Capacity of block Rayleigh fading channels

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Introduction

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**Transmitter** \( h \) **Receiver**
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- Capacity and capacity achieving schemes known for IID fading (\( T_0 = 1 \)) and coherent communication (\( T_0 \rightarrow \infty \))
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- $h$ stays constant for $T_0$ symbol times
- Capacity and capacity achieving schemes known for IID fading ($T_0 = 1$) and coherent communication ($T_0 \to \infty$)

Our contribution
Properties of capacity achieving input distribution for finite $T_0 > 1$
Prior work

IID fading

- Capacity achieving input distribution is discrete and finite in the norm [Abou Faycal, Shamai 2001]

Block fading

- Unitary codes achieve capacity in block fading channels [Hochwald, Marzetta 2000]
Prior work

IID fading

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Block fading

• Unitary codes achieve capacity in block fading channels [Hochwald, Marzetta 2000]
• Capacity achieving distribution/properties known in various regimes [Zheng, Tse 2002] [Lapidoth 2005] . . .

We show
Capacity achieving input distribution for a finite $T_0 > 1$ also assigns a discrete and finite-cardinality measure to the norm
Consider the block fading model

\[
\mathbf{y} = \mathbf{h} \mathbf{x} + \mathbf{\nu}
\]

\[
\begin{align*}
\mathbf{y} & \sim \mathcal{N}(0, \mathbf{I}) \\
\mathbf{x} & \sim \mathcal{N}(0, \mathbf{I}) \\
\mathbf{\nu} & \sim \mathcal{N}(0, \mathbf{I})
\end{align*}
\]

- \( \frac{1}{T_0} \mathbb{E}[\|\mathbf{x}\|^2] \) at transmitter should be less than SNR
Consider the block fading model

\[ \begin{align*}
    y & = h x + \nu \\
    & \in \mathbb{R}^{T_0} \quad \mathcal{N}(0,1) \quad \mathbb{R}^{T_0} \\
    & \in \mathbb{R}^{T_0} \quad \mathcal{N}(0,1)
\end{align*} \]

- \( \frac{1}{T_0} \mathbb{E}[\|x\|^2] \) at transmitter should be less than SNR
- Input distribution on \( x \) is specified by measure \( \mu_X(\cdot) \)
Consider the block fading model

\[
\begin{align*}
\mathbf{y} &\sim \mathbf{h} \mathbf{x} + \mathbf{\nu} \\
\mathbf{R} &\sim \mathcal{N}(0,1) \mathbf{R}^{T_0} &\sim \mathcal{N}(0,1)
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\]

- \( \frac{1}{T_0} \mathbb{E}[\|\mathbf{x}\|^2] \) at transmitter should be less than SNR
- Input distribution on \( \mathbf{x} \) is specified by measure \( \mu_{\mathbf{x}}(\cdot) \)
System model

Consider the block fading model

\[
\begin{align*}
\textbf{y} & = h \textbf{x} + \nu \\
\mathbb{E}[\|\textbf{x}\|^2] & \leq \text{SNR} \\
\text{Input distribution on } \textbf{x} & \text{ is specified by measure } \mu_x(\cdot)
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Consider the block fading model

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\begin{align*}
y & = h x + \nu \\
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\text{Input distribution on } x & \text{ is specified by measure } \mu_x(\cdot)
\end{align*}
\]

**Objective**
Find the capacity; obtain the capacity achieving distribution
System model

Consider the block fading model

\[ \textbf{y} = h \textbf{x} + \nu \]

- \( \frac{1}{T_0} \mathbb{E}[\|\textbf{x}\|^2] \) at transmitter should be less than SNR
- Input distribution on \( \textbf{x} \) is specified by measure \( \mu_X(\cdot) \)

**Objective**

Solve \( \sup_{\mu_X(\cdot)} I(y, x) \) s.t. \( \int \frac{\|x\|^2}{T_0} d\mu_X(x) \leq \text{SNR} \)
Consider the block fading model

\[
\begin{align*}
\mathbf{y} &\sim h \cdot \mathbf{x} + \mathbf{\nu} \\
\mathbf{R}^{T_0} \sim \mathcal{N}(0,1) &\sim \mathbf{R}^{T_0}, \mathcal{N}(0,1)
\end{align*}
\]

- \( \frac{1}{T_0} \mathbb{E}[\|x\|^2] \) at transmitter should be less than SNR
- Input distribution on \( x \) is specified by measure \( \mu_X(\cdot) \)

**Objective**

Solve \( \inf_{\mu_X(\cdot)} I(\mathbf{y}, \mathbf{x}) \) s.t. \( \int \frac{\|x\|^2}{T_0} d\mu_X(x) \leq \text{SNR} \)
System model in more detail

The transition probability $p_{Y|X}$

$$y \sim (2\pi)^{-\frac{T_0}{2}} |\Sigma_x|^{-\frac{1}{2}} e^{-\frac{1}{2} y^T \Sigma_x^{-1} y}$$

Properties of $\Sigma_x$

- $\Sigma_x = I_{T_0} + xx^T$
System model in more detail

The transition probability $p_{Y|X}$

$$
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- Eigenvalues of $\Sigma_x$ are $\{1 + \|x\|^2, 1, \ldots, 1\}$
- Eigenvalues of $\Sigma_x^{-1}$ are $\left\{ \frac{1}{1+\|x\|^2}, 1, \ldots, 1 \right\}$
- The determinant of $\Sigma_x$, i.e., $|\Sigma_x|$, is exactly $1 + \|x\|^2$
Main result

