[\text{for } k = 2, 3, \ldots, L \text{ do }\]

\[\text{Find the smallest } p_k > p_{k-1} \text{ such that } \frac{(p_k-p_{k-1})^2}{2(\sqrt{s(p_k)} + \sqrt{s(p_{k-1})})^2} = t;\]

\[\text{if } k \neq L \text{ then }\]

\[d_{R,k} = \sqrt{2ts(p_k)};\]

\[\text{end if}\]

\[d_{L,k} = \sqrt{2ts(p_k)};\]

\[\mathcal{I}_k = [p_k + \sigma^2 - d_{L,k}, p_k + \sigma^2 + d_{R,k}];\]

\[\text{end for}\]

\[\text{end ConstellationDesign}\]

\[\mathcal{P} = [p_1, p_2, \ldots, p_L];\]

\[\text{return } C = [\mathcal{P}, \mathcal{I}_1, \ldots, \mathcal{I}_L];\]

Note that the suggested design leads to an algorithm which can be employed in very general channel models. The latter is especially important since determining the exact small-scale fading
channel models in some cases, such as millimeter wave frequencies, is still ongoing research, and may not be reliably known beyond the first few moments. Furthermore, note that the approximation gets better as $L$ increases. This is because, as $L$ increases, the transmitted powers will be packed closer together, and the decoding regions will be smaller, i.e., $\{d_{R,k}, d_{L,k}\}$ will be smaller. Fig. 3-(b) shows a comparison between a Monte Carlo estimate of SER ($P_s$) in Rician fading with $K = 0$ dB and $\gamma = 5$ dB with $C_{K,\gamma}^{(1)}$ and $C_{K,\gamma}^{(2)}$ as a function of the constellation size $L$. We see that, with increasing constellation sizes, the approximation (14) gets tighter, which means that both designs lead to similar error exponents, and thus approximately the same SER.

D. Robust constellation design

Perfect knowledge of the first, second and fourth moments of the channel distribution is often unavailable due to changing propagation environments associated with user mobility. This motivates the need for designs which take into account uncertainties in these channel statistics. We build upon the design principles laid out in the previous section to develop a design that performs well even in the face of bounded uncertainties.

Specifically, recall that $E[U_k^2] = s(p_k) = \alpha_1 p_k^2 + \alpha_2 p_k + \alpha_3$. Thus, for a fixed $p_k$, $E[U_k^2]$, and hence the rate function approximation, depends on the channel and noise statistics only through $\alpha_1$ and $\sigma$. We define the following set $\mathcal{F} = \{(\alpha_1, \sigma) : \alpha_{\min} < \alpha_1 < \alpha_{\max}, \sigma_{\min} < \sigma < \sigma_{\max}\}$, and note that for each $f = (\tilde{\alpha}_1, \tilde{\sigma}) \in \mathcal{F}$, we can define $s_f(p) \triangleq \tilde{\alpha}_1 p^2 + \tilde{\alpha}_2 p + \tilde{\alpha}_3$, where $\tilde{\alpha}_2 = 2\tilde{\sigma}^2, \tilde{\alpha}_3 = \tilde{\sigma}^4$. Then, in order to maximize the approximate worst-case rate function for all possible conditions we modify problem (20) in the following way:

$$\maximize_{\{p_k, d_{L,k}, d_{R,k}\} \in \mathcal{L}} \min_{f \in \mathcal{F}, k \in [L]} \left( \frac{d_{L,k}^2}{s_f(p_k)} \cdot \frac{d_{R,k}^2}{s_f(p_k)} \right)$$

$$0 \leq p_1 < p_2 < \cdots < p_L, d_{L,k} \geq 0, d_{R,k} \geq 0$$

(16)

In Appendix C we show how to solve problem (16) and design a constellation which maximizes the error exponent for all statistics in $\mathcal{F}$. The main difference between this design as compared to the previous two algorithms is the need for using power levels and decoding regions which would work well for any channel statistics inside the bounded uncertainty of the channel’s moments. To satisfy this, the consecutive power levels and decoding regions are generally spread as far
apart as the worst channel requires. Note also that if there is no uncertainty, Algorithm 4 reduces to Algorithm 3. An important aspect of this approach is that problem (16), in contrast to the

