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Solving the Neoclassical Growth Model with Quasi-Geometric Discounting: A Grid-Based Euler-Equation Method

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Abstract. The standard neoclassical growth model with quasi-geometric discounting is shown elsewhere (Krusell, P. and Smith, A., CEPR Discussion Paper No. 2651, 2000) to have multiple solutions. As a result, value-iterative methods fail to converge. The set of equilibria is however reduced if we restrict our attention to the interior (satisfying the Euler equation) solution. We study the performance of a grid-based Euler-equation methods in the given context. We find that such a method converges to an interior solution in a wide range of parameter values, not only in the "test" model with the closed-form solution but also in more general settings, including those with uncertainty.

Key words: neoclassical growth model, numerical methods, quasi-geometric (hyperbolic) discounting, time-inconsistency

JEL Classification: C73, D90, E21

1. Introduction

In the recent literature, much attention has been paid to studying the implications of models with quasi-geometric (hyperbolic) discounting, e.g., Laibson (1997), Harris and Laibson (2001), Krusell and Smith (2000, 2003), Krusell, Kuruşçu and Smith (2002), Maliar and Maliar (in press), Judd (2004). Under such discounting, the short-run discount factor (applied between today and tomorrow) differs from the long-run discount factor (applied between tomorrow and the day after tomorrow, and onwards). This assumption leads to time-inconsistent preferences.

Krusell and Smith (2000) incorporate quasi-geometric discounting into the deterministic version of the standard neoclassical growth. They find that in addition to the standard interior (satisfying the Euler equation) solution, the model has multiple step-function equilibria. As a result, value-iterative methods fail to converge. The set of equilibria is however reduced if we restrict our attention to the interior solution, however, it remains unknown whether such a solution is unique. In particular, Krusell et al. (2002) and Judd (2004) solve the same model with different variants of perturbation method and reach different conclusions: the former paper finds that an interior solution is unique while the latter paper finds multiple solutions. A possible explanation for the difference in the results of the two papers is that the programming problem in question may be non-concave. Even if there were a unique interior maximum, there might exist other interior non-maximum critical points. As a result, different numerical methods can converge to different solutions or can fail to converge at all. It is therefore of interest to investigate the performance of other Euler-equation methods in the given context.[1]

In this paper, we study the performance of an algorithm that solves the Euler equation on a grid of prespecified points, and we find that such a method leads to a unique interior solution. We first applied the method to the deterministic version of the neoclassical growth model. We find that the algorithm converges in a wide range of parameter values, provided that the grid is not very fine and that the decision rules are updated slowly enough. The solutions delivered by our grid method proved to be identical to those found by the perturbation method developed in Krusell et al. (2002). We next employ our method to solve for equilibria in the stochastic version of the neoclassical growth model. (To our knowledge, this version of the model has not been studied in the literature yet). Once again we observe that if the short-run discount factor is not very different from the long-run one, and if the algorithm's parameters (the number of grid points and the updating parameter) are appropriately chosen, our algorithm converges to the interior solution.

2. The Model

We consider a neoclassical economy populated by one quasi-geometric agent, see Laibson (1997) and Krusell and Smith (2000). Time is discrete and infinite, $t \in \{0, 1, 2, \ldots\}$. On each date t, the agent chooses a contingency plan for consumption $\{c_t^t, c_{t+1}^t, c_{t+2}^t, \ldots\}$ and for capital $\{k_{t+1}^t, k_{t+2}^{t+1}, k_{t+3}^{t+2}, \ldots\}$, where time superscript and time subscript indicate, respectively, the periods in which and for which consumption and capital are chosen (e.g., consumption c_{t+1}^t is chosen in period t for period $t+1$) subject to the capital-accumulation constraint:

$$\max_{\{c_\tau^t, k_{\tau+1}^t\}_{\tau=t}^{\infty}} \left\{ u\left(c_t^t\right) + E_t \sum_{\tau=t}^{\infty} \beta \delta^{\tau+1-t} u\left(c_{\tau+1}^t\right) \right\} \tag{1}$$

$$\text{s.t.} \qquad c_\tau^t + k_{\tau+1}^t = (1-d) k_\tau^{t-1} + \theta_\tau f\left(k_\tau^{t-1}\right), \tag{2}$$

