AN ANALYTICAL CONSTRUCTION OF CONSTANTINIDES' SOCIAL UTILITY FUNCTION*

Lilia Maliar and Serguei Maliar**

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Corresponding author: L. Maliar. Departamento de Fundamentos del Análisis Económico, Universidad de Alicante, Campus San Vicente del Raspeig, Ap. Correos 99, 03080 Alicante, Spain. E-mail: maliarl@merlin.fae.ua.es.

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** Universidad de Alicante.

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ABSTRACT

This paper studies the properties of the social utility function defined by the planner's problem of Constantinides (1982). We show one set of restrictions on the optimal planner's policy rule, which is sufficient for constructing the social utility function analytically. For such well-known classes of utility functions as the HARA and the CES, our construction is equivalent to Gorman's (1953) aggregation. However, we can also construct the social utility function analytically in some cases when Gorman's (1953) representative consumer does not exist; in such cases, the social utility function depends on "heterogeneity" parameters. Our results can be used for simplifying the analysis of equilibrium in dynamic heterogeneous-agent models.

Keywords: Aggregation of preferences, Planner's problem, Social utility function, Social welfare, Gorman aggregation

Classification numbers: D11, D58, D60, D70

1 Introduction

In general, a characterization of equilibrium in dynamic heterogeneous-agent models is complicated and relies heavily on numerical methods, see, e.g., the model of Krusell and Smith (1998). In the presence of Gorman's (1953) aggregation, this task is simplified considerably, as, in essence, a heterogeneous-agent model is reduced to the familiar one-consumer setup, see, e.g., Chatterjee (1994), Caselli and Ventura (2000). However, by construction, such models do not provide a framework for analyzing the role of heterogeneity in the aggregate behavior of actual economies, which, in fact, is the issue of greatest interest for the current literature on heterogeneous agents.

There are several papers that show examples of the so-called imperfect aggregation, where the aggregate dynamics depends on distributions but in a manner which is relatively easy to characterize and understand. Here, the aggregate dynamics are still described by a one consumer model but such a model has parameters (shocks) whose values (properties) depend on distributions. This class of models proved to be very convenient for empirical work, see Atkeson and Ogaki (1996), Maliar and Maliar (2001, 2003a, 2003b). In this paper, we therefore attempt to establish general results concerning the possibility of describing the aggregate behavior of heterogeneous-agent economies by one-consumer models without assuming Gorman's representative consumer.

A starting point for our analysis is the result of Negishi (1960) who shows that a competitive equilibrium in a multi–consumer economy can be restored by solving the problem of a social planner whose objective is to maximize a weighted sum of individual utilities subject to the economy's feasibility constraint. Constantinides (1982) reformulates a dynamic version of Negishi's (1960) problem in the form of two sub-problems: First, a social (intra-period) utility function is constructed by solving a one-period multi-consumer model, and then, an aggregate allocation is computed by solving a multi-period one-consumer model. In general, the social utility function defined in Constantinides (1982) is a complicated object, which depends on a distribution of welfare weights in an unknown way. There are certain cases, however, in which the social utility function (i.e., the mapping between the distribution of welfare weights and the social preferences) can be constructed analytically. The well-known case is Gorman's (1953) aggregation, in which the preference ordering on the aggregate commodity space is the same for all distributions of welfare weights. However, there are also examples of the analytical construction of the social utility function in the absence of Gorman's (1953) aggregation, see Atkeson and Ogaki (1996), Maliar and Maliar (2001, 2003a, 2003b), and Ogaki (2003).

We start by generalizing the property that is common to all known examples of analytical construction of the social utility function. This property happens to be a particular kind of the optimal planner's sharing rule: First, the amount of a commodity that each agent gets from the planner depends on the total endowment of this given commodity, but not on the endowments of any other commodities. Secondly, a change in the total endowment of each commodity is distributed among agents in fixed proportions that are determined by the distribution of the welfare weights. We refer to the planner's policy rule that satisfies the above two properties as a linear sharing rule.

We show that under the assumption of the linear sharing rule, the social utility function is additive in some partition of commodities, with each sub-function being composed of two multiplicatively separable terms, one depending on the aggregate commodity endowment and another depending on the distribution of the welfare weights. The terms that depend on welfare weights capture all of the effects that the distribution of the welfare weights (wealth) have on the social preference relation. The actual number of such "heterogeneity" parameters does not exceed the number of additive sub-functions in the social utility function. If either the social utility function consists of just one additive component, or if the values of all the heterogeneity parameters are equal, we have Gorman's (1953) representative consumer. Otherwise, the preference relation on the commodity space is not invariant to redistributions of the welfare weights, and Gorman's (1953) representative consumer does not exist.

We illustrate the construction of social utility functions under the planner's linear sharing rule for three classes of utility functions. Our first two classes are the Hyperbolic Absolute Risk Aversion (HARA) and the Constant Elasticity of Substitution (CES); here, the agents' preferences are similarly quasi-homothetic and our construction is equivalent to Gorman's (1953) aggregation. Our third class is defined by assuming that the individual utility functions are given by identical-for-all-agents (up to possibly different translated origins) linear combinations of distinct members from the HARA and the CES classes. The considered preferences are not similarly quasi-homothetic and hence, they are not consistent with Gorman's (1953) aggregation. Still, the planner's sharing rule is linear, and the social utility function can be constructed analytically.

The properties of social preferences have been well investigated in the literature on aggregation. Typically, aggregation is achieved by imposing sufficient restrictions on the distribution of the agents' characteristics, e.g., on the Engel curves (Gorman, 1953, Pollack, 1971), on the distribution of preferences (Freixas and Mas-Colell, 1987, Grandmont, 1992), on the cost function (Muellbauer, 1976), on distribution of wealth (Eisenberg, 1961, Chipman, 1974, Chipman and Moore, 1979, Shafer, 1977), and on the shape of wealth distribution (Hildenbrand 1983).[1] We differ from this literature in two respects: First, we impose restrictions on a different object, namely, on the planner's policy rule. Secondly, we do not require the existence of a representative consumer, (i.e., that the social utility function is independent of distributions), but rather, try to establish cases in which the mapping between the distribution of the welfare weights and the social preferences is simple enough to be characterized analytically. Thus, our results are somewhere in the middle between Gorman's (1953) aggregation when the social preferences are independent of distributions and Constantinides' (1982) implicit construction of the social utility function when the relationship between the social preferences and the distribution of welfare weights is unknown.

The paper is organized as follows: Section 2 describes the market and the planner's economies and defines the social utility function. Section 3 introduces restrictions on the planner's policy rule and characterizes the corresponding social preference relation. Section 4 presents three classes of the individual utility functions for which the social utility function can be constructed analytically, and finally, Section 5 concludes.