Algorithm 4: Constellation design: Robust constellation design

\[ [t^*, C^*] = \text{Bisection}(t); \]

function: \( C = \text{ConstellationDesign}(t) \)

\( p_1 = 0; c_0 = -\infty; c_L = \infty; \)

for \( k = 1, 2, \ldots, L - 1 \) do

\( c_k = \sup_{f \in F} \left( \sigma^2 + t \sqrt{s_f(p_k)} \right) + p_k; \)

Find the smallest \( p_{k+1} > p_k \) such that \( p_{k+1} - c_k - \sup_{f \in F} \left( t^* \sqrt{s_f(p_{k+1})} - \sigma^2 \right) \geq 0; \)

\( \mathcal{I}_k = [c_{k-1}, c_k]; \)

end for

\( \mathcal{I}_L = [c_{L-1}, c_L]; \)

\( \mathcal{P} = [p_1, p_2, \ldots, p_L]; \)

return \( C = [\mathcal{P}, \mathcal{I}_1, \ldots, \mathcal{I}_L]; \)

end ConstellationDesign

problems (13) and (17), may lead to a constellation that is not possible to guarantee any positive error exponent if \( \mathcal{F} \) is very large. Such an extreme example is presented below.

E. Existence of a robust constellation design

In this section we present a simple example that shows that, for a fixed average power constraint \( P \), a very high uncertainty on the channel statistics could lead to infeasibility in the robust constellation design problem (Section IV-D). Consider the case of constructing a constellation with \( L = 2 \), an uncertainty region \( \sigma^2 \in (\epsilon, \frac{1}{\epsilon}) \) for some \( \epsilon > 0 \) and perfectly known \( \alpha_1 = 1 \) for simplicity (the case of Rayleigh fading). Fix \( t^* > 0 \). Then, based on Algorithm 4 we choose \( p_1 = 0 \) and \( c_1 = \frac{1}{\epsilon} + \frac{t^*}{\epsilon} \). We next choose \( p_2 \) to be the smallest \( p > 0 \) that satisfies

\[
p - \frac{1 + t^*}{\epsilon} - \sup_{\sigma^2 \in (\epsilon, \frac{1}{\epsilon})} \left( t^* \sqrt{p^2 + 2\sigma^2 p + \sigma^4} - \sigma^2 \right) \geq 0 \Leftrightarrow p - \frac{1 + t^*}{\epsilon} - \sup_{\sigma^2 \in (\epsilon, \frac{1}{\epsilon})} \left( t^* p + \sigma^2 (t^* - 1) \right) \geq 0.
\]

If \( t^* \geq 1 \), then \( p \geq t^* p + 2 \frac{t^*}{\epsilon} \) which is impossible, and if \( t^* < 1 \), then \( p \geq \frac{1 + t^*}{1 - t^*} \frac{1}{\epsilon} - \epsilon \). Thus, the smallest choice of \( p_2 \) that can be chosen is \( p_2 = \frac{1 + t^*}{1 - t^*} \frac{1}{\epsilon} - \epsilon \). In this case, since

\[
p_1 + p_2 \leq 2P \Rightarrow \frac{1 + t^*}{1 - t^*} \frac{1}{\epsilon} - \epsilon < 2P;
\]
it follows that, no matter how small $t^*$ is, if the uncertainty is so large such that $2P < \frac{1}{\epsilon} - \epsilon$, the robust design problem will be infeasible.

V. NUMERICAL EXAMPLES

This section contains simulation studies which demonstrate and compare the performance of all the constellation designs proposed in this work. Recall that for the constellation designs which depend on an underlying channel, i.e., $C^{(1)}_{K,\gamma}, C^{(1,a)}_{K,\gamma}, C^{(2)}_{K,\gamma}, C^{(2,a)}_{K,\gamma}$, the $(K, \gamma)$ channel is referred to as the nominal channel, whereas any other channel is referred to as a mismatched channel.