The capacity achieving distribution [M.C., Goldsmith 2016]

For any finite $T_0 > 1$, the measure on the norm $\|x\|$ assigned by the optimal $\mu_\star^X(x)$ is supported on a set which is discrete and has a finite cardinality.
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- Also unitarily invariant in $\mathbb{R}^{T_0}$ [Hochwald, Marzetta 2000]
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For low SNRs, cardinality of support found to be 2 numerically
Proof ideas

- Argue that extremum is attained by a measure $\mu^*_X(\cdot)$
- Form Lagrangian
- Write down KKT conditions for optimality
- Prove properties of the support of measure assigned to $\|x\|$
Form Lagrangian

\[ L(\mu X(\cdot), \lambda_1, \lambda_2) = \int_y \left( \int p_{Y|x}(y) \, d\mu_X(x) \right) \log \left( \int p_{Y|x}(y) \, d\mu_X(x) \right) \, dy + \right. \\
+ \left. \int_x 0.5 \log((2\pi e)^{1/2} T_0 |\Sigma_x|) \, d\mu_X(x) + \tilde{\lambda}_1 \left( \int \frac{1}{T_0} \|x\|^2 \, d\mu_X(x) - \text{SNR} \right) + \lambda_2 \left( \int \, d\mu_X(x) - 1 \right) \]
Wherever $\mu^*_x(\mathcal{N}_x) > 0$, $\mathcal{N}_x$ being any ball containing $x$,

$$
\int_y (1 + \log(p^*_Y(y))) p_{Y|x}(y) dy \\
+ 0.5 \log((2\pi e)^{T_0} |\Sigma_x|) + \lambda_1 \|x\|^2 + \lambda_2 \\
\triangleq g(p^*_Y(\cdot), x) = 0,
$$
Discreteness and boundedness of support

First establish **discreteness** in $\|x\|$: 

- Note that $g(p^*_Y(\cdot), x)$ is analytic in $\frac{1}{1 + \|x\|^2}$.
- If not discrete in $\|x\|$, there exists a limit point around which $g(p^*_Y(\cdot), x)$ is zero. For $s = \frac{1}{1 + \|x\|^2}$, we have that

$$
\int_g c_0 \left(1 + \log(p^*_Y(y))\right) y_\parallel \left(-\frac{1}{2} s \frac{1}{2} e^{-c_1 s y_\parallel} - c_2 \|y_\perp\|^2\right) dy
$$

$$
- 0.5 \log((2\pi e)^{T_0} s) + \lambda_1 \left(\frac{1}{s} - 1\right) + \lambda_2 = 0.
$$

Inverting Laplace transform we get $p^*_Y = c_3 \frac{e^{-K y_\parallel}}{y_\parallel} \Rightarrow invalid!$
Discreteness and boundedness of support

Then establish boundedness of support:

- Lower bound $p_Y^*(y)$ by $p_0$ \( p_Y|_{x_0} \) possible function of $\|x_0\|
- Evaluate $g(p_Y^*(\cdot), x) \approx c_6 \log(\|x\|) + \lambda_1 \|x\| + c_7$
- Observe that $g(p_Y^*(\cdot), x)$ is bounded away from 0 as $\|x\| \to \infty$
Discreteness and boundedness of support

Then establish boundedness of support:

- Lower bound $p_Y^*(y)$ by $p_0$ if $p_Y|_{x_0}$ is a possible function of $||x_0||$
- Evaluate $g(p_Y^*(\cdot), x) \approx c_6 \log(||x||) + \lambda_1 ||x|| + c_7$
- Observe that $g(p_Y^*(\cdot), x)$ is bounded away from 0 as $||x|| \to \infty$

Hence proved
Connections to channel estimation

Channel estimation and data transmission

Typical communication systems have two distinct phases

- Estimating $h$ using pilots known at the receiver
- Using estimate of $h$ for data transmission/detection
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Capacity achieving strategy
Involves joint channel estimation and data transmission
- No explicit pilot symbols
- Channel estimation is performed implicitly
- We study implicit channel estimation numerically through $I(h; y)$
Numerical results

Objectives

- Compare the $\mu^*_X(\cdot)$ for $T_0 = 1$ and $T_0 = 2$
- Compare the noncoherent capacity for $T_0 = 1$ and $T_0 = 2$
- Study implicit channel estimation $I(h; y)$
Capacity achieving distribution for SNR = 0dB

\[ T_0 = 1 \] [Abou Faycal, Shamai 2001]
Capacity achieving distribution for SNR = 0dB

\[ T_0 = 1. \text{ Note the invariance to sign changes} \]
Capacity achieving distribution for SNR = 0dB

Product for $T_0 = 2$ from $T_0 = 1$. Not unitarily invariant
Capacity achieving distribution for SNR $= 0\text{dB}$

$T_0 = 2$
Capacity achieving distribution for SNR = 0dB

\[ T_0 = 2. \text{ Note unitary invariance} \]
**Capacity plots**

**Figure:** Plots for $T_0 = 1$ (dotted) [Abou Faycal, Shamai 2001] and $T_0 = 2$
Implicit channel estimation

- Information about channel greater than 0; increases with SNR
- Implicit channel estimation by the capacity achieving scheme
Conclusions

- Capacity achieving input distribution for block Rayleigh fading
  - assigns a discrete and finite measure to the norm and
  - is isotropic in the $T_0$ dimensional space
- Capacity-achieving schemes motivate joint channel estimation and data transmission
  - Pilot based schemes strictly suboptimal
  - Capacity achieving scheme entails implicit channel estimation
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Thank you for your attention!