2 The market and the planner's economies

Time is discrete and is indexed by $t = 0, ..., T$, where $T \geq 1$ can be either finite or infinite. We consider a market economy populated by a set of agents I. The variables of agent $i \in I$ will be denoted by superscript i. The total measure of agents is normalized to one, $\int_{i \in I} di = 1$. Agents are heterogeneous in two dimensions: preferences and income endowments.

[1] The reverse approach is also pursued in the literature. It involves imposing some restrictions at the aggregate level and studying their implications for the underlying multi-consumer economies. Sonnenschein (1974), Mantel (1974) and Geanakoplos and Polemarchakis (1980), for example, impose restrictions on the market excess demand functions; Blackorby and Schworm (1993) impose restrictions on the social preferences.

The income endowment of agent i in period t is denoted by y_t^i. The distribution of income endowments in period t is $Y_t \equiv \{y_t^i\}^{i\in I} \in \Im \subseteq R_{++}^I$. We assume that Y_t follows a stationary first-order Markov process. To be precise, let $\Re$ be the Borel σ-algebra on $\Im$, and let us define a transition function for the distribution of income endowments $\Pi : \Im \times \Re \to [0, 1]$ on the measurable space $(\Im, \Re)$ as follows: for each $z \in \Im$, $\Pi(z, \cdot)$ is a probability measure on $(\Im, \Re)$, and for each $Z \in \Re$, $\Pi(\cdot, Z)$ is a $\Re$-measurable function. The function $\Pi(z, Z)$ yields the probability that the next period's distribution of income endowments lies in the set Z given that the current distribution of income endowments is z, i.e., $\Pi(z, Z) = \Pr\{Y_{t+1} \in Z \mid Y_t = z\}$. The initial distribution of income endowments $Y_0 \in \Im$ is given. Markets are complete, so that agents can trade Arrow securities, contingent on all possible realizations of the distribution of income endowments.

The preferences of agent i are represented by $E_0 \sum_{t=0}^{T} \delta^t U^i(X_t^i)$, where E_0 is the operator of conditional expectation and $\delta \in (0, 1)$ is the discount factor. The momentary utility function, $U^i : H \subseteq R_{++}^K \to R$, is twice continuously differentiable, strictly increasing and strictly concave for all commodity vectors $X_t^i \in H$. The agent i solves the following utility maximization problem:

$$\max_{\left\{X_t^i, \left\{m_{t+1}^i(Z)\right\}_{Z\in\Re}\right\}_{t\in T}} E_0 \sum_{t=0}^{T} \delta^t U^i\left(X_t^i\right) \tag{1}$$

$$\text{s.t.} \quad P_t X_t^i + \int_{Z\in\Re} q_t(Z)\, m_{t+1}^i(Z)\, dZ = y_t^i + m_t^i(Y_t), \tag{2}$$

where $P_t \in R_{++}^K$ is the price vector in period t. The portfolio of Arrow securities bought by the agent in period t is $\left\{m_{t+1}^i(Z)\right\}_{Z\in\Re}$. The price of security $q_t(Z)$ is to be paid in period t for the delivery of one unit of the consumption good in period $t+1$ if $Y_{t+1} \in Z$. Initial income endowment, y_0^i, and initial holdings of Arrow securities, $m_0^i(Y_0)$, are given.

The above model is a partial equilibrium setup where uncertainty arises because income endowments of agents are stochastic. This is the simplest possible setup which contains all relevant features for our analysis: intertemporal choice, heterogeneity, uncertainty and complete markets. However, our aggregation results are not limited to the above setup and will hold in many other models, which are more interesting from the economic point of view. In particular, we can easily extend our benchmark setup to include capital accumulation. Furthermore, we can endogenize prices by introducing production: in the general-equilibrium case, firms maximize profits and pay to the agents the interest rate and wages in exchange for their capital and labor. Finally, we can consider other sources of uncertainty, for example, aggregate uncertainty in the form of shocks to production technology as in Maliar and Maliar (2001) or idiosyncratic uncertainty in the form of shocks to individual labor productivities and discount factors as in Maliar and Maliar (2003a).

Negishi (1960) demonstrates that a competitive equilibrium allocation in a deterministic one-period market economy can be restored by solving the problem of a social planner who maximizes the weighted sum of the individual utilities subject to the economy's feasibility constraint. Under the assumption of complete markets, this result also holds for dynamic stochastic economies like ours. Indeed, the First Order Condition (FOC) of the agent's utility maximization problem (1), (2) with respect to Arrow securities is

$$\lambda_t^i q_t(Z) = \delta \lambda_{t+1}^i \Pr\{Y_{t+1} \in Z \mid Y_t = z\}, \tag{3}$$

where λ_t^i is the Lagrange multiplier associated with the budget constraint (2). Note that equation (3) implies that for any two agents $i', i'' \in I$, we have

$$\frac{\lambda_t^{i'}}{\lambda_t^{i''}} = \frac{\lambda_{t+1}^{i'}}{\lambda_{t+1}^{i''}} \text{ for all } Z \in \Re \qquad \Rightarrow \qquad \lambda_t^i = \lambda_t / \lambda^i. \tag{4}$$

That is, we can rewrite each agent's Lagrange multiplier as a ratio of a common-for-all-agents time-dependent variable λ_t and an agent-specific time-invariant parameter λ^i. This result is the standard consequence of the complete markets assumption that the ratio of marginal utilities of any two agents remains constant in all periods and states of nature. Let us assume that the planner weighs the utility of each agent i by λ^i and solve the following problem:

$$\max_{\{X_t^i\}_{t=0}^T} \left\{ E_0 \sum_{t=0}^T \delta^t \left[\int_{i \in I} \lambda^i U^i\left(X_t^i\right) di \right] \mid \int_{i \in I} X_t^i di = X_t,\ P_t X_t = y_t \right\}, \tag{5}$$

where $y_t \equiv \int_{i \in I} y_i^i di$. The constraint $P_t X_t = y_t$ follows by aggregating (2) across agents and by imposing market clearing conditions for Arrow securities $\int_{i \in I} m_t^i(Y_t) di = 0$ and $\int_{i \in I} m_{t+1}^i(Z) di = 0$ for all $Z \in \Re$. It is easy to check that the FOCs describing the equilibrium allocation in the market economy (1), (2) and those describing the optimal allocation in the planner's economy (5) are equivalent.

Constantinides (1982) shows that the social planner's problem (5) has an equivalent representation in the form of two sub-problems. The first one is to distribute the economy's endowment of commodities X among agents to maximize the weighted sum of the individual utilities. This sub-problem defines the (momentary) social utility function:

$$V(X,\lambda) \equiv \max_{\{X^i\}^{i\in I}} \left\{ \int_{i\in I} \lambda^i U^i\left(X^i\right) di \;\;\middle|\;\; \int_{i\in I} X^i di = X \right\}, \tag{6}$$

where $\lambda \equiv \left\{\lambda^i\right\}^{i\in I}$. We assume that the solution to the problem (6), $X^i : H \times \Lambda \to R_+^K$ for all $i \in I$, is unique interior and continuously differentiable in the region $H \times \Lambda$.