A. Comparison with a pilot-based system with PAM and a noncoherent system with ASK

Consider a block-fading Rician fading channel $h_i \sim \mathcal{CN}(\mu, \sigma^2)$ with coherence time $T$ slots and $n$ antennas at the receiver. We assume that both the transmitter and the receiver know the channel statistics but not the exact channel realization. In the first numerical example we compare the performance of a noncoherent system that uses the suggested energy-based architecture with $C^{(1)}_{K,\gamma}$ and ASK constellation, and with a system that uses a PAM constellation, referred to as PAM system, assuming a binary reflected Gray Code (BRGC) [27]. In the PAM system the transmitter uses the first $T_1$ slots of each coherence interval to transmit pilot symbols. Based on the received signals in these slots, the receiver derives the MMSE channel estimates $\{\hat{h}_i\}$ at the end of the $T_1$ learning slots. Using these estimates it decodes the symbols transmitted during the remaining $T - T_1$ slots of the coherence interval. Note that, assuming a constellation size of $L$, the effective rate of such a system is $\frac{T - T_1}{T} \log_2(L)$. The noncoherent system that uses ASK i.e., amplitudes that are equally spaced apart, performs decoding using an energy-based ML receiver.

Fig. 4-(a) to 4-(c) plot the minimum number of antennas needed to achieve an uncoded BER $= 10^{-3}$ for different $K$ and $\gamma = 10$ dB for different coherence times $T$. We make the following observations: First, the suggest constellation performs significantly better than the system with an ASK constellation. For example, in Rayleigh fading the suggested constellation needs approximately half of the number antennas to achieve the same BER performance. Second, even in the case of high $K$, 4-(b),4-(c), which is known to all the receivers, the suggested system performs better than the noncoherent PAM system (i.e., $T_1 = 0$) which exploits the phase of the LOS component of the channel. Note that this is not the case with our energy-based system which only uses the $K$–factor to decode the symbols. Also, note that in Rayleigh fading 4-(a),
the PAM system cannot reach the BER target for any number of antennas since the phase of the transmitted symbol is completely destroyed. We also observe that, for short coherence times, the suggested system still requires smaller number of antennas to reach the BER threshold than the PAM system with $T_l = 1$. On the other hand, for higher coherence times, the PAM system achieves a better performance as it was expected since the gains of learning are more than the corresponding decrease in the effective rate. Yet, observe that for small effective rates, e.g., $1 - 2$ bits/symbol, the additional number of antennas needed by the energy-based system to achieve the same BER compared is not more than 20. This shows that even a simple energy-based architecture design at the receiver, which requires only envelope detectors, could be enough to transmit information as reliably as a typical pilot-based system, especially in channels with small coherence times and high LOS, without the need for significantly more antennas.

Fig 4-(d) plots the error exponent $I_e$ for different values of $\gamma$, and Rician channels with $K = \{\infty, 0\}$ dB and $L = 4$, for two noncoherent systems that use ASK and $C_{K,\gamma}^{(1)}$. We observe that for all channel conditions the suggested constellation achieves a much higher $I_e$ than ASK constellation. Also, for high SNR, the error exponent that uses ASK constellation is not increasing.
as fast as the system with $C_{K,\gamma}^{(1)}$, which is due to the fact that the power levels are fixed, and do not adapt to the channel conditions. This is not the case with the $C_{K,\gamma}^{(1)}$ constellation.

B. SER performance comparison of $C_{K,\gamma}^{(1)}$, $C_{K,\gamma}^{(2)}$, $C_{K,\gamma}^{(\text{min})}$, ASK

In the second numerical example (Fig. 6) we demonstrate a Monte Carlo SER estimate for a 3-bit constellation ($L = 8$) of $C_{K,\gamma}^{(1)}$, $C_{K,\gamma}^{(2)}$, $C_{K,\gamma}^{(\text{min})}$, ASK for channels with $K = -\infty$ dB, i.e., Rayleigh fading, for $\gamma = \{5, 10\}$ dB as a function of $n$. As expected, $C_{K,\gamma}^{(1)}$ achieves better SER performance than all the remaining designs. Yet, the difference of the approximate design $C_{K,\gamma}^{(2)}$ from $C_{K,\gamma}^{(1)}$ is not significant, especially at low SNR. Also, the minimum distance design $C_{K,\gamma}^{(\text{min})}$ is significantly worse than any other designs, except for very low SNRs, where the gap in the performance is smaller.

