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Heterogeneity in capital and skills in a neoclassical stochastic growth model☆

Lilia Maliar*, Serguei Maliar

Universidad de Alicante, Departamento de Fundamentos del Análisis Económico, Campus San Vicente del Raspeig, Ap. Correos 99, 03080 Alicante, Spain

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Abstract

Does a heterogeneous agents version of a neoclassical model with labor–leisure choice replicate the distributions of consumption and working hours observed in the cross-sectional data? Does incorporating heterogeneity enhance the aggregate performance of the representative agent model? We address these questions in a complete market model economy with two sources of heterogeneity: initial endowments and non-acquired skills. We find positive answers to both questions. © 2001 Elsevier Science B.V. All rights reserved.

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* Corresponding author.

E-mail addresses: maliar-l@upf.es (L. Maliar), maliars@upf.es (S. Maliar).

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1. Introduction

This paper studies a complete market heterogeneous agents version of a standard neoclassical model by Kydland and Prescott (1982). We assume that instead of many identical agents the economy is populated by a number of agents who differ along two dimensions: initial endowments and non-acquired skills. The first question we ask is: Can the heterogeneous model replicate the distributions of individual quantities observed in the cross-sectional data such as consumption and hours worked?

The representative agent version of Kydland and Prescott's (1982) model has been extensively studied in the literature. Although it is successful at matching most of the features of aggregate fluctuations of the real economies, this model has serious drawbacks regarding the labor markets. In particular, it predicts that the correlation between hours worked and labor productivity is in excess of 0.9, while the actual correlation is close to zero. The latter empirical fact is often referred to in the literature as the 'Dunlop–Tarshis observation' after Dunlop (1938) and Tarshis (1939). The capability to account for this observation 'continues to play a central role in assessing the empirical plausibility of different business-cycle models' (Christiano and Eichenbaum, 1992). One can reasonably hope that neglecting heterogeneity contributes to the above shortcoming, as there exists an empirical evidence that aggregate measures of fluctuations are affected by cyclical variation in the average skill level (please see Hansen (1993) and Kydland and Prescott (1993)). Therefore, the second question we address is: Does incorporating heterogeneity help to improve the aggregate predictions of the model?

The two questions we ask have been addressed in previous studies. Kydland (1984) analyzes a real business cycle model with two types of agents who are heterogeneous in skills; he finds that allowing for the heterogeneity has a minor positive effect on the aggregate predictions. Garcia-Milá et al. (1995) (further, GMV) consider a similar two-agents setup and reach a different conclusion, namely, that heterogeneity deteriorates the aggregate performance dramatically: the heterogeneous agents version not only fails to account for the correlation between productivity and hours worked but it also counterfactually predicts extremely low volatility of hours worked, a negative correlation between hours worked and output, etc. They also find that the model generates the failure at the individual level: it counterfactually predicts that rich (skilled) individuals spend less time working in the market compared to the poor (unskilled). The inability of the heterogeneous model to generate the appropriate predictions is referred to in GMV (1995) as a puzzle.

The studies by Kydland (1984) and GMV (1995) differ in several dimensions. In particular, the first paper assumes the preferences of the Cobb–Douglas type, constructs two heterogeneous agents by splitting the panel data into two educational groups and calibrates the model to reproduce the groups' average hours worked. On the other hand, the second considers the preferences of the addilog type, distinguishes two agents according to their level of wealth (wealth to wage ratio) and calibrates the parameters to match the groups' wealth holdings. A significant discrepancy in the findings of Kydland (1984) and GMV (1995) indicates that some of the above assumptions play a determinant role for the model's implications. Also, GMV (1995) argues that the model's predictions are likely to be sensitive to the number of the heterogeneous agents introduced in the model. Given these differences in results and the implied uncertainty about the effects of heterogeneous agents, further study of the model is of interest.

Unlike the previous literature, we characterize the equilibrium in the heterogeneous economy by using aggregation theory. Aggregation allows us to achieve two objectives. First, it simplifies qualitative analysis and makes it possible to describe in a simple way the relation between distributions and aggregate dynamics. In particular, aggregation results allow us to gain intuition into the effect of heterogeneity on labor markets, to understand the origin of the puzzle in GMV (1995) and to elaborate a modification to the calibration procedure which resolves the problems encountered by these authors. Further, aggregation simplifies substantially the numerical analysis and enables us to extend the model to include any number of heterogeneous agents without having the corresponding increase in computational cost. It is worth noting that the concept of aggregation used in the paper is different from the standard one by Gorman (1953). The latter requires that the preferences are quasi-homothetic. We consider two examples of preferences, the Cobb–Douglas and the addilog. The addilog preferences are not quasi-homothetic; they lead to demand which is not linear in wealth and imply aggregate dynamics which depend on the joint distribution of capital and skills.

In the paper, we present several analytical results of interest. We show that the model's time-series performance is directly linked to its distributive implications. In such a way, we establish that in order to generate the appropriate time series predictions on labor markets, it is necessary that the model is able to account for the empirical observation that hours worked by the agents increase in the level of skills (wealth). The numerical results reported in GMV (1995) suggest that such regularity is difficult to generate in the model. We provide an analytical argument in support of this finding: we show that assumptions about preferences which are standard for macro literature are inconsistent with cross-sectional observations. Specifically, under the Cobb–Douglas utility, the heterogeneous agents version is never able to reproduce the joint distribution of capital, skills, consumption and hours worked observed in the data. Under the addilog utility, the model's implications are primarily determined by the value of the intertemporal elasticity of substitution for consumption; however, under the standard range of values for this parameter (smaller than or equal to one) the model fails to account for the distributions as well. Our analysis has also a positive implication, namely, we show that if the intertemporal elasticity of substitution for consumption in the addilog utility is higher than one, then it is likely that the heterogeneous agents version is successful in matching both time series and distributions.

The results from simulations confirm this conjecture. We calibrate and solve the model with eight heterogeneous consumers to match the aggregate quantities and the distributions of productivity and endowments in the U.S. economy. We find that if the utility parameters are chosen so that the distributions of consumption and hours worked in the model are consistent with patterns observed in the data, heterogeneity improves substantially the model's performance at the aggregate level. In particular, the heterogeneous agents version can account for several time-series facts which cannot be reconciled within a similar representative agent setup; for example, the weak correlation between productivity and hours worked. Our results are in contrast with conclusion of most papers on heterogeneity, which do not find a significant difference in the predictions of the heterogeneous and the representative agent versions of the studied models, e.g., Cho (1995), Krusell and Smith (1995), Ríos-Rull (1996), etc.