The second sub-problem is to compute the aggregate optimal allocation given the social utility function:

$$\max_{\{X_t\}_{t=0}^T} \left\{ E_0 \sum_{t=0}^{T} \delta^t V(X_t,\lambda) \;\;\middle|\;\; P_t X_t = y_t \right\}. \tag{7}$$

We must draw attention to an important difference between the notion of the social utility function of Constantinides (1982) and the one of Bergson (1938) and Samuelson (1947) that is standard in the social choice literature.[2] In the Bergson-Samuelson case, the planner owns all commodities and distributes them across agents to maximize social welfare; the planner's choice defines the socially-optimal distribution of wealth across agents. In the Constantinides case, the planner's solution should replicate a competitive equilibrium in the underlying market economy (1), (2) and hence, should be consistent with a given wealth distribution or equivalently, with a given set of welfare weights; the distribution of wealth / welfare weights need not be socially optimal. Consequently, the planner in the sense of Constantinides (1982) faces an additional set of restrictions (i.e., a fixed wealth / welfare weights distribution) compared to the planner in the sense of Bergson-Samuelson.

Constantinides (1982) argues that one can interpret $V(X,\lambda)$ with a fixed set of the welfare weights as the utility function of a representative consumer. Indeed, under our assumptions, the function $V(X,\lambda)$ is single-valued and

[2] See Mas-Colell et al. (1995) for a formal definition of the Bergson-Samuelson welfare function. The literature studying the Bergson-Samuelson planner's problem includes, e.g., Eisenberg (1961), Chipman (1974), Chipman and Moore (1979).

twice continuously differentiable on $H \times \Lambda$. Moreover, for any fixed $\lambda \in \Lambda$ and all $X \in H$, $V(X, \lambda)$ is strictly increasing and strictly concave. Therefore, for a fixed $\lambda \in \Lambda$, the function $V(X, \lambda)$ induces a binary (transitive and convex) preference relation on the aggregate commodity space H. This aggregation concept is often referred to in the literature as "aggregation in equilibrium point" because the constructed composite consumer represents the economy only for just one fixed set of welfare weights.

The construction of the planner's problem in Constantinides (1982) has an advantage over the one in Negishi (1960), since it allows us to explicitly separate the intra-temporal and the inter-temporal aspects of the planner's choice. In other words, instead of solving the original multi-consumer multi-period problem (1), (2), we can first construct the social utility function by solving a multi-consumer but one-period problem (6), and then compute the aggregate quantities from a multi-period but one-consumer problem (7). This result is particularly useful for empirical applications if the social utility function can be constructed analytically. The well-known case is Gorman's (1953) aggregation, where the preference relation induced by $V(X, \lambda)$ on H is the same for all sets of weights (see Blackorby and Schworm, 1993, for a detailed discussion). However, there are also examples of economies that are not consistent with aggregation in the sense of Gorman (1953), but for which the social utility function can be constructed analytically, see Atkeson and Ogaki (1996), Maliar and Maliar (2001, 2003a, 2003b), and Ogaki (2003). Our subsequent objective, therefore, is to distinguish the property that is common to all of the above examples and to provide general results concerning the possibility of constructing the social utility function analytically.

3 Constructing the social utility function

Let us first illustrate the construction of the social utility function, $V(X, \lambda)$, on the example of Atkeson and Ogaki (1996) where the agents' momentary utility functions are of the addilog type.[3]

Example 1 Assume $U^i(X^i) = \sum_{k=1}^{K} (x_k^i - b_k)^{c_k}$, where $0 < c_k < 1$ and

[3]The fact that the addilog class of utility functions satisfy the weak axiom of revealed preferences was pointed out in Shafer (1977).

$b_k < x_k^i$ for $k = 1, ..., K$. The FOC of (6) with respect to x_k^i is

$$\lambda^i \left(x_k^i - b_k\right)^{c_k - 1} = \mu_k, \tag{8}$$

where μ_k is the Lagrange multiplier associated with the constraint on the kth commodity, $\int_{i \in I} x_k^i di = x_k$. By expressing x_k^i from (8) and integrating across agents, we obtain that the individual and the aggregate quantities are related by

$$x_k^i = b_k + \frac{\left(\lambda^i\right)^{1/(1-c_k)}}{\int_{i \in I} \left(\lambda^i\right)^{1/(1-c_k)} di} \left(x_k - b_k\right). \tag{9}$$

Substituting x_k^i for $k = 1, ..., K$ in the objective function in (6) yields the social utility function

$$V\left(X, \lambda\right) = \sum_{k=1}^{K} \xi_k\left(\lambda\right) \left(x_k - b_k\right)^{c_k}, \qquad \xi_k\left(\lambda\right) = \left(\int_{i \in I} \left(\lambda^i\right)^{1/(1-c_k)} di\right)^{1-c_k}. \tag{10}$$

Note that if $c_k = c$ for all $k = 1, ..., K$, the individual preferences are identical quasi-homothetic. In this case, we have $\xi_k\left(\lambda\right) = \xi\left(\lambda\right)$ and $V\left(X, \lambda\right) = \xi\left(\lambda\right) \sum_{k=1}^{K} \left(x_k - b_k\right)^c$, i.e., the social utility function is identical to the individual utility functions (up to a multiplicative constant $\xi\left(\lambda\right)$), which is the case of Gorman's (1953) aggregation. However, if c_k's differ across commodities, the individual preferences are not quasi-homothetic, and Gorman's (1953) representative consumer does not exist. Still, we have an analytical expression for the social utility function although the parameters of such a function, $\xi_k\left(\lambda\right)$, depend on a specific distribution of welfare weights. ■

The introspection of all known examples of the analytical construction of the social utility function in Atkeson and Ogaki (1996), Maliar and Maliar (2001, 2003a, 2003b), and Ogaki (2003) reveals that they all have the planner's sharing rule of the type (9), one which is linear in aggregate commodities (for a fixed set of welfare weights). We therefore proceed in two steps: We first postulate a general form of the planner's linear sharing rule and we then construct the corresponding social utility function.[4]

[4]In fact, a sharing rule $X^i\left(X, \lambda\right)$ in our planner's problem is an analogue of the individual demand functions in a market economy. Our approach is therefore similar to that of Gorman (1953), which imposes a restriction on the individual demand functions by assuming linear Engel curves, i.e., $X^i\left(P, y^i\right) = \alpha^i\left(P\right) + \beta\left(P\right) y^i$, where $\alpha^i\left(P\right)$ and $\beta\left(P\right)$ are the agent-specific and the common-for-all-agents functions of prices, respectively, and then identifies the preference classes that are consistent with such demand functions.