C. Performance of the robust constellation designs on the nominal and mismatched channels

In the third numerical example we demonstrate the inefficiency of the $C_{K,\gamma}^{(2)}$ ($C_{K,\gamma}^{(1)}$) constellation in a mismatched channel and the ability of $C_{K,\gamma}^{(2,\alpha)}$ ($C_{K,\gamma}^{(1,\alpha)}$) to sustain good performance. Specifically, we consider the case of a user with $2$ dB of uncertainty in both $K$ and $\gamma$ values and that the center of the uncertainty interval corresponds to the $(-10, 10)$ channel; approximately Rayleigh fading ($K$ is very low) with a high SNR value, using $L = 8$. In Fig.s 6-(a), 6-(b), 6-(c), 6-(d) we plot the Monte Carlo SER estimate of the $C_{-10,10}^{(2)}$ and $C_{-10,10}^{(2,2)}$ designs on the $\{-9, 9\}, \{-9, 11\}, \{-11, 9\}, \{-11, 11\}$ channels respectively. Observe the huge performance loss that could occur due to the overestimation of the SNR. Smaller performance loss is observed due to the uncertainty on the value of $K$, or when the SNR is underestimated.

---

Fig. 5: SER performance comparison of $C_{K,\gamma}^{(1)}$, $C_{K,\gamma}^{(2)}$, $C_{K,\gamma}^{(\text{min})}$. 

(a) $K = -\infty$ dB, $\gamma = 5$ dB 

(b) $K = -\infty$ dB, $\gamma = 10$ dB
Fig. 6: SER performance of the robust constellation designs in mismatched channel.

(a) $K = -11$ dB, $\gamma = 9$ dB  
(b) $K = -9$ dB, $\gamma = 9$ dB  
(c) $K = -9$ dB, $\gamma = 11$ dB  
(d) $K = -11$ dB, $\gamma = 11$ dB

Fig. 7: (a) SER performance of the robust constellation designs in nominal channel (b) Nakagami-m fading channel

(a) $K = -10$ dB, $\gamma = 10$ dB  
(b) $K = 6$ dB, $\gamma = 0$ dB

Fig 7-(c) presents the SER performance of $C^{(2,2)}_{-10,10}$ and $C^{(2)}_{-10,10}$ when used on the $(-10, 10)$ channel to show that even in the nominal statistics, the performance of the robust design is close to the performance of the design that is explicitly optimized for the nominal statistics. This shows that the maximum performance loss due to the robust design compared to an optimized constellation, is tolerable, especially because not taking into account the uncertainty could lead
to significant performance deterioration as presented in the previous numerical example.

D. Performance on a Nakagami-$m$ fading channel

We now show an example in which using $C_{K,\gamma}^{(1)}$ designed for a Rician fading channel leads to a worse performance compared to a $C_{K,\gamma}^{(2)}$ in a Nakagami-$m$ fading channel. This shows that not taking into account the uncertainty in the channel distribution, and over-optimizing the constellations for the Rician channel, could lead to worse performance than a much simpler constellation design which is based only on the first four moments of the channel. Specifically, consider the case of a channel for which it holds that $E[h_i] = \sqrt{\frac{K}{K+1}}$, $E[|h_i|^2] = 1$, $E\left[|h_i - \sqrt{\frac{K}{1+K}}|^2\right] = \frac{1}{1+K}$. This channel could correspond to a Rician channel, i.e., $h_i \sim \mathcal{CN}\left(\sqrt{\frac{K}{K+1}}, \frac{1}{1+K}\right)$, or a Nakagami-$m$ channel with $\Omega = 1$ and $m$ such that $\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m)} \frac{1}{\sqrt{m}} = \sqrt{\frac{K}{K+1}}$. Fig. 7-(d) plots $P_s$ for $\gamma = 0$ dB and $K = 6$ dB in a Nakagami-$m$ fading channel using $L = 8$, for the following two scenarios:

1) assume a Rician fading channel model and use the $C_{6,0}^{(1)}$ constellation design
2) assume the fourth moment is perfectly estimated and use the $C_{6,0}^{(2)}$ design.