The paper is organized as follows. Section 2 contains the description of the model economy and derives the results from aggregation. Section 3 analyzes qualitative implications of the model. Section 4 describes the calibration procedure. Section 5 discusses the numerical predictions. Finally, Section 6 concludes.

2. The model

We start by analyzing a competitive equilibrium in an economy populated by a set of utility-maximizing heterogeneous consumers and a single profit-maximizing firm. Subsequently, we construct a planner's problem generating the optimal allocation which is identical to the competitive equilibrium in the decentralized economy. Finally, we show how to simplify the characterization of the equilibrium in the model by using aggregation theory.

2.1. The economy

The consumer side of the economy consists of a set of agents S. The measure of agent s in the set S is denoted by $d\omega^s$, where $\int_S d\omega^s = 1$. The agents are heterogeneous in skills and initial endowments. The skills are intrinsic, permanent characteristics of the agents. We denote the skills of an agent $s \in S$ by e^s, assume $e^s > 0$ for $\forall s \in S$ and normalize the aggregate skills to one, $\int_S e^s d\omega^s = 1$. The initial endowment of the individual $s \in S$ is denoted by κ_0^s. The timing is discrete, $t \in T$, where $T = 0, 1, \ldots$.

An infinitely-lived agent $s \in S$ seeks to maximize the expected sum of momentary utilities $u(c_t^s, l_t^s)$, discounted at the rate $\delta \in (0, 1)$, by choosing a path for consumption, c_t^s, and leisure, l_t^s. The utility function $u(\cdot)$ is continuously differentiable, strictly increasing in both arguments, and strictly concave. In period t the agent owns capital stock k_t^s and rents it to the firm at the rental price r_t. Also, he supplies to the firm n_t^s units of labor in exchange on income $n_t^s e^s w_t$, where w_t is the wage paid for one unit of efficiency labor. The total time endowment of the agent is normalized to one, $n_t^s + l_t^s = 1$. Capital depreciates at the rate $d \in (0,1]$. When making the investment decision, the agent faces uncertainty about the future returns on capital. We assume that markets are complete: the agent can insure himself against uncertainty by trading state contingent claims, $\{m_t^s(\theta)\}_{\theta \in \Theta}$, where Θ denotes the set of all possible realizations of productivity shocks. The claim of type $\theta \in \Theta$ costs $p_t(\theta)$ in period t and pays one unit of consumption good in period $t+1$ if the state $\theta \in \Theta$ occurs and zero otherwise.

Consequently, the problem solved by agent $s \in S$

$$\max_{\{c_t^s,\, n_t^s,\, k_{t+1}^s,\, m_{t+1}^s(\theta)\}_{\theta \in \Theta,\, t \in T}} \mathrm{E}_0 \sum_{t=0}^{\infty} \delta^t u(c_t^s, 1 - n_t^s) \tag{1}$$

s.t.

$$c_t^s + k_{t+1}^s + \int_{\Theta} p_t(\theta) m_{t+1}^s(\theta)\,\mathrm{d}\theta = (1 - d + r_t)k_t^s + n_t^s e^s w_t + m_t^s(\theta_t). \tag{2}$$

Initial holdings of capital and contingent claims, k_0^s and m_0^s, are given.

The production side of the economy consists of a single representative firm. Given the prices, r_t and w_t, the firm rents capital $\{k_t^s\}^{s \in S}$ and hires labor $\{n_t^s\}^{s \in S}$ to maximize period-by-period profits. Capital and labor inputs are given by aggregate capital in the economy and efficiency hours worked by all consumers, $k_t = \int_S k_t^s \,\mathrm{d}\omega^s$ and $h_t = \int_S n_t^s e^s \,\mathrm{d}\omega^s$.

Therefore, the problem of the firm is the following:

$$\max_{k_t,\, h_t} \pi_t = \theta_t f(k_t, h_t) - r_t k_t - w_t h_t. \tag{3}$$

The aggregate technology shock θ_t follows a first-order Markov process with transitional probabilities $\Pr\{\theta_{t+1} = \theta \mid \theta_t = \theta'\}_{\theta, \theta' \in \Theta}$. The value θ_0 is given. The production function $f(\cdot)$ has constant returns to scale, is strictly concave, continuously differentiable, strictly increasing with respect to both arguments and satisfies the appropriate Inada conditions.

We define initial endowment of agent $s \in S$ as the value of initial capital and security payment measured in terms of output in period $t = 0$

$$\kappa_0^s = (1 - d + r_0)k_0^s + m_0^s(\theta_0).$$

The function $\{\kappa_0^s\}^{s \in S}$ will be referred to as (initial) wealth distribution. The rest of the economy's characteristics such as the distribution of skills, the initial condition (k_0, θ_0), etc. will be summarized by the set $\mathfrak{I}$.

A competitive equilibrium in the economy (1)–(3) is defined as a set $\Im$, a distribution of wealth $\{\kappa_0^s\}^{s\in S}$ and a sequence of contingency plans for the consumers' allocation $\{c_t^s, n_t^s, k_{t+1}^s, m_{t+1}^s(\theta)\}_{\theta\in\Theta,\, t\in T}^{s\in S}$, for the firm's allocation $\{k_t, h_t\}_{t\in T}$ and for the prices $\{r_t, w_t, p_t(\theta)\}_{\theta\in\Theta,\, t\in T}$ such that given the prices, the sequence of plans for the consumers' allocation solve the utility maximization problem (1), (2) of each agent $s\in S$, the sequence of plans for the firm's allocation leads to zero profit solution to (3) for $\forall t\in T$ and all markets clear. Moreover, the equilibrium plans are to be such that $c_t^s \geq 0$, and $1 \geq n_t^s \geq 0$ for $\forall s\in S$, $t\in T$ and $w_t, r_t, k_t \geq 0$ for $\forall t\in T$. We assume that the equilibrium exists, is interior and is unique. The assumption of the uniqueness refers only to the equilibrium plans for prices and for allocations other than the individual holdings of capital and state contingent claims. The last two are not uniquely defined because the number of assets traded exceeds the number of the economy's states.