Definition *The linear sharing rule is a planner's policy rule such that the optimal allocation of each consumer $i \in I$ is given by*

$$X^i(X,\lambda) = \Omega^i(\lambda) + \Phi^i(\lambda) X, \tag{11}$$

for all $X \in H$, $\lambda \in \Lambda$, where $\Omega^i(\lambda)$ and $\Phi^i(\lambda)$ are defined as

$$\Omega^i(\lambda) \equiv \begin{bmatrix} \omega_1^i(\lambda) \\ ... \\ \omega_K^i(\lambda) \end{bmatrix}, \qquad \Phi^i(\lambda) \equiv \begin{bmatrix} \phi_1^i(\lambda) & ... & 0 \\ ... & ... & ... \\ 0 & ... & \phi_K^i(\lambda) \end{bmatrix},$$

with $\int_{i\in I} \omega_k^i(\lambda)\, di = 0$ and $\int_{i\in I} \phi_k^i(\lambda)\, di = 1$ for $k = 1, ..., K$.

The linear sharing rule (11) has two characteristic features. First, the planner's optimal policy for distributing a kth commodity across the population does not depend on the economy's endowment of the other commodities. Secondly, an increase in the endowment of a kth commodity is always distributed among agents in fixed proportions, $\left\{\phi_k^i(\lambda)\right\}^{i\in I}$. The linear sharing rule (11) is sufficiently general: in particular, in Section 4, we will show that such well-known classes of utility functions as the HARA and the CES lead to linear sharing rules.[5]

For the purpose of our analysis, we shall express the individual preferences in the (strongly) additive form. Let us consider a partition of the commodity vector X^i into N sub-vectors $X^i = \left\{(X^i)^{[1]}, ..., (X^i)^{[n]}, ..., (X^i)^{[N]}\right\}$ such that the utility function of each individual can be represented as a direct sum of N sub-functions in the same partition, thus:

$$U^i\left(X^i\right) = \sum_{n=1}^{N} \left(U^i\right)^{[n]} \left(\left(X^i\right)^{[n]}\right), \tag{12}$$

where each $(X^i)^{[n]}$ is a vector of the dimension $\#n \geq 1$. The partition we consider is maximal in the sense that no partition with more than N additive components exists. We shall also notice that the above representation does not impose the additivity restriction on the individual preferences but merely

[5] Restrictions on sharing rules have been used previously in the context of the (surplus-) cost-sharing problem, see, e.g., a survey on cooperative decision making theory in Moulin (1988).

makes the additive structure of the individual preferences explicit if such a structure is in fact present.

The following theorem states the main result of the paper.

Theorem 1 *Assume that $X^i(X,\lambda)$ is a linear sharing rule given by* (11). *The social preferences corresponding to* (12) *can therefore be represented by:*

$$V(X,\lambda) = \xi^{[0]}(\lambda) + \sum_{n=1}^{N} \xi^{[n]}(\lambda) W^{[n]}\left(X^{[n]}\right), \tag{13}$$

where $W^{[n]} : R_{++}^{\#n} \to R$, $\xi^{[0]} : \Lambda \to R$ and $\xi^{[n]} : \Lambda \to R_{++}$, $n = 1, ..., N$.

Proof. See Appendix. ■

The constructed social utility function is additive in the same partition as the individual utility functions, and it can depend on $N+1$ "heterogeneity" parameters, $\xi^{[n]}(\lambda)$, $n = 0, ..., N$. The values of the heterogeneity parameters are determined by a specific distribution of the welfare weights. Since an increasing linear transformation of preferences has no effect on the optimal allocation, the social preferences can be equivalently represented by $\frac{V(X,\lambda)-\xi^{[0]}(\lambda)}{\xi^{[1]}(\lambda)}$. Under the latter representation, the number of the heterogeneity parameters is reduced from $N+1$ to $N-1$.

We finally discuss the implications of our results for the existence of Gorman's (1953) representative consumer, which is the case when the preference relation induced by $V(X,\lambda)$ on H is the same one for all sets of welfare weights.

Corollary 1 *Gorman's (1953) aggregation.*
Assume (13) *and let $W(X) \equiv \sum_{n=1}^{N} W^{[n]}\left(X^{[n]}\right)$.*
If $N = 1$, then $V(X,\lambda) \sim W(X)$.[6]
If $N > 1$, then $V(X,\lambda) \sim W(X) \Longleftrightarrow \xi^{[n]}(\lambda) \equiv \xi(\lambda)$ for all $n = 1, ..., N$.

Indeed, if the individual utility functions have only one additive component, $N = 1$, we therefore obtain $V(X,\lambda) = \xi^{[0]}(\lambda) + \xi^{[1]}(\lambda) W(X) \sim W(X)$, i.e., the linearity of the sharing rule is sufficient for Gorman's (1953)

[6]Notation " $\sim$ " and " $\nsim$ " mean "identical" and "not identical", respectively, up to an increasing linear transformation.

aggregation. However, if there is more than one additive component, $N > 1$, then the linear sharing rule does not necessarily imply the existence of Gorman's (1953) representative consumer, since there is the possibility that not all of the heterogeneity parameters are equal and thus, $V(X, \lambda) \nsim W(X)$.

4 Three classes of utility functions

In this section, we illustrate the construction of the social utility function for three different classes of utility functions that lead to linear sharing rules of type (11). The first two classes are the HARA and the CES; here, the agents' preferences are similarly quasi-homothetic, and we have Gorman's (1953) aggregation. Our third class is composed of identical-for-all-agents (up to possibly different translated origins) linear combinations of HARA and CES members. Such preferences are not similarly quasi-homothetic, and Gorman's (1953) representative consumer does not exist. Still, the planner's sharing rule is linear, so that the social utility function takes the form of (13) and can be constructed analytically.