where (k_t^{t-1}, θ_t) is given. Here, θ_τ is the technology shock, u is the period utility function, f is the production function, E_t is the operator of the conditional expectation, $d \in (0, 1]$ is the depreciation rate of capital, and $\beta > 0$ and $\delta \in (0, 1)$ are the discounting parameters. We assume that u and f are strictly increasing, strictly concave, continuously differentiable and satisfy the Inada conditions and that the random variable $\ln \theta_{t+1}$ follows $AR(1)$ process, $\ln \theta_{t+1} = \rho \ln \theta_t + \varepsilon_{t+1}$ with $\rho \in [0, 1)$ and $\varepsilon_{t+1} \sim N(0, v^2)$.

The standard case of geometric discounting corresponds to $\beta = 1$. If $\beta > 1$ ($\beta < 1$), then the short-run discount factor, $\beta\delta$, is higher (lower) than the long-run one, δ; such discounting is called quasi-geometric (hyperbolic). The assumption of quasi-geometric discounting leads to time-inconsistency in preferences in the sense that the relative value of consumption in any two adjacent periods t and $t + 1$ depends on the date on which the evaluation is performed. We assume that the agent is fully aware of her preference inconsistency and also, that she cannot commit herself to fulfilling her plans. In the presence of time-inconsistency, consumption chosen at t, c_{t+1}^t, is not equal to the one chosen at $t + 1$, c_{t+1}^{t+1}. The "true" consumption at $t + 1$ is c_{t+1}^{t+1}. Therefore, the "true" lifetime stream of consumption is $\{c_0^0, c_1^1, \ldots\} \equiv \{c_0, c_1, \ldots\}$. Similarly, the "true" sequence of capital is given by $\{k_1^0, k_2^1, \ldots\} \equiv \{k_1, k_2, \ldots\}$. Note that if commitment was possible at any time t, then a sequence $\{c_t^t, c_{t+1}^t, c_{t+2}^t, \ldots\}$ solving (1), (2) at t would be the "true" one.

We restrict our attention to the recursive first-order Markov equilibrium. We assume that the agent chooses the next period's capital stock k_{t+1} according to a time-invariant policy function, $k_{t+1} = g(k_t, \theta_t)$. If such a solution exists and is interior, then it satisfies the Euler equation,

$$u'(c_t) = \delta E_t \left\{ u'(c_{t+1}) \left(\beta(1 - d + \theta_{t+1} f'(k_{t+1})) + (1 - \beta) \frac{\partial g(k_{t+1}, \theta_{t+1})}{\partial k_{t+1}} \right) \right\}. \quad (3)$$

A distinctive feature of the Euler Equation (3), compared to the standard one, is the appearance of the last term on the right-hand side: it contains the derivative of the unknown decision rule, $\frac{\partial g(k_{t+1}, \theta_{t+1})}{\partial k_{t+1}}$. The deterministic steady state in such a model satisfies

$$1 = \delta \left(\beta(1 - d + f'(\bar{k})) + (1 - \beta) \frac{\partial g(\bar{k}, 1)}{\partial k} \right), \quad (4)$$

where $\bar{k}$ denotes the steady-state level of capital. In the standard model ($\beta = 1$), Equation (4) delivers $\bar{k}$ straightforwardly. With quasi-geometric discounting ($\beta \neq 1$), however, this is not the case. Here, we have only one equation but two unknowns: the steady-state level of the function g (since $\bar{k} = g(\bar{k})$ by definition) and its first derivative, $\frac{\partial g(\bar{k}, 1)}{\partial k}$, at this point. The consequence of this fact is that we cannot compute the steady state without solving for the function g.

3. The Model with a Closed-Form Solution

Let us assume that the period utility function is logarithmic, $u(c) = \ln(c)$, that the production function is Cobb–Douglas, $f(k) = k^{\alpha}$ with $\alpha \in (0, 1)$ and that capital depreciates fully in each period, $d = 1$. Then, the model admits a closed-form solution

$$k_{t+1} = \frac{\beta\delta\alpha}{1 - \delta\alpha + \beta\delta\alpha}\theta_t k_t^{\alpha}. \tag{5}$$

Krusell and Smith (2000) study the deterministic version of the model ($\theta_t = 1$ for all t) and find that numerical algorithms iterating on value function fail to converge to the closed-form solution. They explain the failure of the value-iterative approach by the fact that the model has multiple solutions. "The multiplicity takes two forms. First, there is a continuum of stationary points for the consumer's asset holdings. Second, for each stationary point, there is a continuum of paths leading into this stationary point" (Krusell and Smith, 2000, p. 17). The interval of possible stationary points (steady states) for the capital stock is given by

$$\bar{k} \in \left((f')^{-1}\left(\frac{1}{\beta\delta}\right), (f')^{-1}\left(\frac{1 - \delta(1 - \beta)}{\beta\delta}\right)\right). \tag{6}$$

The paths leading to each steady state are discontinuous (they take the form of step functions).