2.2. The planner's problem

To simplify the analysis of the equilibrium, we exploit two fundamental theorems of welfare economics. Specifically, consider an otherwise identical economy except that it is ruled by a planner who maximizes the weighted sum of the agents' preferences subject to the economy's resource constraint

$$\max_{\{c_t^s,\, n_t^s,\, k_{t+1}\}_{t\in T}^{s\in S}} \mathrm{E}_0 \sum_{t=0}^{\infty} \delta^t \int_S \lambda^s u(c_t^s, 1-n_t^s)\, \mathrm{d}\omega^s \tag{4}$$

s.t.

$$c_t + k_{t+1} = (1-d)k_t + \theta_t f(k_t, h_t), \tag{5}$$

where $c_t = \int_S c_t^s \,\mathrm{d}\omega^s$ and $h_t = \int_S n_t^s e^s \,\mathrm{d}\omega^s$ are aggregate consumption and aggregate efficiency hours and $\{\lambda^s\}^{s\in S}$ is a set of welfare weights.

In addition, the planner's choice is restricted to satisfy the expected lifetime budget constraint of each agent $s\in S$

$$\mathrm{E}_0\left[\sum_{\tau=0}^{\infty} \delta^\tau \frac{u_1(c_\tau^s, n_\tau^s)}{u_1(c_0^s, n_0^s)} (c_\tau^s - n_\tau^s e^s w_\tau)\right] = \kappa_0^s, \tag{6}$$

where $\{\kappa_0^s\}^{s\in S}$ is the distribution of wealth and $w_\tau = \theta_\tau \partial f(k_\tau, h_\tau)/\partial h_\tau$ is the equilibrium wage in decentralized economy (1)–(3).

A solution to problem (4)–(6) is defined as a set $\Im$, a distribution of wealth $\{\kappa_0^s\}^{s\in S}$, a set of welfare weights $\{\lambda^s\}^{s\in S}$ and a sequence of contingency plans for the consumers' allocation $\{c_t^s, n_t^s\}_{t\in T}^{s\in S}$ and for aggregate allocation $\{c_t, h_t, k_t\}_{t\in T}$ such that, given the welfare weights, the sequence of plans for the allocations solves problem (4), (5) and satisfies the constraint (6) of each agent $s\in S$.

Moreover, the weights are strictly positive, $\lambda^s > 0$ for $\forall s \in S$, and the allocations are such that $c_t^s \geq 0$, and $1 \geq n_t^s \geq 0$ for $\forall s \in S$, $t \in T$ and $k_t \geq 0$ for $\forall t \in T$.

Proposition 1. If a competitive equilibrium in the decentralized economy (1)–(3) *exists and is interior and if the equilibrium sequence of contingency plans for* $\{c_t^s, n_t^s\}_{t \in T}^{s \in S}$ *and for* $\{c_t, h_t, k_{t+1}\}_{t \in T}$ *is unique, then such a sequence is uniquely determined by* (4)–(6).

Proof. The fact that the equilibrium allocation in the decentralized economy is a solution to (4), (5) and satisfies the constraint (6) for each $s \in S$ follows by the first welfare theorem and the results of Appendix A, respectively. Therefore, (4)–(6) are necessary for the equilibrium. The sufficiency follows by the second welfare theorem and by the fact that for any set of weights for which a solution to (4), (5) exists, is interior and is unique, the distribution of wealth $\{\kappa_0^s\}^{s \in S}$ is uniquely defined by the constraints (6). □

Let us comment on this result. According to the first welfare theorem, under complete markets and in the absence of externalities and other distortions, a competitive equilibrium is Pareto optimal and therefore, can be calculated as a solution to planner's problem (4), (5). However, to make use of this result, it is necessary to find a set of weights which corresponds to a given distribution of wealth. To identify such weights, we exploit the transversality conditions or, equivalently, the expected lifetime budget constraints of the agents in the decentralized economy. Constraints (6) give us S restrictions which are sufficient to identify S unknown welfare weights.

In the decentralized economy, the agents's lifetime budget constraint allows for a simple economic interpretation. Specifically, it restricts the value of commodities consumed by the agent over the lifetime to be equal to his endowment of wealth and the value of his lifetime labor income.

The fact that welfare weights are endogenous to the model complicates substantially the analysis of the equilibrium. An example of numerical algorithm which solves for weights is described by GMV (1995). It is roughly as follows: fix the weights to some values, find a solution to planner's problem (4), (5), calculate the left side of the constraints (6) and compare the result to the given distribution of wealth; subsequently, iterate on weights until a solution to problem (4), (5), which is consistent with the constraints (6), is found. In fact, this algorithm is costly as each iteration on weights requires finding a solution to the planner's problem and evaluating the expectations in the lifetime budget constraint. Moreover, the cost depends on the number of agents and increases significantly if there are more than two agents. Therefore, we will not resort to simulations from the outset but will begin with exploring the possibility to simplify the description of the equilibrium.

2.3. Aggregation

In this section, we show how to employ aggregation theory in order to simplify characterization of the equilibrium under the assumption of the Cobb–Douglas and addilog preferences. Given that both of these preferences are commonly used in macroeconomics, it is of interest to study their implications for the economy with heterogeneous consumers. Another reason for our choice is that these two types of preference are assumed in Kydland (1984) and GMV (1995), respectively. Considering both Cobb–Douglas and addilog preferences will enable us to evaluate whether the discrepancy in findings of these studies is explained by the preference choice.

2.3.1. Quasi-homothetic preferences

Gorman's (1953) theorem implies that if the consumers differ only in wealth and have identical quasi-homothetic preferences, then demand is linear in wealth and, as a result, at the aggregate level, the economy behaves as if there was a single representative consumer. Examples of Gorman's (1953) aggregation in neoclassical economies with a single consumption commodity are discussed in Chatterjee (1994) and Caselli and Ventura (1996).