Observe that over-optimizing the constellation for the case of Rician fading leads to worse performance in Nakagami-$m$ fading than using a constellation design which takes into account only the fourth moment of the channel.

VI. Conclusions and Future Work

In this work we formulate and solve the single-shot constellation design problem for a noncoherent SIMO system with a large number of antennas and an average energy-detection-based receiver. We present asymptotically optimal constellation designs with respect to the achieved error exponent when the system has perfect knowledge of the channel statistics. Then we present an approximate constellation design which requires only the knowledge of the first, second and fourth moments of the fading statistics. Lastly, we present a robust counterpart of the latter design which takes into account the uncertainty of the channel statistics. We exemplify the performance of all the proposed constellations, and compare them with existing symbol-by-symbol noncoherent schemes in typical scenarios. The proposed system asks for a very simple encoding and decoding and for a receiver which only senses the received energy.

Our findings here suggest that simple receiver architectures are very promising within the large antenna systems of the not-too-distant future. We did not however explore the full range of
optimizations that could potentially be carried out in such a setup. We list some directions for future research in the following: (1) Antenna correlation and how this affects the performance. This is especially relevant as antenna form factors go down with increasing numbers of antennas. (2) Constellation designs for a multiuser noncoherent SIMO system. Initial results towards this direction appear in [32].

APPENDIX A

To begin, without loss of generality, index the constellation points such that \(0 \leq p_1 < p_2 < \cdots < p_L\). Then, fix any codebook \(\mathcal{P}\) which satisfies the average power constraint, and solve (13) over only the decoding regions \(\{\mathcal{I}_i\}\), i.e., over \(\{d_{L,k}, d_{R,k}\}_{k=1}^L\). This subproblem can be written as

\[
\begin{align*}
\text{maximize} & \quad \min_{k \in [L]} (I_{L,k}(d_{L,k}), I_{R,k}(d_{R,k})) \\
\text{subject to} & \quad d_{L,k+1} + d_{R,k} = p_{k+1} - p_k, \quad \forall k \in [L-1], \\
& \quad d_{L,k} \geq 0, d_{R,k} \geq 0, \quad \forall k \in [L],
\end{align*}
\]

(17)

Observe that (17) is separable to \(L - 1\) optimization problems, one for every \(k = [L-1]\), since for each \(k\), the constraints are separable. To see this, the constraint for \(k\) is \(d_{L,k+1} + d_{R,k} = p_{k+1} - p_k\) and for \(k+1\) is \(d_{L,k+2} + d_{R,k+1} = p_{k+2} - p_{k+1}\); the former is a linear constraint between \(d_{L,k+1}, d_{R,k}\) and the latter another constraint between \(d_{L,k+2}, d_{R,k+1}\). Each one of the resulting subproblems identifies the boundary between the \(p_k\) and \(p_{k+1}\) constellation point such that the minimum between those two is maximum\(^1\):

\[
\begin{align*}
\text{maximize} & \quad \min_{d_{L,k+1}, d_{R,k}} (I_{R,k}(d_{R,k}), I_{L,k+1}(d_{L,k+1})) \\
\text{subject to} & \quad d_{L,k+1} + d_{R,k} = p_{k+1} - p_k, \\
& \quad d_{L,k+1} \geq 0, d_{R,k} \geq 0.
\end{align*}
\]

(18)

Each of the above problems is solved for \(d_{L,k+1}, d_{R,k}\) such that

\[I_{R,k}(d_{R,k}) = I_{L,k+1}(d_{L,k+1}) \Rightarrow I_{R,k}(d_{R,k}) = I_{L,k+1}(p_{k+1} - p_k - d_{R,k}).\]

(19)