Krusell et al. (2002) argue that the closed-form solution is a unique interior solution to the model because all discontinuous solutions are ruled out by the assumption that the equilibrium is interior (i.e., satisfies the Euler equation). However, the numerical results in Judd (2004) indicate that multiple solutions are still present. It is therefore of interest to investigate whether other numerical algorithms iterating on the Euler equation yield a unique interior solution.

4. Euler-Equation Method

In this section, we investigate the performance of a grid-based Euler-equation projection method under quasi-geometric discounting. We restrict the domain of the capital stock to the interval $[k_{\min}, k_{\max}] = [0.25\bar{k}^*, 4\bar{k}^*]$, where $\bar{k}^*$ is the steady-state value of capital in the model with standard geometric discounting. We consider an equally spaced grid of N points. To evaluate the policy function outside the grid, we use Matlab's cubic polynomial interpolation, which cubically interpolates four points to find the maximum value. To solve the stochastic version of the model, we approximate the autoregressive process for the logarithms of shocks by a Markov chain with seven states, $\Theta \equiv \{0, \pm\frac{5v}{3}, \pm\frac{5v}{2}, \pm 5v\}$, as in Tauchen (1986). For each state $\ln\theta \in \Theta$, we parametrize the next period's capital stock as a function of the current capital stock.

By substituting consumption from the Euler Equation (3) in budget constraint (2), we obtain

$$\tilde{g}(k,\theta) \equiv k' = (1-d)k + \theta k^{\alpha}$$
$$- \left\{ \delta \sum_{\ln \theta' \in \Theta} \left[\frac{\beta(1-d+\theta' \alpha g(k,\theta)^{\alpha-1}) + (1-\beta)(\partial g(g(k,\theta),\theta'))/(\partial g(k,\theta))}{((1-d)g(k,\theta)+\theta' g(k,\theta)^{\alpha} - g(g(k,\theta),\theta'))^{\sigma}} \right] \pi(\theta' \mid \theta) \right\}^{-1/\sigma},$$

where $\pi(\theta' \mid \theta)$ is the probability of θ' conditional on θ.

We implement the following iterative procedure: Fix some policy function on the grid, $g(k,\theta)$, and use it to re-calculate $\tilde{g}(k,\theta)$ in each point of the grid. Compute the policy function for the next iteration by using updating, $\eta\tilde{g}(k,\theta) + (1-\eta)g(k,\theta)$, where $\eta \in (0,1]$. Iterate until $\tilde{g}(k,\theta) = g(k,\theta)$ with a given precision.

For all numerical experiments, we assume the constant relative risk aversion (CRRA) utility function, $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, where $\sigma > 0$, and the Cobb–Douglas production function, $f(k) = k^{\alpha}$, and we fix $\delta = 0.95$, $\alpha = 0.36$. In the stochastic case, we parameterize the process for shock by $\rho = 0.95$ and $v = 0.01$. If $\sigma = 1$ and $d = 1$, we obtain the model with the closed-form solution.

We begin by presenting the numerical results obtained for the model with the closed-form solution, and we then discuss the results obtained for more general variants of the model and compare them to those presented in Krusell et al. (2002).

4.1. NUMERICAL RESULTS UNDER THE CLOSED-FORM SOLUTION

We find that whether the algorithm converges to the closed-form solution or not depends on specific values of the model's and the algorithms' parameters, such as the number of grid points for capital, N, and the value of β. We shall also mention that in order to ensure convergence, the policy function should be updated much more slowly than in the usual geometric discounting case, e.g., $\eta = 0.01$.