It turns out that with heterogeneity in both wealth and skills, the aggregation result changes. Precisely, demand for physical hours worked n_t^s is not linear in wealth any longer; however, demand for efficiency hours worked $n_t^s e_s^s$ is. Consequently, the preferences of a representative consumer depend on aggregate efficiency hours worked, h_t, and not on aggregate physical hours worked, $n_t = \int_S n_t^s \, d\omega^s$. Rather than elaborate a strict proof of this fact, we illustrate the aggregation results by using a particular example of quasi-homothetic preferences.

Assume that the agent's momentary utility is of the Cobb–Douglas type

$$u(c_t^s, n_t^s) = \frac{((c_t^s)^{\mu}(1 - n_t^s)^{1-\mu})^{1-\eta} - 1}{1 - \eta}, \quad 1 > \mu > 0, \ \eta > 0. \tag{7}$$

Then, the equilibrium in the model can be described by a single-agent utility maximization problem

$$\max_{\{c_t, h_t, k_{t+1}\}_{t \in T}} \ \mathrm{E}_0 \sum_{t=0}^{\infty} \delta^t \frac{(c_t^{\mu}(1 - h_t)^{1-\mu})^{1-\eta} - 1}{1 - \eta} \quad \text{s.t. } RC, \tag{8}$$

a set of equations which relates individual and aggregate allocations

$$c_t^s = c_t f^s, \qquad n_t^s = 1 - (1 - h_t)\frac{f^s}{e^s}, \tag{9}$$

and a set of equations which determines the agent-specific parameters $\{f^s\}^{s \in S}$

$$f^s = \frac{\kappa_0^s + e^s \mathrm{E}_0 \sum_{\tau=0}^{\infty} \delta^\tau \frac{u_1(c_\tau, h_\tau)}{u_1(c_0, h_0)} w_\tau}{\mathrm{E}_0 \sum_{\tau=0}^{\infty} \delta^\tau \frac{u_1(c_\tau, h_\tau)}{u_1(c_0, h_0)} (c_\tau + w_\tau(1 - h_\tau))}, \tag{10}$$

where notation RC is used to denote the economy's resource constraint (5) and f_s is the share of consumption of agent $s \in S$ in aggregate consumption, $f_s = (\lambda^s)^{1/\eta}(e^s)^{-(1-\mu)(1-\eta)/\eta} / \int_S (\lambda^s)^{1/\eta}(e^s)^{-(1-\mu)(1-\eta)/\eta} \,\mathrm{d}\omega^s$.

Proposition 2. Under utility (7), (4)–(6) *and* (8)–(10) *are equivalent.*

Proof. See Appendix B. □

The above characterization of the equilibrium has two important advantages compared to (4)–(6). First, the equilibrium relations between individual and aggregate variables are defined explicitly. This fact will make it possible to deduce some qualitative properties of the equilibrium without calculating the exact numerical solution. Second, the quantitative analysis can be carried out without computationally costly iteration on weights. Precisely, the equilibrium can be computed in three steps: solve model (8), compute the shares from (10) and restore the agents' consumption and hours worked by using (9). The cost of calculating a numerical solution by using this algorithm does not depend on the number of heterogeneous agents and is comparable to that of finding the equilibrium in the associated representative agent model.

2.3.2. Addilog preferences

In fact, a representative consumer can be constructed under some preferences which are not quasi-homothetic, though in this case demand will not be linear in wealth. An example of preferences which lead to such 'imperfect' aggregation is the addilog utility. This type of preferences is introduced in the literature by Houthakker (1960). The fact that addilog preferences are consistent with aggregation is pointed out by Shafer (1977). Atkeson and Ogaki (1996) consider a neoclassical economy with two consumption commodities and exogenous production and exploit the property of aggregation for estimating the parameters in the addilog utility function.

Assume that the agents have momentary utility of the addilog type

$$u(c_t^s, n_t^s) = \frac{(c_t^s)^{1-\gamma} - 1}{1-\gamma} + B\frac{(1 - n_t^s)^{1-\sigma} - 1}{1-\sigma}, \quad \gamma, \sigma, B > 0. \tag{11}$$

Then, the equilibrium in the model can be characterized by a utility maximization problem of a single-agent

$$\max_{\{c_t,\, h_t,\, k_{t+1}\}_{t\in T}} \mathrm{E}_0 \sum_{t=0}^{\infty} \delta^t \left\{ \frac{c_t^{1-\gamma}-1}{1-\gamma} + XB\frac{(1-h_t)^{1-\sigma}-1}{1-\sigma} \right\} \quad \text{s.t. } RC, \tag{12}$$

a condition which identifies the parameter X

$$X = \left(\int_S (e^s)^{1-1/\sigma} (f^s)^{\gamma/\sigma} \, \mathrm{d}\omega^s \right)^{\sigma}, \tag{13}$$

a set of equations which relates individual and aggregate allocations

$$c_t^s = c_t f^s, \qquad n_t^s = 1-(1-h_t)X^{-1/\sigma}(e^s)^{-1/\sigma}(f^s)^{\gamma/\sigma}, \tag{14}$$

and conditions which determine the agents' shares of consumption $\{f^s\}^{s\in S}$

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \delta^{\tau} \frac{u_1(c_\tau, h_\tau)}{u_1(c_0, h_0)} [c_\tau f^s - w_\tau e^s(1-(1-h_\tau)X^{-1/\sigma}(e^s)^{-1/\sigma}(f^s)^{\gamma/\sigma})] = \kappa_0^s, \tag{15}$$

where $f^s = (\lambda^s)^{1/\gamma} / \int_S (\lambda^s)^{1/\gamma} \, \mathrm{d}\omega^s$.

Proposition 3. Under utility (11), (4)–(6) *and* (12)–(15) *are equivalent.*

Proof. See Appendix B. □

Here, as in the case of quasi-homothetic preferences, the preferences of the representative consumer depend on aggregate efficiency hours worked and the equilibrium relation between individual and aggregate quantities is explicit. There are two differences, however. First, the utility of the representative consumer depends on the joint distributions of wealth and skills through the parameter X and, second, constraint (15) does not allow for a closed-form solution with respect to the consumption share, unless $\gamma = \sigma$.