Note that such \(d_{R,k}\) always exists since, for any \(p_k < p_{k+1}\), \(I_{R,k}(d)\) and \(I_{L,k+1}(p_{k+1} - p_k - d)\) are increasing and decreasing functions of \(d\), respectively, with \(I_{R,k}(0) = 0, I_{L,k+1}(0) = 0\) and \(I_{L,k+1}(p_{k+1} - p_k) > 0, I_{R,k}(p_{k+1} - p_k) > 0\). In other words, increasing \(d_{R,k}\) increases the right error exponent of the \(k^{th}\) power level, but decreases the left error exponent of the \((k+1)^{th}\) level, and there always exists a \(d_{R,k}\) for which both are equal. Therefore, for any fixed \(\mathcal{P}\), the best decoding regions between any consecutive constellation points can be calculated by (19).

\(^1\)assuming \(d_{L,1} = \infty\) and \(d_{R,L} = \infty\).
Then, the following optimization problem finds the optimal $\mathcal{P}$:

$$
\begin{align*}
\text{maximize} & \quad t \\
\text{subject to} & \quad I_{R,k}(d_{R,k}) = I_{L,k+1}(p_{k+1} - p_k - d_{R,k}), \forall k \in [L-1], \\
& \quad I_{R,k}(d_{R,k}) \geq t, \forall k \in [L-1], \\
& \quad \frac{1}{L} \sum_{k=1}^{L} p_k = 1, \ 0 \leq p_k < p_{k+1}, \ 0 < d_{R,k} < p_{k+1} - p_k, \forall k \in [L-1].
\end{align*}
$$

The solution of (20) corresponds the largest $t^*$ such that the following problem is feasible:

$$
\begin{align*}
\text{find} & \quad \{p_k\}_{k=1}^{L}, \{d_{R,k}\}_{k=1}^{L-1} \\
\text{subject to} & \quad I_{R,k}(d_{R,k}) = I_{L,k+1}(p_{k+1} - p_k - d_{R,k}), \forall k \in [L-1], \\
& \quad I_{R,k}(d_{R,k}) \geq t^*, \forall k \in [L-1], \\
& \quad \frac{1}{L} \sum_{k=1}^{L} p_k = 1, \ 0 \leq p_k < p_{k+1}, \ 0 < d_{R,k} < p_{k+1} - p_k, \forall k \in [L-1].
\end{align*}
$$

(21)

Observe that for $t^* = 0$ the above problem is always feasible since $I_{R,k}(d) \geq 0$ and $I_{L,k}(d) \geq 0$. Also observe that for $t^* = \infty$ it is infeasible due to the finite power constraint and the fact that $I_{R,k}(d), I_{L,k}(d)$ are increasing functions of $d$.

The problem now is to find the largest $t^*$ for which (21) is feasible. We are going to describe in detail the algorithm that finds whether problem (21) has a feasible solution for any fixed and finite $t^* > 0$. The basic idea of this construction is that, for any fixed $t^*$, we should find the constellation with the smallest average power constraint, as this is the only constraint that could lead to infeasibility of (21). To start, fix $t^* > 0$ and choose $p_1^* = 0$. Choosing a higher value for $p_1$ can only make the problem more difficult since $I_{R,1}(d)$ is decreasing in $p_1$, and $p_2 > p_1$, thus also the rate functions for $p_2$ and the rest constellation points will be lower. Then, find $d_{R,1} > 0$, denoted as $d_{R,1}^*$, such that

$$
I_{R,1}(d_{R,1}) = t^*.
$$

(22)

The above equation has always only one solution since $I_{R,1}(d)$ is an increasing function of $d$, $I_{R,1}(0) = 0$ and $t^* > 0$. The solution of (22) leads to the closest point to the right of $p_1^* = 0$ which should be used as a boundary point for the first constellation point. Using a smaller boundary point would lead to a smaller rate function than $t^*$ since $I_{R,1}(d)$ is increasing on $d$. Until now we have specified $p_1^*, d_{R,1}^*$. Now, we find the smallest $p_2 > p_1^* + d_{R,1}^*$, denoted as $p_2^*$, such that

$$
I_{L,2}(p_2 - p_1^* - d_{R,1}^*) = t^*.
$$

(23)

