Uncoded transmission in MAC channels achieves arbitrarily small error probability

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Motivation: Uplink communication in large networks

Often limited by transmitter side constraints
- Power/energy requirements
- Delay constraints
- Processing capability

Examples
- Cellular uplink
- Sensor networks
- Low power radio
$H_{ij} \sim \mathcal{N}(0, 1), \nu_i \sim \mathcal{N}(0, \sigma^2)$

$y = Hx + \nu$

$N_U$ users, each with 1 antenna, $N_R$ receive antennas
System model

\[ x_1, x_2, \ldots, x_{N_U}, H_{N_R1}, H_{N_R2}, \ldots, H_{N_RN_U}, v_1, v_2, \nu_1, \nu_2, \nu_{N_R}, y_1, y_2, y_{N_R} \]

\( N_U \) single antenna users
\( N_R \) antenna receiver

- \( H_{ij} \sim \mathcal{N}(0, 1), \nu_i \sim \mathcal{N}(0, \sigma^2) \)
- No CSI at transmitters, perfect CSI at receiver
- Receiver does perfect ML decoding

\[ \hat{x} = \arg\min_{x \in \{-1, +1\}^{N_U}} \| y - Hx \| \]
We ask

- $N_U$ users, each user transmitting at a fixed rate $R$
- Given $P_e$
- How many receive antennas are required?

We show

As $N_U \to \infty$, $P_e \to 0$ for any ratio $\frac{N_R}{N_U}$
Background

For MIMO point-to-point channel
- Multiplexing gain $\frac{1}{2} \min(N_R, N_U)$
- Achieved by coding across time
- $P_e \to 0$ with larger blocklengths

For a MAC channel
- $N_U$ transmitters each with 1 antenna
- $N_R$ receiver antennas
- Total transmit power grows like $N_U$
- Multiplexing gain $\frac{1}{2} \min(N_R, N_U) \log N_U$
- Achieved by coding over large blocklengths
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Both results rely on coding across time to get low $P_e$
Main question

Is coding necessary in large systems to achieve

- Reliability in communication ($P_e \to 0$) ?
- Optimal number of receiver antennas per user ?
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Intuition

Large systems already have sufficient degrees of freedom to achieve low error probability
Main question

Is coding necessary in large systems to achieve

- Reliability in communication \((P_e \to 0)\) ?
- Optimal number of receiver antennas per user?

Intuition

Large systems already have sufficient degrees of freedom to achieve low error probability

To investigate this, from now on, we consider BPSK transmissions without coding
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Reliability in communication

**Theorem**

For $N_R = \alpha N_U$ for any $\alpha > 0$ and a large enough $n_0$, there exists a $c > 0$ such that the probability of block error goes to zero exponentially fast with $N_U$, i.e.

$$P(\hat{x} \neq x) \leq 2^{-cN_U} \text{ for all } N_U > n_0$$

**In other words**

- Equal rate transmission
- Any $\frac{N_R}{N_U}$ ratio
- Error probability $P_e \to 0$ with $N_U \to \infty$
Reliability in communication

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This is due to a combination of receiver diversity and spatial multiplexing
Minimum number of required Rx antennas per user

Let us consider coding across time

- Every user needs a rate of 1 bit per channel use on average
- Every user has average power 1
- Sum rate required from the system is \( N_U \)
- Equal rate capacity of the system is \( \frac{1}{2} \log | (I + \frac{HH^T}{\sigma^2}) | \)
Minimum number of required Rx antennas per user

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- Sum rate required from the system is $N_U$
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Thus, the lowest $N_R$ to support reliable transmissions is

$$N_R \geq \frac{2N_U}{2c + \log N_U}$$
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Smallest $\frac{N_R}{N_U}$ ratio achievable with coding is

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Smallest $\frac{N_R}{N_U}$ ratio achievable with coding is

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Can we achieve similar ratios for reliable uncoded systems also?
Minimum number of required Rx antennas per user:

**Theorem**

For \( \frac{N_R}{N_U} = \frac{2+\epsilon}{\log N_U} \) for any \( \epsilon > 0 \) and a large enough \( n_0 \), there exists a \( c > 0 \) such that the probability of block error goes to zero exponentially fast with \( N_U \), i.e.

\[
P(\hat{x} \neq x) \leq 2^{-cN_U \log N_U} \text{ for all } N_U > n_0
\]

**In other words**

- Coding does not reduce the number of required antennas per user for 1 bit per channel use
- Scaling behaviour of minimum \( \frac{N_R}{N_U} \) is \( \Theta\left(\frac{1}{\log N_U}\right) \)
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View of the $2^{NU}$ points in the projected space
Proof outline: Union bound