As the utility (12) depends on the endogenous parameter X, a numerical solution cannot be computed without iterations. An example of algorithm which calculates the equilibrium is as follows: fix X to some value, solve model (12), compute the shares from (15) and restore X according to (13); iterate on X until a fixed-point solution is found. Note that, unlike the algorithm iterating on welfare weights, the above procedure iterates on a single parameter X independently of the number of heterogeneous agents. As a result, with any number of agents, the cost of calculating a numerical solution by using this algorithm is roughly equal to that of finding the equilibrium in the two-agent model by using iterating on welfare weights.

3. Model's implications

This section is dedicated to qualitative analysis of the model's predictions. Specifically, we study the relation between the implications of the model at the individual and aggregate levels and identify the factors which play a determinant role for the properties of the equilibrium.

3.1. Aggregate dynamics

Let us first discuss a potential effect of the parameter X which appears in the utility of the representative consumer under the assumption of the addilog utility. According to (12) this parameter premultiplies the utility parameter B and, therefore, has the same effect on the properties of the model as a variation in B. Normally, the parameter B is chosen to match average hours worked in the real economy. The estimate of average hours worked, however, largely depends on whether the time endowment is measured in terms of 'real' or 'discretionary' time (discretionary time is total time minus time spent on personal care such as sleeping, eating, etc.). This implies a substantial degree of freedom in the choice of the parameter B. For instance, when calibrating identical models, Hansen (1985) assumes 'discretionary' time and obtains $B = 2$, while Christiano and Eichenbaum (1992) match 'real' time and get $B = 2.99$, a value which is about 50% higher. This example suggests that the effect associated with X is of little interest (at least, for the second moments of aggregate variables) unless it differs from one by a very large order of magnitude. Therefore, the parameter X will be disregarded in this section.

Given the supposition above, under both Cobb–Douglas and addilog types of preferences, we would expect the following. First, the time series properties of variables $\{c_t, k_t, w_t, r_t\}$ in the model are not affected by the assumed types of heterogeneity. Second, the variable efficiency hours worked $\{h_t\}$ in the heterogeneous model behaves as physical hours worked in the representative agent case. Therefore, the only implication of heterogeneity which is of potential interest is its effect on time series properties of physical hours worked $\{n_t\}$. Consequently, in the rest of the section, we concentrate on the model's implications regarding labor markets.

Integrating individual hours worked, we get that under both the Cobb–Douglas and addilog utilities, efficiency hours and 'simple' hours are related as

$$h_t = 1 - (1 - n_t) \cdot \xi, \tag{16}$$

where under Cobb–Douglas utility the parameter ξ is computed from (9)

$$\xi = \int_S \frac{f^s}{e^s} \, d\omega^s, \tag{17}$$

while in the case of addilog utility it is obtained from (14)

$$\xi = \frac{\int_S (e^s)^{1-1/\sigma}(f^s)^{\gamma/\sigma}\, \mathrm{d}\omega^s}{\int_S (e^s)^{-1/\sigma}(f^s)^{\gamma/\sigma}\, \mathrm{d}\omega^s}. \tag{18}$$

We call the parameter ξ *labor input bias.* We will denote by σ_x and $corr(x, y)$ the volatility of a variable x and the correlation of variables x and y.

Consider two empirical regularities on labor market behavior:

(i) physical hours worked fluctuate more than efficiency hours worked;
(ii) average productivity and hours worked are weakly correlated.

The first fact is reported, for example, by Kydland and Prescott (1993). Using the PSID data, they construct aggregate labor-input series by summing individual hours worked weighted by average hourly wage and find that the resulting series fluctuate less than the unadjusted hours worked. Hansen (1993) obtains a similar result using monthly data on age-sex groups from the U.S. Bureau of Labor Statistics' household survey. Note that in order for the model to be consistent with this empirical observation, it is necessary that $\xi < 1$. Indeed, according to (16), we have $\sigma_h = \xi^2\sigma_n$.

The second fact is well-known to the literature and is often referred to as the Dunlop–Tarshis observation. Christiano and Eichenbaum (1992) document this regularity for the U.S. economy. Also, they show that the representative agent version of the model insistently predicts that this correlation is close to one and, therefore, does not reproduce fact (ii). Below we demonstrate that by taking into account the effect of heterogeneity on labor markets, we can improve the model's predictions in this dimension.

It follows from (16) that physical and efficiency hours worked are perfectly correlated in the model, $corr(n, h) = 1$, and, thus,

$$corr(y/n, n) = corr(y/h \cdot h/n, h),$$

where y/n is average labor productivity. Given that in the heterogeneous model, h_t behaves as n_t in the representative agent model and also, that the representative agent model generates almost perfect positive correlation between productivity and hours worked, we have $corr(y/h, h) \simeq 1$. Therefore, $corr(y/n, n)$ in the heterogeneous case depends on how the variable h_t/n_t behaves over the business cycle. Eq. (16) implies that whether this variable is procyclical or countercyclical depends on the value of the parameter ξ

$$\frac{\mathrm{d}(h_t/n_t)}{\mathrm{d}h_t} = \frac{1}{n_t}\left(1 - \frac{\mathrm{d}n_t/n_t}{\mathrm{d}h_t/h_t}\right) = -\frac{1-\xi}{\xi n_t^2}. \tag{19}$$

It follows by the last result that if $\xi < 1$, then the correlation between productivity and hours worked in the heterogeneous model will be smaller than this statistic in the representative agent setup.

3.2. Individual dynamics

In this section, we focus on the relation between aggregate and distributive predictions of the model. For the purpose of subsequent analysis, we assume that there is a continuum of agents in the economy so that decisions of a single individual have no effect on aggregate allocation.

Kydland (1984) and Ríos-Rull (1993) document two empirical regularities of the individual labor choice:

(iii) individual hours worked are increasing in skills;
(iv) the volatility of individual hours worked is decreasing in skills.

In terms of our model, empirical facts (iii) and (iv) imply $\mathrm{d}n_t^s/\mathrm{d}e^s > 0$ and $\mathrm{d}\varepsilon_t^s/\mathrm{d}e^s < 0$ respectively, where

$$\varepsilon_t^s = \left(\frac{\mathrm{d}n_t^s}{\mathrm{d}w_t}\frac{w_t}{n_t^s}\right)$$

is the period elasticity of agent's labor supply with respect to wage.