4.1 The HARA Class

Pollack (1971) shows that all additive utility functions leading to linear Engel curves are members of the generalized Bergson family, which is also referred to in the literature as the HARA class:

$$U^{i(\wedge)}\left(X^i\right) = \sum_{n=1}^{N} a_n \left(\gamma_n \left(x_n^i - b_n^i\right)\right)^c, \quad \begin{array}{llll} \gamma_n = 1, & c < 0, & a_n < 0, & x_n^i > b_n^i; \\ \gamma_n = 1, & 0 < c < 1, & a_n > 0, & x_n^i > b_n^i; \\ \gamma_n = -1, & c > 1, & a_n < 0, & x_n^i < b_n^i; \end{array}$$

$$U^{i(\ln)}\left(X^i\right) = \sum_{n=1}^{N} a_n \ln\left(x_n^i - b_n^i\right), \qquad a_n > 0, \quad x_n^i > b_n^i;$$

$$U^{i(\exp)}\left(X^i\right) = \sum_{n=1}^{N} b_n^i \exp\left(a_n x_n^i\right), \qquad a_n < 0, \quad b_n^i < 0. \tag{14}$$

Let us analyze the implications of the utility functions from the HARA class for the planner's economy. Suppose that the agents have preferences given by either the power utility functions, $U^{i(\wedge)}\left(X^i\right)$, the logarithmic utility functions, $U^{i(\ln)}\left(X^i\right)$, or the exponential utility functions, $U^{i(\exp)}\left(X^i\right)$. By expressing the individual optimal allocation, x_n^i, from the first-order conditions, by computing $x_n = \int_{i\in I} x_n^i di$ and combining the formulas for x_n^i and x_n to eliminate the Lagrange multiplier, for $n = 1, ..., N$, we obtain

$$\begin{aligned}
\left(x_n^i\right)^{(\wedge)} &= b_n^i + \frac{\left(\lambda^i\right)^{1/(1-c)}}{\int_{i\in I}\left(\lambda^i\right)^{1/(1-c)} di}\left(x_n - b_n\right), & b_n &\equiv \int_{i\in I} b_n^i di; \\
\left(x_n^i\right)^{(\ln)} &= b_n^i + \frac{\lambda^i}{\int_{i\in I}\lambda^i di}\left(x_n - b_n\right), & b_n &\equiv \int_{i\in I} b_n^i di; \\
\left(x_n^i\right)^{(\exp)} &= \frac{\ln\left(b_n/b_n^i\right) - \ln\left(\lambda^i\right) + \int_{i\in I}\ln\left(\lambda^i\right) di}{a_n} + x_n, & \ln\left(b_n\right) &\equiv \int_{i\in I} \ln\left(b_n^i\right) di.
\end{aligned} \tag{15}$$

Given that the planner's sharing rule is linear, by Theorem 1, we have that the social utility function takes the form (13). Substituting X^i into (6) yields

$$\begin{aligned}
V^{(\wedge)}(X,\lambda) &= \xi^{(\wedge)} \sum_{n=1}^{N} a_n \left(\gamma_n\left(x_n - b_n\right)\right)^c, & \xi^{(\wedge)} &= \left(\int_{i\in I}\left(\lambda^i\right)^{1/(1-c)} di\right)^{1-c}; \\
V^{(\ln)}(X,\lambda) &= \xi^{(\ln)} + \sum_{n=1}^{N} a_n \ln\left(x_n - b_n\right), & \xi^{(\ln)} &= \int_{i\in I}\lambda^i \ln\left(\frac{\lambda^i}{\int_{i\in I}\lambda^i di}\right) di; \\
V^{(\exp)}(X,\lambda) &= \xi^{(\exp)} \sum_{n=1}^{N} b_n \exp\left(a_n x_n\right), & \xi^{(\exp)} &= \exp\left(\int_{i\in I}\ln\lambda^i di\right).
\end{aligned} \tag{16}$$

In none of the above cases does the heterogeneity parameter influence the social preference relationship induced by $V(X,\lambda)$ on the aggregate commodity space H. Hence, we have Gorman's (1953) representative consumer.

4.2 The CES Class

In a one-period market economy, an increasing non-linear transformation of the individual utility function does not affect the solution, i.e., the maximization of $U^i(X^i)$ leads to the same optimal allocation as does the maximization of $F\left[U^i(X^i)\right]$, where $F : R \to R$ with $F' > 0$. However, such a transformation does affect the individual optimal allocation in the planner's economy since the maximization of $\int_{i\in I}\lambda^i U^i(X^i)\, di$ and $\int_{i\in I}\lambda^i F\left[U^i(X^i)\right] di$ leads to different solutions. As a result, the linearity of the planner's sharing rule does not, in general, survive a non-linear transformation of the individual utility functions, although in certain cases, it does. An example of such a case is discussed below.

Consider a planner's economy, in which all agents possess preferences given by a power transformation of the CES utility function:

$$U^{i(CES)}\left(X^i\right)=\frac{1}{\sigma}\left(\sum_{k=1}^{K}a_k\left(x_k^i-b_k^i\right)^{\rho}\right)^{\sigma/\rho}, \tag{17}$$

where $\rho\leq 1$, $\rho\neq 0$, $\sigma<1$, $\sigma\neq 0$ and $a_k>0$, $\sum_{k=1}^{K}a_k=1$, $x_k^i>b_k^i$, $k=1,...,K$. The limiting case of the transformed CES utility function under $\rho\to 0$ is the Cobb-Douglas utility function, thus:

$$U^{i(CD)}\left(X^i\right)=\frac{1}{\sigma}\left(\prod_{k=1}^{K}\left(x_k^i-b_k^i\right)^{a_k}\right)^{\sigma}. \tag{18}$$

The utility functions (17) and (18) are transformations of members of the HARA class.[7]

In the case of the CES class, by following the same procedure we employed in Example 1, we shall now show that the individual optimal allocations is given by a linear sharing rule:

$$\left(x_k^i\right)^{(CES)}=b_k^i+\frac{\left(\lambda^i\right)^{1/(1-\sigma)}}{\int_{i\in I}\left(\lambda^i\right)^{1/(1-\sigma)}di}\left(x_k-b_k\right),\qquad b_k\equiv\int_{i\in I}b_k^i di, \tag{19}$$

where $k=1,...,K$. Theorem 1 implies that the social utility function is given by (13). After substituting (19) into (6), we obtain:

$$V^{(CES)}\left(X,\lambda\right)=\frac{\xi^{(CES)}}{\sigma}\left(\sum_{k=1}^{K}a_k\left(x_k-b_k\right)^{\rho}\right)^{\sigma/\rho},\quad \xi^{(CES)}\equiv\left(\int_{i\in I}\left(\lambda^i\right)^{1/(1-\sigma)}di\right)^{1-\sigma}. \tag{20}$$

Regarding the Cobb-Douglas case, the results are similar. The individual optimal allocations are also given by (19), i.e., $\left(x_k^i\right)^{(CD)}=\left(x_k^i\right)^{(CES)}$. The

[7] Although the CES and the Cobb-Douglas utility functions under $\sigma=1$ are strictly quasi-concave, they do not satisfy our assumption of strict concavity. As a result, the individual optimal allocations in the planner's economy are either indeterminate or non-interior. In the market economy, the property of strict quasi-concavity is sufficient, however, for unique interior optimal allocations, see Maliar and Maliar (2003b) for a discussion.

social utility function is

$$V^{(CD)}(X,\lambda)=\frac{\xi^{(CD)}}{\sigma}\left(\prod_{k=1}^{K}(x_k-b_k)^{a_k}\right)^{\sigma},\qquad \xi^{(CD)}=\xi^{(CES)},$$

i.e., the formula for $\xi^{(CD)}$ is the same as the one for $\xi^{(CES)}$.