Let’s say $x_0$ is transmitted.

$$P_e \leq \sum_{x \neq x_0} P(\hat{x} = x)$$
Proof outline: Union bound

Let’s say $x_0$ is transmitted.

$$P_e \leq \sum_{x \neq x_0} P(\hat{x} = x)$$

Idea

Group wrong codewords by the number of positions in which they differ from $x_0$

Thus

$$P_e \leq \sum_{i=1}^{NU} \sum_{x : d(x, x_0) = i} P(\hat{x} = x),$$

where $d(\cdot, \cdot)$ computes the hamming distance between its two arguments.
Proof outline: Computing pairwise error probabilities

Let $x_i$ be a codeword with $i$ positions different from $x_0$. Then

$$P(\hat{x} = x_i) = Q\left(\frac{||H(x_i - x_0)||}{2\sigma}\right)$$

$$= Q\left(\frac{||\sum_{j=1}^{i} 2h_{b(j)}||}{2\sigma}\right)$$

$b(j)$ is the $j^{th}$ position where $x_i$ and $x_0$ differ,

$h_b$ is the $b^{th}$ column of $H$$$

$$\leq \frac{1}{2} \exp\left(-\frac{||\sum_{j=1}^{i} 2h_{b(j)}||^2}{8\sigma^2}\right)$$
Proof outline: Average pairwise error probability

Idea

Average over channel realizations

\[
\mathbb{E}_H(P(\hat{x} = x_i)) \leq \mathbb{E}_H \left( \frac{1}{2} \exp \left( - \frac{\left\| \sum_{j=1}^{i} 2h_{b(j)} \right\|^2}{8\sigma^2} \right) \right)
\]

This is just the moment generating function of an appropriately scaled chi squared distribution.

Closed form expression

\[
\mathbb{E}_H(P(\hat{x} = x_i)) \leq \frac{1}{2} \left( 1 + \frac{i}{\sigma^2} \right)^{-\frac{NR}{2}}
\]
Proof outline: Average pairwise error probability

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Average over channel realizations

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\mathbb{E}_H(P(\hat{x} = x_i)) \leq \mathbb{E}_H \left( \frac{1}{2} \exp \left( - \frac{\| \sum_{j=1}^{i} 2h_{b(j)} \|^2}{8\sigma^2} \right) \right)
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\mathbb{E}_H(P(\hat{x} = x_i)) \leq \frac{1}{2} \left( 1 + \frac{i}{\sigma^2} \right)^{-\frac{NR}{2}}
\]

Depends only on \( i \), i.e. number of columns, not on particular columns considered
Proof outline: Upper bound on total error probability

Let

\[ \mathbb{E}_H(P(\hat{x} = x_i)) = P_{e,i}, \]

since it is independent of \( x_i \). Then the total error probability is

\[
P_e \leq \sum_{i=1}^{NU} \sum_{\mathbf{x}:d(\mathbf{x}, \mathbf{x}_0) = i} P(\hat{x} = x) = \sum_{i=1}^{NU} \frac{1}{2} \binom{NU}{i} P_{e,i}
\]

\[
\leq \sum_{i=1}^{NU} \frac{1}{2} \binom{NU}{i} \left(1 + \frac{i}{\sigma^2}\right)^{-\frac{NR}{2}}
\]

\[
\leq NU \max_{i \in \{1, \ldots, NU\}} \frac{1}{2} \binom{NU}{i} \left(1 + \frac{i}{\sigma^2}\right)^{-\frac{NR}{2}}
\]

\[
\leq \frac{NU}{2} \max_{i \in \{1, \ldots, NU\}} 2^{NUH_2\left(\frac{i}{NU}\right)} \left(1 + \frac{i}{\sigma^2}\right)^{-\frac{NR}{2}}
\]
Proof outline: Asymptotic limits

Define

\[ \alpha = \frac{N_R}{N_U} \]

Decoding error probability

\[ P_e \leq 2^{N_U g(N_U)} \]

where

\[ g(n) \triangleq \max_{1 \leq i \leq n} H_2 \left( \frac{i}{n} \right) - \frac{\alpha}{2} \log \left( 1 + \frac{i}{\sigma^2} \right) + \frac{\log n - \log 2}{n} \]

\[ \triangleq \max_{1 \leq i \leq n} g(i, n) \]
Proof outline: Behaviour of $g(n)$, constant $\alpha$

\[ \alpha = 0.3, \sigma^2 = 0.5, n = 50, g(n) = \max_{1 \leq i \leq n} g(i, n) = 0.26 \]
Proof outline: Behaviour of $g(n)$, constant $\alpha$