First, we demonstrate that if regularity (iii) is satisfied in our model, then (iv) will also be satisfied. Using the formulas for individual working hours given in (9) and (14) for the Cobb–Douglas and addilog utilities, one can show that

$$\frac{\mathrm{d}\varepsilon_t^s}{\mathrm{d}e^s} = -\frac{\varepsilon_t h_t}{n_t^s(1-h_t)}\frac{\mathrm{d}n_t^s}{\mathrm{d}e^s}, \tag{20}$$

where

$$\varepsilon_t = \left(\frac{\mathrm{d}h_t}{\mathrm{d}w_t}\frac{w_t}{h_t}\right) > 0.$$

The latter follows because ε_t is the same as the elasticity of labor with respect wage in the representative agent model.

Let us analyze how distributive facts (iii) and (iv) are related to previously discussed aggregate regularities (i) and (ii). Using formula (19), the change in supply of physical and efficiency hours worked in response to wage increase can be written as

$$\mathrm{d}n_t = \frac{\mathrm{d}w_t}{w_t}\int_S n_t^s\varepsilon_t^s\,\mathrm{d}\omega^s, \qquad \mathrm{d}h_t = \frac{\mathrm{d}w_t}{w_t}\int_S n_t^s\varepsilon_t^s e^s\,\mathrm{d}\omega^s.$$

Together with (19), these formulas imply

$$\frac{\mathrm{d}(h_t/n_t)}{\mathrm{d}h_t} = \frac{\int_{S\times S} n_t^s n_t^{s'}(e^s - e^{s'})(\varepsilon_t^s - \varepsilon_t^{s'})\,\mathrm{d}\omega^s\,\mathrm{d}\omega^{s'}}{n_t^2\cdot\int_S n_t^s\varepsilon_t^s e^s\,\mathrm{d}\omega^s}. \tag{21}$$

Observe that if individual elasticities decrease in skills, $\mathrm{d}\varepsilon_t^s/\mathrm{d}e^s < 0$, then each term in the numerator of the above expression is negative and, therefore, $\mathrm{d}(h_t/n_t)/\mathrm{d}h_t < 0$. As shown in the previous section, the last result corresponds to $\xi < 1$ which is consistent with aggregate facts (i) and (ii).

The elasticity considerations allows us to gain some intuition on why the correlation between productivity and hours worked in the heterogeneous model might be lower than in the representative agent model. In response to a positive shock, all individuals supply more hours on the market. However, if the elasticity of labor supply decreases in skills, the increase in hours of low skilled workers is larger than that of high skilled workers. As a result, a fraction of low skilled hours in total hours worked is larger after the shock than it was before the shock and the average productivity of labor, y_t/n_t, goes down. If the difference in the elasticities across agents is high enough, then it could be the case that $corr(y/n, n) < 0$.

3.3. Working hours

The analysis of the previous section suggests that the model can produce the appropriate aggregate and distributive predictions only if it is capable of generating hours worked which increase in the level of skills (fact (iii)). To evaluate the model's ability to account for this observation, we will employ an additional empirical regularity at the individual level, namely:

(v) wage differentials across agents do not exceed wealth differentials.

In terms of our model, this implies $\mathrm{d}\log\kappa_0^s/\mathrm{d}\log e^s \geq 1$, since this ratio gives us the percentage change in wealth relative to one percent change in skills. GMV (1995) divide the PSID sample into two groups according to two alternative criteria, the wage to wealth ratio and the level of wealth. Depending on the criterion, they obtain that the difference in the wages of two groups amount to 2 while the differences in wealth range from 5 to 30. We find that a quantitative expression of fact (v) is highly sensitive to a criterion which is used for dividing the population into groups. For example, in the next section, we will show that if the PSID data are divided according to the educational level, then the distributions of wealth and wages across group do not differ significantly.

Let us analyze the relation between facts (iii) and (v). Formulas (10), (15) imply that under both Cobb–Douglas and addilog utility functions, the lifetime budget constraint can be written as follows:

$$\mathrm{E}_0\left[\sum_{\tau=0}^{\infty}\delta^{\tau}\frac{u_1(c_\tau,n_\tau)}{u_1(c_0,n_0)}(c_\tau f^s - n_\tau^s e^s w_\tau)\right] = \kappa_0^s. \tag{22}$$

Differentiating (22) with respect to skills and subtracting from the resulting condition Eq. (22), previously divided by skills, we get

$$1-\frac{\mathrm{d}\log f^s}{\mathrm{d}\log e^s}=\frac{\dfrac{\kappa_0^s}{e^s}\left(1-\dfrac{\mathrm{d}\log\kappa_0^s}{\mathrm{d}\log e^s}\right)-\mathrm{E}_0\sum_{\tau=0}^{\infty}\delta^{\tau}\dfrac{u_1(c_\tau,n_\tau)}{u_1(c_0,n_0)}e^s w_\tau\dfrac{\mathrm{d}n_\tau^s}{\mathrm{d}e^s}}{\dfrac{f^s}{e^s}\mathrm{E}_0\sum_{\tau=0}^{\infty}\delta^{\tau}\dfrac{u_1(c_\tau,n_\tau)}{u_1(c_0,n_0)}c_\tau}. \tag{23}$$

Under the assumption of Cobb–Douglas utility function, the condition for individual hours provided in (9) yields

$$1-\frac{\mathrm{d}\log f^s}{\mathrm{d}\log e^s}=\frac{\mathrm{d}n_t^s}{\mathrm{d}e^s}\frac{1}{1-n_t^s}. \tag{24}$$

Observe that if the model is to reproduce fact (iii) and fact (v), then the right-hand side of (23) must be negative. However, the condition (24) demonstrates that the model can account for fact (iii) only if the left-hand side of (23) is positive. Thus, under the assumption of Cobb–Douglas utility, the model cannot explain facts (iii) and (v) simultaneously. In other words, if such model is calibrated to reproduce the distribution of wealth in the data, then it counterfactually predicts that rich (skilled) agents supply less labor on the market than poor (unskilled). Equivalently, if the model is calibrated to match the difference in hours worked across productivity groups, then the implied distribution of wealth is such that regularity (v) is violated.[1]

For the addilog type of preferences, using (14), we obtain

$$1-\frac{\mathrm{d}\log f^s}{\mathrm{d}\log e^s}=\frac{\mathrm{d}n_t^s}{\mathrm{d}e^s}\frac{e^s\sigma}{\gamma(1-n_t^s)}-\frac{1}{\gamma}+1. \tag{25}$$

Note that Eq. (25) together with (23) implies that, similar to the Cobb–Douglas case, the model will also fail with respect to facts (iii) and (v), if the agents' preferences are of the addilog type with the standard elasticity of intertemporal substitution for consumption, $1/\gamma \leq 1$.[2] The above results suggest that the assumptions about preferences which are standard for macroeconomic literature are inconsistent with basic cross-sectional observations.