In all of the above cases, we again have Gorman's (1953) aggregation.

4.3 Linear combinations of HARA and CES members

We shall now construct a class of the utility function that is consistent with the linear sharing rule but not with Gorman's (1953) representative consumer. We shall assume that the individual utility functions have the form (12) with $N>1$, where each sub-function $(U^i)^{[n]}\left((X^i)^{[n]}\right)$ is given by a CES- or HARA-class member that is identical for all agents (up to the value of the parameters b_k^i). Therefore, by Theorem 1, we have that the social utility function takes the form (13), where $\left\{\xi^{[n]}(\lambda)\right\}_{n=1}^{N}$ are the heterogeneity parameters from (20) and (15) corresponding to the given CES and HARA members, and that each sub-function $W^{[n]}\left(X^{[n]}\right)$ is identical to $(U^i)^{[n]}\left((X^i)^{[n]}\right)$ (again, up to the value of the parameters b_k). Below, we elaborate another related example.

Example 2 Let the agents have the preferences given by

$$\begin{aligned} U\left(X^i\right) &= a_1\left(x_1^i-b_1^i\right)^{c_1}+b_2^i\exp\left(-a_2x_2^i\right) \\ &\quad+\frac{1}{\sigma}\left(a_3\left(x_3^i-b_3^i\right)^{\rho}+a_4\left(x_4^i-b_4^i\right)^{\rho}\right)^{\sigma/\rho}, \end{aligned} \tag{21}$$

where the parameters satisfy the corresponding restrictions outlined in Sections 4.1 and 4.2. From the individual optimality conditions, we obtain that the individual optimal allocations x_1^i and x_2^i are given, respectively, by the formulas for $(x_n^i)^{(\wedge)}$ and $(x_n^i)^{(\exp)}$ in (15), and x_3^i and x_4^i are given by the formulas for $(x_n^i)^{(CES)}$ in (19). The planner's sharing rule is, therefore, linear. By Theorem 1, we have that the social utility function is of the form (13). By substituting X^i into (6), we obtain

$$\begin{aligned} V(X,\lambda) &= \xi^{(\wedge)}a_1(x_1-b_1)^{c_1}+\xi^{(\exp)}b_2\exp(-a_2x_2) \\ &\quad+\frac{\xi^{(CES)}}{\sigma}\left(a_3(x_3-b_3)^{\rho}+a_4(x_4-b_4)^{\rho}\right)^{\sigma/\rho}, \end{aligned} \tag{22}$$

where $\xi^{(\wedge)}$, $\xi^{(\exp)}$ are given in (16), and $\xi^{(CES)}$ is given in (20). Since the heterogeneity parameters $\xi^{(\wedge)}$, $\xi^{(\exp)}$ and $\xi^{(CES)}$ are given by different functions of the welfare weights, the social preference relation induced by $V(X, \lambda)$ on H will depend on the distribution of the welfare weights assumed. We therefore do not have Gorman's (1953) aggregation. ■

5 Recovering the competitive equilibrium

In this section, we discuss how to recover the competitive equilibrium in the decentralized heterogeneous-agent economy (1), (2) by using the associated planner's problem (6), (7). We argue that aggregation results considerably simplify the task of recovering the competitive equilibrium.

To characterize the relation between the distribution of initial endowments in the decentralized heterogeneous-agent economy and the distribution of welfare weights in the planner's economy, we use the agents' expected life-time budget constraints. To derive such constraints, we proceed as follows. First, we use the FOC (3) to show that $E_{t-1}\left[\delta \frac{\lambda_t^i}{\lambda_{t-1}^i} m_t^i(Y_t)\right] = \int_{Z\in\Re} q_{t-1}(Z)\, m_t^i(Z)\, dZ$. Second, with this result, we re-write the individual budget constraint (2) taken at $t = 0$ as

$$m_0^i(Y_0) = P_0 X_0^i - y_0^i + \int_{Z\in\Re} q_0(Z)\, m_1^i(Z)\, dZ =$$

$$P_0 X_0^i - y_0^i + E_0\left[\delta \frac{\lambda_1^i}{\lambda_0^i}\left(P_1 X_1^i - y_1^i + \int_{Z\in\Re} q_1(Z)\, m_2^i(Z)\, dZ\right)\right]. \quad (23)$$

Finally, by using forward recursion and the law of iterative expectation, and by imposing transversality condition $\lim_{\tau\to T} E_0\left[\delta^\tau \lambda_\tau^i \int_{Z\in\Re} q_\tau(Z)\, m_{\tau+1}^i(Z)\, dZ\right] = 0$, we obtain

$$E_0\left[\sum_{\tau=0}^{T} \delta^\tau \frac{\lambda_\tau}{\lambda_0}\left(P_\tau X_\tau^i - y_\tau^i\right)\right] = m_0^i(Y_0). \quad (24)$$

In formula (24), we replace the the ratio of the individual Lagrange multipliers, $\frac{\lambda_\tau^i}{\lambda_0^i}$, with the aggregate ratio $\frac{\lambda_\tau}{\lambda_0}$ by using the result (4).[8]

[8] A similar expected life-time budget constraint also holds in more sophisticated economic environments. For example, Maliar and Maliar (2001) derive such a constraint in a general-equilibrium neoclassical growth model with capital accumulation and valued leisure.

By computing a solution to the planner's problem under all possible sets of welfare weights, we obtain a set of Pareto optimal allocations. The Pareto optimal allocation that coincides with the competitive equilibrium in the decentralized economy is one that satisfies the expected life-time budget constraint (24) for each agent $i \in I$.

In the absence of aggregation, we can solve for the individual welfare weights by using the following iterative procedure: fix some set of welfare weights $\left\{\lambda^i\right\}_{i\in I}$, find the solution $X_t^i\left(\left\{\lambda^i\right\}_{i\in I}\right)$ to the planner's problem (6), (7) and check whether the obtained solution is consistent with the expected life-time budget constraint (24) for each agent $i \in I$; iterate on welfare weights until the fixed-point values are found.[9] When the number of agents is large, we have many welfare weights to iterate on, so that the above iterative procedure is burdensome.