Note that for $p_2 = p_1^* + d_{R,1}^*$, $I_{L,2}(0) = 0$. If there is no $p_2 > p_1^* + d_{R,1}^*$ that solves (23), then problem (21) is infeasible for this $t^*$ and we need to repeat the construction for a larger $t^*$. Note that if this is the case, i.e., if
Equation (19) can now be written as follows to simplify the algorithm needed for a constellation design that uses only the first, second and fourth moments. The feasibility problem (21) is simplified to observe that for a transmitted power (infinite power if the problem is infeasible). This is true because, at each step of the constellation design, finding a smaller right error exponent than \( t^* \) for the \( k^{th} \) point, or a smaller left error exponent than \( t^* \) for the \( (k + 1)^{th} \) point. Then, if it holds that \( \frac{1}{L} \sum_{k=1}^{L} p_k^* \leq 1 \), problem (21) is feasible for \( t^* \). Identifying the largest \( t^* \) for which (21) is feasible solves (13).

To efficiently perform this procedure we can employ a simple bisection algorithm (Algorithm 2). To see this, observe that for a \( i \), such that \( t^* < i \), the corresponding constellation design leads to higher (or equal) average transmitted power (infinite power if the problem is infeasible). This is true because, at each step of the constellation design, finding \( \tilde{d}_{R,k} \) that satisfies \( I_{R,k}(\tilde{d}_{R,k}) = i \) will lead to a \( \tilde{d}_{R,k} \) with \( \tilde{d}_{R,k} > d_{R,k}^* \), and finding \( \tilde{p}_k \) which satisfies \( I_{L,k}(\tilde{p}_k - \tilde{p}_{k-1} - \tilde{d}_{R,k-1}) = i \) will lead to a \( \tilde{p}_k \) with \( \tilde{p}_k > p_k^* \).

**APPENDIX B**

In this appendix we take into account the approximation of the left and right rate functions shown in (14) to simplify the algorithm needed for a constellation design that uses only the first, second and fourth moments. Equation (19) can now be written as follows \( \frac{d_{L,k+1}^2}{2s(p_{k+1})} = \frac{d_{L,k}^2}{2s(p_k)} = \frac{(p_{k+1} - p_k)^2}{2\left(\sqrt{s(p_{k+1})} + \sqrt{s(p_k)}\right)^2} \), which means that the feasibility problem (21) is simplified to

\[
\text{find } \{p_k\}_{k=1}^{L} \\
\text{subject to } p_{k+1} - p_k \geq 2t^* \left( \sqrt{s(p_{k+1})} + \sqrt{s(p_k)} \right) \\
\frac{1}{L} \sum_{k=1}^{L} p_k \leq 1.
\]

Then the procedure described in Appendix A is now simplified to the following: Fix \( t^* \) and choose \( p_1^* = 0. \) Then, iteratively choose the smallest \( p_{k+1} > p_k^* \) for \( k = 1, 2, \ldots, L - 1 \), such that \( \frac{(p_{k+1} - p_k)^2}{2\left(\sqrt{s(p_{k+1})} + \sqrt{s(p_k)}\right)^2} = t^* \). If no \( p_{k+1} > p_k^* \) exists, then problem (25) is infeasible.
APPENDIX C

In this appendix, we show the details of the robust constellation design problem. This problem is simplified if we denote a constellation using \( \{p_k\}_{k=1}^L \) and \( \{c_k\}_{k=1}^{L-1} \), where \( c_k \) is the boundary of the decoding region between the \( p_k \) and the \( p_{k+1} \) constellation point. Then, using the approximation shown in (14), the problem of maximizing the worst case approximate rate functions for all the channels inside the uncertainty region \( \mathcal{F} \) is expressed as follows:

\[
\begin{align*}
\text{maximize} & \quad t, \{p_k\}_{k=1}^L, \{c_k\}_{k=1}^{L-1} \\
\text{subject to} & \quad c_k - p_k \geq \sup_{f \in \mathcal{F}} \left( \sigma^2 + t \sqrt{s_f(p_k)} \right), \forall k \in [L], \\
& \quad p_{k+1} - c_k \geq \sup_{f \in \mathcal{F}} \left( t \sqrt{s_f(p_{k+1})} - \sigma^2 \right), \forall k \in [L-1], \\
& \quad \frac{1}{L} \sum_{k=1}^L p_k = 1, p_k \geq 0, \forall k \in [L].
\end{align*}
\]