[1] Kydland (1984) does not analyze the model's implications with respect to the distribution of wealth and, therefore, does not pin down this failure of the model.

[2] Precisely this feature of the model accounts for GMV's (1995) puzzle discussed in the introduction. This paper assumes the addilog utility with $1/\gamma = 1$, calibrates the model to reproduce the distribution of wealth (fact (v)), and finds that such model generates undesirable predictions at both individual and aggregate levels. Indeed, our analysis implies that in this case regularities (i)–(iv) are violated.

There is another implication that follows from (25). Specifically, if the utility is of the addilog type and if the intertemporal elasticities of substitution for consumption and leisure, $1/\gamma$ and $1/\sigma$, are large enough, then the model might be able to reproduce facts (iii) and (v) simultaneously. Our previous analysis suggests that in this case, the model's predictions will also be consistent with the remaining empirical regularities (i), (ii) and (iv). Given that the model can account for all empirical facts of interest only under the assumption of the addilog preferences, we will study the quantitative implications of the model only under this type of preferences.

4. Calibration procedure

This section discusses the calibration procedure. To compute numerical predictions, we use the model which is adjusted to growth as shown in Appendix C. The solution algorithm is described in Appendix D.

For studying quantitative implications of the model, specific values need to be assigned to the model parameters. We calibrate the model to reproduce the capital to output ratio, π_k, the consumption to output ratio, π_c, and aggregate hours worked, n, in the U.S. economy. Output is produced according to the Cobb–Douglas production function, $f(k_t, h_t) = k_t^{\alpha} h_t^{1-\alpha}$. The parameter choice is summarized in Table 1.

Here, $\tilde{\delta}$ is the discount rate in the model with growth, and g is the rate of technological progress. The parameters $\tilde{\delta}$, d, B are computed from the optimality conditions evaluated in the steady state as it is shown in Appendix C. We assume that initially the economy is in the steady state, i.e. we set $\theta_0 = 1$ and choose aggregate capital, k_0, to be equal to the steady-state value. The remaining parameters are set to the values standard in the RBC literature. The ratios π_k and π_c and the values of the parameters g and α are adopted from Christiano and Eichenbaum (1992); following their paper, we define consumption in π_c as a sum of private and government consumption. The value of n is taken from the micro study by Juster and Stafford (1991). Finally, the aggregate technology shock θ_t has the time series representation $\theta_t = \theta_{t-1}^{\rho} \exp(\varepsilon_t)$, $\rho \in [0, 1]$, where $\varepsilon_t \sim \mathrm{N}(0, \sigma_\varepsilon^2)$ for $\forall t \in T$. The parameters ρ and σ_ε are calibrated as in Hansen (1985).

Table 1
Model parameters

Parameter	π_k	π_c	n	g	δ	α	d	ρ	σ_ε
Value	10.62	0.727	0.31	1.004	0.993	0.339	0.0217	0.95	0.00712

Table 2
Heterogeneity parameters generated by the household data

	Groups							
Parameter	1	2	3	4	5	6	7	8
κ_s	0.268	0.730	0.419	0.600	0.796	1.270	1.853	1.904
β_s	0.123	0.272	0.624	0.771	0.946	1.129	1.643	2.120
$d\omega_s$	0.173	0.640	1.274	1.781	0.806	1.624	1.188	0.513

Note: Each heterogeneity parameter is the average of corresponding individual variable in the subsample. The share, $d\omega_s$, is the relative weight of the subsample in the total population. The shares and the heterogeneity parameters weighted by the shares are normalized to 8.

To calibrate the heterogeneity parameters, we use the PSID sample (1989). This data set contains the observations on 7114 U.S. households. Following Kydland (1984), we split the population into the groups by the level of education of the head of the household. We remove from the PSID sample 96 households for which the level of education of the head is not available. As a proxy for skills, we use average hourly earnings of the head of the household. Initial endowment is calibrated according to the total wealth of the household. To compute the averages, we adjust the households' observations to PSID sample weights. Exact labels of the cross sections used are $V17545$, $V17536$, $V17389$, $V17612$. Table 2 summarizes the estimates.

The groups 1–8 distinguished in the table correspond to subsamples of the households, whose heads completed: (1) grades 0–5; (2) grades 6–8; (3) grades 9–11; (4) grade 12 grade (high school); (5) grade 12 plus non-academic training; (6) college but no degree; (7) college BA but no advanced degree; (8) college and advanced or professional degree.

5. Results

The quantitative implications of the model to a large extent depend on the choice of the parameters γ and σ. To illustrate the tendencies, we report the results from simulations for several pairs of γ and σ.

5.1. Distributive predictions

This section reports the model's predictions on consumption and working hours of eight heterogeneous groups distinguished in Section 4 and compares them with the corresponding quantities in the U.S. economy. As a proxy for consumption, we use monetary income of households. Working hours in the

U.S. economy are calibrated according to those worked by the head of the household. The groups' quantities in the U.S. economy are the averages of individual variables in the subsamples. To compute the averages, we adjust the households' observations to PSID sample weights. Exact labels of the cross sections used are $V16335$, $V17533$. The predictions of the model are computed using Eqs. (14) and (16) which are evaluated in the steady state. Table 3 presents the results.

In Section 3.3, we show qualitatively that the values of the parameters γ and σ play a decisive role in distributive predictions of the model. The results in Table 3 allow us to evaluate their effect quantitatively. As we can see, the tendency is that, if $\gamma > 1$, the model fits relatively well the empirical distribution of consumption; however, it counterfactually predicts that working hours decrease with the level of skills. If $\gamma < 1$, the model generates the appropriate predictions for hours worked; however, it fails to produce the appropriate variability of consumption across groups.