In contrast, with Gorman's (1953) aggregation, computing the individual welfare weights is trivial. First, we can solve for the aggregate equilibrium allocation from a representative agent model. Then, by using formulas relating the individual and aggregate allocations and the individual expected life-time budget constraint (24), we can derive a closed-form expression for the individual welfare weights and also, a closed-form expression for individual allocation in terms of the aggregate allocations.[10] In particular, by using (15), we can show that individual and aggregate allocations for the HARA class (14) of utility functions are related by

$$\left(x_{nt}^i\right)^{(\wedge)} = b_n^i + \left(x_{nt} - b_n^i\right)^{\frac{\left(w_0^i - E_0 \sum_{\tau=0}^T \sum_{n=1}^N \delta^\tau \frac{\lambda_\tau}{\lambda_0} p_{n\tau} b_n^i\right)}{E_0 \sum_{\tau=0}^T \sum_{\tau=1}^N \delta^\tau \frac{\lambda_\tau}{\lambda_0} p_{n\tau}\left(x_{n\tau} - b_n^i\right)}},$$

$$\left(x_{nt}^i\right)^{(\ln)} = \left(x_{nt}^i\right)^{(\wedge)}, \tag{25}$$

$$\left(x_{nt}^i\right)^{(\exp)} = x_{nt} + \frac{w_0^i - E_0 \sum_{\tau=0}^T \sum_{n=1}^N \delta^\tau \frac{\lambda_\tau}{\lambda_0} p_{n\tau} b_n^i}{E_0 \sum_{\tau=0}^T \sum_{\tau=1}^N \delta^\tau \frac{\lambda_\tau}{\lambda_0} p_{n\tau}},$$

where $w_0^i \equiv m_0^i\left(Y_0\right) + E_0\left[\sum_{\tau=0}^T \delta^\tau \frac{\lambda_\tau}{\lambda_0} y_\tau^i\right]$ is the expected life-time wealth of the agent and $p_{n\tau}$ is price of commodity n in period t. Under the CES and

[9] This kind of iterative procedure is used in den Haan (1997).

[10] This procedure based on the roperty of Gorman's (1953) was employed by Chatterjee (1994) and Caselli and Ventura (2000).

CD utility functions, (17) and (18), the formula relating the individual and aggregate allocations is the same as one for $(x^i_{nt})^{(\wedge)}$ except that subscript n should be formally replaced by k.

Our "imperfect" aggregation based on the linear sharing rule does not allow us to entirely avoid the iterative procedure, however, it can reduce dramatically the number of parameters to iterate on. We shall demonstrate this fact by using the economy in the example 2.

Example 2 (cont.) Equations (15) and (19) allow us to rewrite the expected life-time wealth of agents (24) as follows:

$$E_0 \sum_{t=0}^{T} \delta^\tau \frac{\lambda_\tau}{\lambda_0} \left[\left(b_1^i + \frac{\left(\lambda^i\right)^{1/(1-c_1)} (x_{1\tau} - b_1^i)}{\left(\xi^{(\wedge)}\right)^{1/(1-c_1)}} \right) p_{1\tau} + \left(\frac{1}{a_2} \ln \left(\frac{\xi^{(\exp)} b_2}{\lambda^i b_2^i} \right) + x_{2\tau} \right) p_{2\tau} \right.$$
$$\left. + \left(b_3^i + \frac{\left(\lambda^i\right)^{1/(1-\sigma)} (x_{3\tau} - b_3^i)}{\left(\xi^{(CES)}\right)^{1/(1-\sigma)}} \right) p_{3\tau} + \left(b_4^i + \frac{\left(\lambda^i\right)^{1/(1-\sigma)} (x_{3\tau} - b_4^i)}{\left(\xi^{(CES)}\right)^{1/(1-\sigma)}} \right) p_{4\tau} \right] = w_0^i. \tag{26}$$

Consequently, in order to solve for competitive equilibrium, we use the following iterative algorithm: fix $\xi^{(\wedge)}$, $\xi^{(\exp)}$ and $\xi^{(CES)}$ to some values, substitute the social utility function V given in (22) into the planner's problem (6), (7), solve for the aggregate quantities, restore the weights on individual utilities from (26) and recompute the values of the parameters $\xi^{(\wedge)}$, $\xi^{(\exp)}$ and $\xi^{(CES)}$ according to formulas given in (16) and (20); iterate on the above parameters until fixed-point values are found. Therefore, in this case we have to iterate on three parameters, no matter how many agents we have. Thus, imperfect aggregation reduces the computational cost substantially when solving models with a small number of commodities and a large number of agents.

6 Final comments

In this paper, we propose to look for restrictions on individual characteristics that are weaker than the ones required for Gorman's (1953) aggregation but which are sufficient for characterizing the relationship between heterogeneity and aggregate dynamics in a relatively simple way. We describe one such restriction, the linear planner's sharing rule. This restriction provides a sufficient condition for constructing the social utility function analytically. In our case, the social utility function can depend on a set of "heterogeneity" parameters that summarize the effect of the distribution of welfare weights (wealth) on social preferences.

Obviously, our restriction of a linear planner's sharing rule is not a necessary condition for constructing the social utility function. In fact, the term "necessary condition" does not have a precise meaning in our context. Indeed, there is no underlying fundamental property behind our construction such as there is in Gorman's (1953) requirement that the aggregate allocation be independent of the distribution of wealth. The property we target may be loosely described as "an easy characterization of aggregate behavior". We expect the class of economies, in which aggregate dynamics depend on distributions in a relatively simple way, to be much broader than the Gorman's (1953) economies, where aggregate dynamics is independent of distributions. This direction seems to be merit further exploration.

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7 Appendix

We will use the following notation. F_k and F_{ks} denote the first-order and second-order partial derivatives of a function $F(z_1, ..., z_k, ..., z_K)$ with respect to the kth argument and with respect to the kth and the sth arguments, correspondingly; $\nabla F \equiv \begin{bmatrix} F_1 \\ ... \\ F_K \end{bmatrix}$; $\triangle F \equiv \frac{\partial \log \nabla F}{\partial X} = \begin{bmatrix} \frac{F_{11}}{F_1} & ... & \frac{F_{1K}}{F_1} \\ ... & ... & ... \\ \frac{F_{K1}}{F_K} & ... & \frac{F_{KK}}{F_K} \end{bmatrix}$ and $\triangle^{-1} F \equiv [\triangle F]^{-1}$ is the inverse of a square matrix $\triangle F$.

Proof of Theorem 1.