This problem is equivalent to finding the largest \( t^* > 0 \) which gives a feasible point in this formulation:

\[
\begin{align*}
\text{find} & \quad \{p_k\}_{k=1}^L, \{c_k\}_{k=1}^{L-1} \\
\text{subject to} & \quad c_k - p_k \geq \sup_{f \in \mathcal{F}} \left( \sigma^2 + t^* \sqrt{s_f(p_k)} \right), \forall k \in [L], \\
& \quad p_{k+1} - c_k \geq \sup_{f \in \mathcal{F}} \left( t^* \sqrt{s_f(p_{k+1})} - \sigma^2 \right), \forall k \in [L-1], \\
& \quad \frac{1}{L} \sum_{k=1}^L p_k = 1, p_k \geq 0, \forall k \in [L].
\end{align*}
\]

Solving the above feasibility problem can be done as follows: Fix a small \( t^* > 0 \) and choose \( p_1^* = 0 \) and \( d_{L,1}^* = \infty \) so that \( \tilde{I}_1(d_{L,1}^*) = \infty \). Using \( p_1^* > 0 \) would lead to a sub-optimal solution since the transmitter has an average power constraint and \( s_f(p) \) is an increasing function of \( p \) for every \( f \in \mathcal{F} \). Then, choose \( c_1^* \) which satisfies

\[
c_1^* = \sup_{f \in \mathcal{F}} \left( \sigma^2 + t^* \sqrt{s_f(p_1^*)} \right) + p_1^*,
\]

and as \( p_2^* \), the minimum \( p \) that satisfies

\[
p - c_1^* - \sup_{f \in \mathcal{F}} \left( t^* \sqrt{s_f(p)} - \sigma^2 \right) \geq 0.
\]

Note that for \( 0 < t^* \leq \inf_{f \in \mathcal{F}} \frac{1}{\sqrt{s_f(0)}} \), there always exists a \( p \geq c_1^* - \inf_{f \in \mathcal{F}} \sigma^2 \) that satisfies the above equation. To see this, define the following auxiliary function \( w_f(p) = p - c_1^* - t^* \sqrt{s_f(p)} + \sigma^2 \), for which, for any fixed \( f \in \mathcal{F} \), it holds that \( w_f(c_1^* - \sigma^2) < 0 \) and \( \lim_{p \to \infty} \frac{w_f(p)}{p} = 1 - t^* \sqrt{\alpha_f} > 0 \).

Note that choosing a higher value for \( c_1^* \) would only make \( p_2^* \) larger (or infinity) and thus, use more transmit power than necessary (or make the problem infeasible). Using the same procedure we can sequentially specify all \( \{p_k^*\}_{k=1}^L \) and \( \{c_k^*\}_{k=1}^{L-1} \). Then, if \( \frac{1}{L} \sum_k p_k^* \leq 1 \), the problem is feasible. However, if the average power constraint is not satisfied, it is not possible to guarantee this error exponent for all channels in \( \mathcal{F} \), since in our construction, we pack the decoding regions and constellation points as closely as possible. To see this, if in the above construction we choose any value \( \tilde{c}_k > c_k^* \), then the corresponding \( p \) which satisfies (28) would be larger than \( p_k^* + \tilde{c}_k \) since

\[
p - \tilde{c}_k - \sup_{f \in \mathcal{F}} \left( t^* \sqrt{s_f(p)} - \sigma^2 \right) \leq p - c_k^* - \sup_{f \in \mathcal{F}} \left( t^* \sqrt{s_f(p)} - \sigma^2 \right), \forall p > p_k^* + \tilde{c}_k.
\]
REFERENCES