In the deterministic case, for example, if $N = 100$, the algorithm converges to the closed-form solution under $\beta \in [0.4, 1.6]$. When the grid is refined, the range of values of β leading to convergence narrows down: if $N = 300$, the algorithm converges under $\beta \in [0.8, 1.2]$; if $N = 1000$, the convergence range is $\beta \in [0.95, 1.05]$ and, finally, if $N = 10,000$, the algorithm diverges even under $\beta \in [0.99, 1.01]$.[2] In the first panel of Table I, we compare the exact and approximate solutions for the steady-state value of the capital stock under $N = 100$ and $\beta \in \{0.8, 0.9, 1.0, 1.1, 1.2\}$. We observe that the algorithm delivers a relatively high degree of precision, even when the number of nodes is not very large.

In the stochastic case, the performance of the grid algorithm is similar. The range of β, under which the algorithm converges to the closed-form solution for each particular N, is however somewhat larger. For example, if $N = 100$, the convergence is achieved under $\beta \in [0.3, 2.0]$.

Table I. The steady-state value of capital in the deterministic model.

β	0.8	0.9	1	1.1	1.2
The model with the closed-form solution ($d = 1, \sigma = 1$)					
Exact solution	.147426	.167507	.187032	.205955	.224254
approximation	.147405	.167492	.187025	.205955	.224254
The model with $d = 0.1$					
$\sigma = 0.5$	1.9986	2.8734 (2.87)[a]	3.8219	4.8013	5.7838
$\sigma = 1$	2.3900	3.0902 (3.09)[a]	3.8219	4.5690	5.3205
$\sigma = 2$	2.6960	3.2536 (3.25)[a]	3.8219	4.3943	4.9667
$\sigma = 3$	2.8373	3.3282 (3.33)[a]	3.8219	4.3149	4.8049
$\sigma = 4$	2.9226	3.3729 (3.37)[a]	3.8219	4.2672	4.7076
$\sigma = 5$	2.9810	3.4035 (3.40)[a]	3.8219	4.2345	4.6411
$\sigma = 6$	3.0240	3.4260 (3.43)[a]	3.8219	4.2106	4.5922
$\sigma = 7$	3.0574	3.4435 (3.44)[a]	3.8219	4.1919	4.5543

Note. Parameter values: $\alpha = 0.36$, $\delta = 0.95$, $N = 100$.
[a]The numbers in parenthesis correspond to the solution reported by Krusell et al. (2002).

The results in Krusell and Smith (2003) allow us to gain intuition on why the model's and the algorithms' parameters can affect the convergence in the quasi-geometric discounting case. This paper specifically shows that the multiple discontinuous solutions described in Krusell and Smith (2000) satisfy a difference (non-differentiable) analogue of the Euler Equation (3). The consequence is that within the multiplicity interval, the numerical methods fail to distinguish the true (closed-form) solution to the Euler equation from a bunch of nearby discontinuous "pseudo solutions".