1. By deriving the first-order condition of (6) with respect to X^i and combining it with the envelope condition for X, we obtain

$$\lambda^i \nabla U^i \left(X^i\right) = \nabla V(X, \lambda). \tag{27}$$

By taking logarithms and computing the total differential of (27), we get

$$\triangle U^i \left(X^i\right) dX^i = \triangle V(X, \lambda) dX - \overline{1}\, d \log \lambda^i + \int_{i \in I} \frac{\partial \log \nabla V(X, \lambda)}{\partial \log \lambda^i} d \log \lambda^i di, \tag{28}$$

where $\overline{1}$ is a $K \times 1$ vector with all elements equal to 1. Under the assumption of strict concavity of the individual utility function, $\triangle U^i(X^i)$ is invertible. This fact together with (28) implies

$$\frac{\partial X^i}{\partial X} \equiv \begin{bmatrix} \partial x_1^i / \partial x_1 & ... & \partial x_1^i / \partial x_K \\ ... & ... & ... \\ \partial x_K^i / \partial x_1 & ... & \partial x_K^i / \partial x_K \end{bmatrix} = \triangle^{-1} U^i \left(X^i\right) \triangle V(X, \lambda). \tag{29}$$

2. Expressing dX^i from (28), summing up across agents and imposing the restriction $dX = \int_{i \in I} dX^i di$ yields

$$\left\{ \mathbf{I}_{K \times K} - \left[\int_{i \in I} \triangle^{-1} U^i \left(X^i\right) di \right] \triangle V(X, \lambda) \right\} dX +$$
$$\int_{i \in I} \left\{ \triangle^{-1} U^i \left(X^i\right) \overline{1} - \left[\int_{i \in I} \triangle^{-1} U^i \left(X^i\right) di \right] \frac{\partial \log \nabla V(X, \lambda)}{\partial \log \lambda^i} \right\} d \log \lambda^i di = 0,$$

where $\mathbf{I}_{K \times K}$ is a $K \times K$ identity matrix. Since the above equality must hold for any differential of independent variables, the coefficients on dX and each

$d\log\lambda^i$ must be equal to zero. Thus, for all $X \in H$, $\lambda \in \Lambda$, we must have

$$\triangle V(X,\lambda) = \left[\int_{i\in I} \triangle^{-1} U^i\left(X^i\right) di\right]^{-1}, \tag{30}$$

$$\frac{\partial \log \nabla V(X,\lambda)}{\partial \log \lambda^i} = \left[\int_{i\in I} \triangle^{-1} U^i\left(X^i\right) di\right]^{-1} \triangle^{-1} U^i\left(X^i\right)\overline{1}. \tag{31}$$

3. The linear sharing rule (11) implies $\frac{\partial X^i}{\partial X} = \Phi^i(\lambda)$. Combining (29) – (31) yields

$$\frac{\partial \log \nabla V(X,\lambda)}{\partial \log \lambda^i} = \triangle V(X,\lambda)\,\Phi^i(\lambda)\,\triangle^{-1} V(X,\lambda)\,\overline{1} = \Phi^i(\lambda)\,\overline{1}.$$

The fact that $\frac{\partial \log \nabla V(X,\lambda)}{\partial \log \lambda^i}$ is independent of X implies that $\nabla V(X,\lambda)$ is multiplicatively separable in X and λ. Hence, there exist a function $W : R^K_{++} \to R$ and k functions $\theta^{(k)} : \Lambda \to R_{++}$ such that for $k = 1, ..., K$,

$$V_k(X,\lambda) = \theta^{(k)}(\lambda)\, W_k(X). \tag{32}$$

Note that we have $\theta^{(k)}(\lambda) \neq 0$ and $W_k(X) \neq 0$ for all k since $V(X,\lambda)$ is strictly increasing in each commodity, i.e., $V_k(X,\lambda) > 0$.

4. According to (12), all agents have utility functions that are additive in the same partition. Therefore, $(U^i)^{[n]}_{ks}\left((X^i)^{[n]}\right) \neq 0$ for all k, s such that x^i_k, $x^i_s \in (X^i)^{[n]}$ and $(U^i)^{[n]}_{ks}\left((X^i)^{[n]}\right) = 0$ for all k, s such that at least one of x^i_k, x^i_s does not belong to $(X^i)^{[n]}$. Hence, $\triangle U^i(X^i)$ is block-diagonal:

$$\triangle U^i\left(X^i\right) = \begin{bmatrix} \triangle (U^i)^{[1]}\left((X^i)^{[1]}\right) & ... & 0 \\ ... & ... & ... \\ 0 & ... & \triangle (U^i)^{[N]}\left((X^i)^{[N]}\right) \end{bmatrix}.$$

It follows from (30) that $\triangle V(X,\lambda)$ is also block-diagonal:

$$\triangle V(X,\lambda) = \begin{bmatrix} \left[\int_{i\in I} \triangle^{-1} (U^i)^{[1]}\left((X^i)^{[1]}\right) di\right]^{-1} & ... & 0 \\ ... & ... & ... \\ 0 & ... & \left[\int_{i\in I} \triangle^{-1} (U^i)^{[N]}\left((X^i)^{[N]}\right) di\right]^{-1} \end{bmatrix}.$$

That is, $V(X,\lambda)$ is additive in the same partition as the utility function of each agent, $U^i(X^i)$, and thus, it can be represented by

$$V(X,\lambda) = \sum_{n=1}^{N} V^{[n]}\left((X)^{[n]},\lambda\right). \tag{33}$$

5. Differentiating (32) with respect to some $x_s \in X$, $s \neq k$, yields

$$V_{ks}(X,\lambda) = \theta^{(k)}(\lambda)\, W_{ks}(X). \tag{34}$$

Given that $\theta^{(k)}(\lambda) \neq 0$ for all k, we have that $V_{ks}(X,\lambda) = 0$ if and only if $W_{ks}(X) = 0$. That is, the additivity of $V(X,\lambda)$ is sufficient for the additivity of $W(X)$ in the same partition,

$$W(X) = \sum_{n=1}^{N} W^{[n]}\left((X)^{[n]}\right). \tag{35}$$

6. Consider any k, s such that x_k, $x_s \in X^{[n]}$. Then, (33) and (35) imply $V_{ks}(X,\lambda) \neq 0$ and $W_{ks}(X,\lambda) \neq 0$, respectively. Further, by (32), we have

$$V_{sk}(X,\lambda) = \theta^{(s)}(\lambda)\, W_{sk}(X). \tag{36}$$

Given that $V_{ks}(X,\lambda) = V_{sk}(X,\lambda)$ and $W_{ks}(X) = W_{sk}(X)$, we obtain that

$$\theta^{(k)}(\lambda) = \theta^{(s)}(\lambda) \equiv \xi^{[n]}(\lambda). \tag{37}$$

That is, for all commodities belonging to the same partition $X^{[n]}$, the corresponding $\theta^{(k)}(\lambda)$ are equal.

7. By combining (32), (35) and (37), we get

$$V_k(X,\lambda) = \xi^{[n]}(\lambda)\, W_k^{[n]}\left(X^{[n]}\right), \tag{38}$$

for each k such that $x_k \in X^{[n]}$. Therefore, $V(X,\lambda)$ is given by (13). ■