$\alpha = 0.3, \sigma^2 = 0.5, n = 100, g(n) = \max_{1 \leq i \leq n} g(i, n) = 0.08$
Proof outline: Behaviour of $g(n)$, constant $\alpha$

$\alpha = 0.3, \sigma^2 = 0.5, n = 200, g(n) = \max_{1 \leq i \leq n} g(i, n) = -0.1$
Proof outline: Behaviour of $g(n)$, constant $\alpha$

\[ \alpha = 0.3, \sigma^2 = 0.5, n = 700, g(n) = \max_{1 \leq i \leq n} g(i, n) = -0.37 \]
Proof outline: Behaviour of $g(n)$, constant $\alpha$

Message

For fixed $\alpha > 0$ and large enough $n$,

$$g(n) \leq c(\alpha) < 0$$
Proof outline: Behaviour of $g(n)$, $\alpha = \frac{2+\epsilon}{\log n}$

$\epsilon = 0.1, \sigma^2 = 0.5, n = 100, g(n) = \max_{1 \leq i \leq n} g(i, n) = 0.03$
Proof outline: Behaviour of $g(n)$, $\alpha = \frac{2+\epsilon}{\log n}$

$\epsilon = 0.1, \sigma^2 = 0.5, n = 300, g(n) = \max_{1 \leq i \leq n} g(i, n) = -0.01$
Proof outline: Behaviour of $g(n)$, $\alpha = \frac{2+\epsilon}{\log n}$

Message

For fixed $\alpha > 0$ and large enough $n$,

$$g(n) \leq c(\epsilon) < 0,$$

$c(\epsilon)$ goes down as $\Theta\left(-\frac{1}{\log n}\right)$

Thus

$$P_e \leq 2^{-\frac{cN_U}{\log N_U}}$$
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So far discussion focused on
- BPSK constellations: Each user transmits from \{-1, +1\}
- Gaussian channel statistics (Rayleigh fading)
Arbitrary finite constellation $\mathcal{C}$

- Pairwise error probability can be bounded by

$$P(\hat{x} = x_i) \leq \frac{1}{2} \exp \left( -\frac{\| \sum_{j=1}^{i} d h_{b(j)} \|^2}{8\sigma^2} \right),$$

where $d$ is the minimum distance of the constellation i.e.

$$d = \min_{x \in \mathcal{C}, y \in \mathcal{C}, x \neq y} \| x - y \|$$

- In general loose
- Same scaling $\Theta \left( \frac{1}{\log NU} \right)$ for the number of receiver antennas per user
Arbitrary fading statistics

Idea

Central limit theorem and rate of convergence of distributions!

Specifically

- Error due to $i$ mismatches depends on the statistics of $\sum_{j=1}^{i} h_{b(j)}$
- Berry Esseen bound guarantees $\Theta(\frac{1}{\sqrt{n}})$ convergence, i.e.

$$\sup_{x} |F_{n}(x) - F_{g}(x)| \leq \frac{\tilde{C}}{\sqrt{n}}$$

- Can be shown that

$$P_{e,i} \leq C i^{-\frac{N_{R}}{2}},$$

for all $i \geq i_{0}$

Thus

Asymptotic behaviour of $P_{e,i}$ in $i$ same as that for Gaussian fading
Arbitrary fading statistics

Idea
Central limit theorem and rate of convergence of distributions!

Specifically
- Error due to $i$ mismatches depends on the statistics of $\sum_{j=1}^{i} h_b(j)$
- Berry Esseen bound guarantees $\Theta(\frac{1}{\sqrt{n}})$ convergence, i.e.
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  \sup_x |F_n(x) - F_g(x)| \leq \frac{\tilde{C}}{\sqrt{n}}
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  Can be shown that
  \[
  P_{e,i} \leq C i^{-\frac{N_R}{2}},
  \]
  for all $i \geq i_0$

But $P_e = \sum_{i=1}^{i_0} P_{e,i} + \sum_{i=i_0+1}^{N_U} P_{e,i}$?
Terms with less than $i_0$ mismatches $\to 0$ with large $N_R$
Key insights

- Sufficient degrees of freedom already present in large systems
- Receiver diversity allows reliable communication
- Positive rate possible for every user \textit{without coding}

\textbf{Ongoing work:} Extensions to suboptimal decoders, imperfect CSI, correlated channels
Thank you for your attention
Questions/Comments?