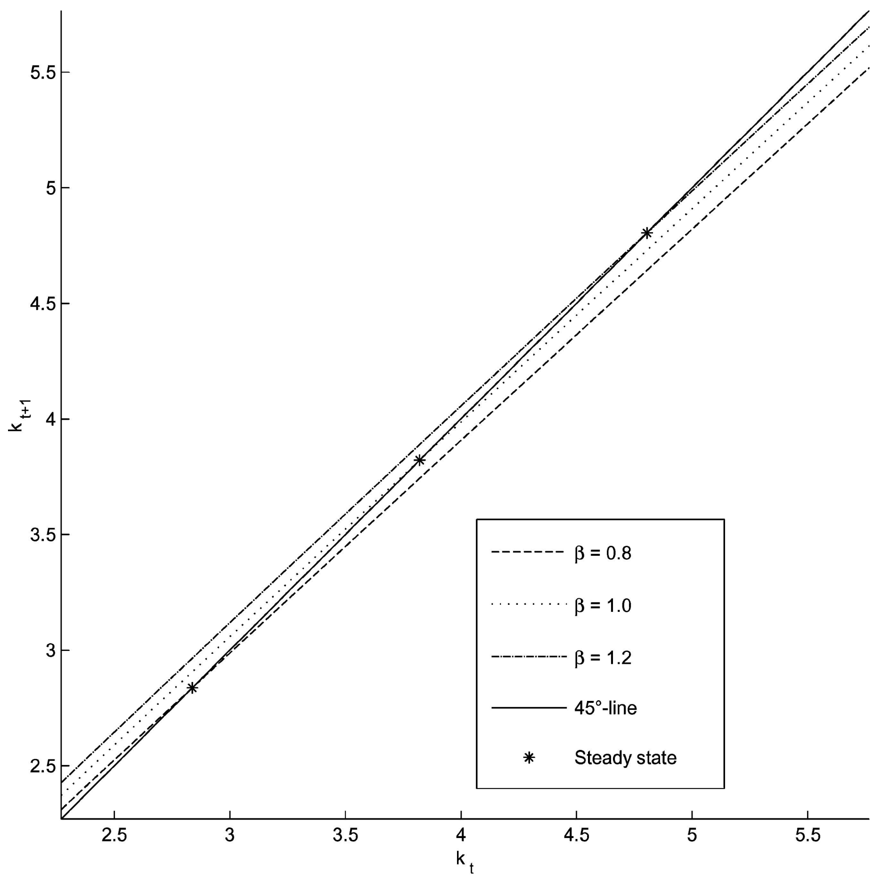
Why does the number of grid points affect the convergence? Given our choice of cubic interpolation, decreasing the number of grid points imposes more structure on the solution candidate. That is, solving the functional Equation (3) on a coarse grid is close to solving this equation subject to an additional constraint that the decision function $g(\cdot)$ is close to a smooth cubic function. As the number of grid points increases, the solution candidate $g(\cdot)$ is allowed to have more curvature parameters and diverges from a cubic function. This may reinstate the multiplicity inherent to the functional Equation (3), leading to divergence.

The role of the value of β in the convergence is as follows. If β is not very different from one, then the multiplicity interval is relatively small, and there are few nodes in this interval. In such a case, our smooth cubic interpolating function converges to a closed-form solution. When the value of β deviates significantly from one, the multiplicity interval increases, and so does the number of nodes in this interval. When the number of nodes lying in the multiplicity interval becomes large, the algorithm fails to converge.[3]

4.2. NUMERICAL RESULTS FOR A GENERAL CASE

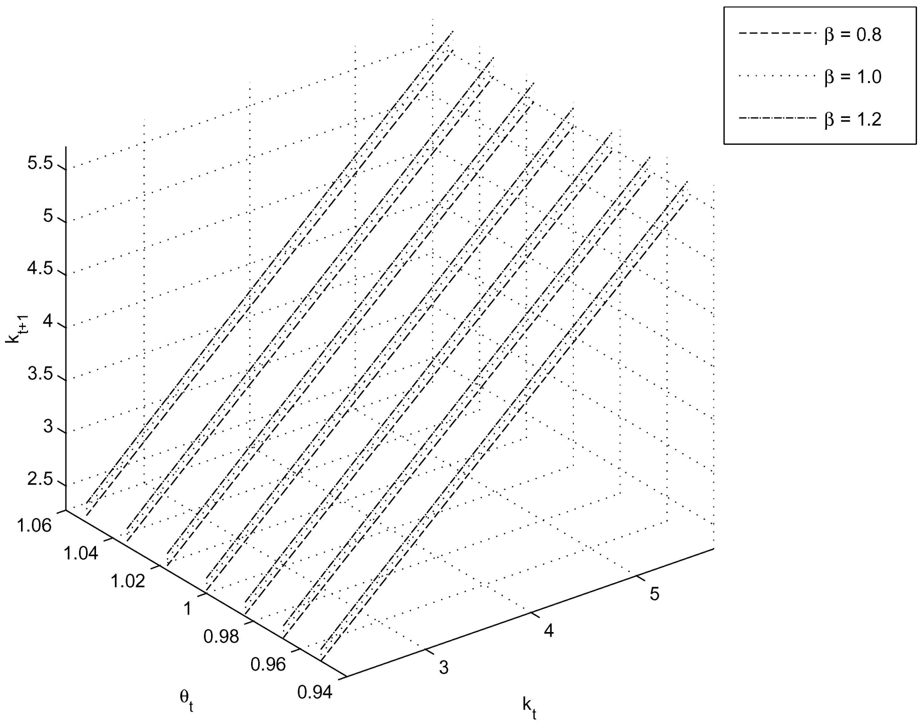
The convergence properties of our computational method in the general model with $\sigma \neq 1$ and $d \neq 1$ proved to be very similar to those in the model with the closed-form solution. Specifically, our grid algorithm converges to a unique interior solution provided that the value of β is not very different from one and that the policy function is updated slowly enough. To achieve the convergence under the grid algorithm, we should use the grid which is not very fine.

Krusell et al. (2002) compute the solution to the deterministic version of the model under $\beta = 0.9$, $d = 0.1$, and $\sigma \in \{0.5, 1, 2, 3, 4, 5, 6, 7\}$. We consider the same values of the parameters d and σ, and explore several values of β, namely, $\beta \in \{0.8, 0.9, 1.0, 1.1, 1.2\}$. In Table I, we report the steady-state values of the capital stock computed by our grid algorithm for the deterministic model. For the



[a]Parameter values: α=0.36, d=0.1, δ=0.95, σ=3.

Figure 1. The grid algorith: the policy function in the deterministic model.



[a]Parameter values: α=0.36, d=0.1, δ=0.95, σ=3, ρ=0.95, v=0.01.

Figure 2. The grid algorith: the policy function in the stochastic model.

sake of comparison, we also provide the results obtained by Krusell et al. (2002). The main thing to be noted here is that our solutions are identical to those computed by the perturbation method in Krusell et al. (2002). Regarding the properties of the solutions, we can observe the following tendencies: The steady-state value of capital increases (decreases) with σ for a given value of β when $\beta < 1$ ($\beta > 1$), and it increases with β for a given value of σ. The latter tendency is illustrated in Figure 1, where we plot the computed decision rules and the corresponding steady states under $\beta \in \{0.8, 1.0, 1.2\}$ and $\sigma = 3$.

We finally investigate the properties of the solutions to the stochastic version of the neoclassical growth model with $\sigma \neq 1$ and $d \neq 1$. (This model was not studied in the literature yet.) We report the results under $\beta \in \{0.8, 1.0, 1.2\}$ and $\sigma = 3$. In Figure 2, we plot the policy functions computed by the grid algorithm. The noteworthy finding in the figures is that the solutions under all three values of β are very similar. The main difference is that an agent with $\beta > 1$ ($\beta < 1$) holds more (less) capital than the one with $\beta = 1$, i.e., the short-run patient (impatient) agent tends to over-save (under-save) relative to the one with $\beta = 1$.[4]

5. Conclusion

This paper studies the possibility of using non-linear Euler-equation methods for computing equilibrium in the neoclassical growth model with quasi-geometric discounting. Our method systematically converges to a unique interior solution although its performance is not entirely satisfactory. First, for a model with a closed-form solution, the considered method allows us to find the solution for only a limited range of values of the discounting parameter β, even though the solution exists for any nonnegative value of this parameter. Secondly, we cannot achieve an arbitrary accuracy by refining the grid, because the method fails to converge when the grid becomes too fine. Finally, to enforce convergence, we have to update the decision rules very slowly (much more slowly than in the usual geometric discounting case). It is possible that the above numerical problems are a consequence of multiplicity of equilibrium encountered in Judd (2004). Thus, alternative methods for solving models with quasi-geometric discounting should be developed. Yet, the Euler-equation methods, like one we studied here, can be a simple and useful alternative in many empirical applications, in spite of all their limitations. Indeed, we have been able to find the solution to the model in a wide range of parameter values. This is not only true for the "test" model with the closed-form solution but also for more general settings.

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Notes

[1] Judd (2004) provides an extensive review of numerical methods proposed in the literature for the problems with time-inconsistency.

[2] The fact that the accuracy of approximation can affect the convergence is also observed by Krusell and Smith (2000) for value-iterative methods: "The algorithm may converge if g is approximated with very low accuracy (with few grid points, or with an inflexible functional form)".

[3] This suggests the following modification of the algorithm. Construct the grid so that all nodes are placed outside the multiplicity interval and compute the decision rules in the multiplicity interval by using interpolation. We find that this method performs very well if β is not very different from one, e.g., $\beta \in [0.4, 1.6]$, however it also fails when β differs from one significantly, and the multiplicity interval is very large.

[4] We also find that the simulation-based parameterized expectation algorithm by den-Hann and Marcet (1990) also converges to a unique interior solution in the stochastic version of the model, see Maliar and Maliar (2003).

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