The effect of the parameter σ on the groups' consumption is determined by the value of γ. In such a way, the variability of consumption across groups increases with σ if $\gamma > 1$, and it decreases with σ if $\gamma < 1$. The variability of hours worked decreases with σ under any value of γ; this implies that if $\gamma > 1$, an increment in σ induces the low skilled agents to work less and high skilled agents to work more, while if $\gamma < 1$, the opposite is true. If $\gamma \simeq 1$, the effect of the parameter σ on the groups' quantities is ambiguous and small.

Only in two cases from all those reported in Table 3, namely, $\gamma = 0.6$, $\sigma = 1.0$ and $\gamma = 0.6$, $\sigma = 0.2$, does the model generate the distributional patterns for both consumption and working hours as in the U.S. economy. The fit of the model is somewhat better under $\sigma = 1.0$ than under $\sigma = 0.2$, although in both cases the variability of consumption across groups is too large. In particular, the model severely underpredicts the value of consumption for two bottom-skill groups.

In fact, the model's failure with respect to individual consumption of unskilled individuals is not surprising. In the real economies, a large portion of the expenditures of such individuals comes from government transfers and public services which are not included in the model. For example, Díaz-Giménez et al. (1996) divide the U.S. population into three groups according to the level of education; they find that the share of transfers in the total income of the bottom group (28%) is about six times as high as that of the top group (4.7%).

5.2. Aggregate predictions

Table 4 contains the estimates of the parameters X and ξ, and selected second moments computed using the series for efficiency hours, h_t, and physical hours worked, n_t. With slight abuse of notation, in this section, σ_x, $corr(x, y)$ will

Table 3
Groups' consumption and working hours for U.S. and artificial economies

	Heterogeneous model													
	$\gamma = 1.5$, $\sigma = 10.0$		$\gamma = 1.5$, $\sigma = 1.0$		$\gamma = 1.0$, $\sigma = 1.0$		$\gamma = 0.6$, $\sigma = 1.0$		$\gamma = 0.6$, $\sigma = 0.2$		$\gamma = 0.6$, $\sigma = 0.15$		U.S. economy	
#	c_s	n_s	c_s	n_s	c_s	n_s	c_s	n_s	c_s	n_s	c_s	n_s	c_s	n_s
1	0.15	0.35	0.21	0.48	0.13	0.29	0.05	0.08	0.03	0.03	0.03	0.02	0.35	0.05
2	0.34	0.33	0.39	0.38	0.29	0.28	0.16	0.14	0.12	0.07	0.11	0.12	0.51	0.12
3	0.64	0.32	0.71	0.34	0.62	0.32	0.49	0.28	0.43	0.25	0.43	0.25	0.64	0.26
4	0.78	0.32	0.84	0.32	0.77	0.32	0.66	0.30	0.60	0.28	0.60	0.28	0.78	0.30
5	0.95	0.31	0.98	0.31	0.94	0.31	0.88	0.32	0.83	0.32	0.83	0.32	0.95	0.34
6	1.14	0.30	1.12	0.28	1.13	0.31	1.13	0.34	1.11	0.35	1.10	0.35	1.10	0.36
7	1.61	0.29	1.48	0.25	1.65	0.31	1.86	0.39	1.97	0.44	1.98	0.44	1.55	0.39
8	2.00	0.29	1.78	0.24	2.11	0.31	2.56	0.43	2.88	0.51	2.92	0.52	2.00	0.39

Note: Consumption, c_s, and hours worked, n_s, weighted by the shares are normalized to 8 and to 8×0.31 respectively.

Table 4
Endogeneous parameters and selected labor statistics for U.S. and artificial economies

	Heterogeneous model[a]							
	$\gamma = 1.5$ $\sigma = 1.0$	$\gamma = 1.0$ $\sigma = 1.0$	$\gamma = 1.0$ $\sigma = 0.2$	$\gamma = 0.6$ $\sigma = 1.0$	$\gamma = 0.6$ $\sigma = 0.3$	$\gamma = 0.6$ $\sigma = 0.2$	$\gamma = 0.6$ $\sigma = 0.15$	U.S. economy
X	1.145	1.000	0.992	0.894	0.897	0.898	0.898	—
ξ	1.031	0.998	0.997	0.951	0.930	0.926	0.923	—
σ_h	0.57	0.70	1.15	0.83	1.33	1.47	1.58	—
	(0.07)	(0.09)	(0.14)	(0.11)	(0.16)	(0.18)	(0.20)	
σ_n	0.51	0.70	1.16	0.97	1.65	1.85	2.01	1.66[b]
	(0.06)	(0.09)	(0.14)	(0.13)	(0.20)	(0.23)	(0.26)	
$\sigma_{y/h}$	0.73	0.69	0.55	0.64	0.50	0.45	0.43	—
	(0.09)	(0.09)	(0.08)	(0.09)	(0.07)	(0.07)	(0.07)	
$\sigma_{y/n}$	0.78	0.69	0.54	0.51	0.25	0.23	0.25	1.18[b]
	(0.10)	(0.09)	(0.08)	(0.07)	(0.05)	(0.05)	(0.05)	
$corr(y/h, h)$	0.91	0.94	0.86	0.96	0.90	0.87	0.83	—
	(0.02)	(0.02)	(0.03)	(0.01)	(0.02)	(0.03)	(0.03)	
$corr(y/n, n)$	0.92	0.94	0.86	0.93	0.48	0.05	− 0.29	− 0.20[c]
	(0.02)	(0.02)	(0.03)	(0.02)	(0.05)	(0.05)	(0.09)	
$corr(y/h, y)$	0.98	0.98	0.94	0.99	0.94	0.92	0.89	—
	(0.00)	(0.00)	(0.01)	(0.00)	(0.01)	(0.01)	(0.02)	
$corr(y/n, y)$	0.99	0.98	0.99	0.97	0.59	0.17	− 0.17	0.42[b]
	(0.00)	(0.00)	(0.00)	(0.01)	(0.04)	(0.06)	(0.11)	

[a]The standard deviations and correlations are sample averages of statistics computed for each of 400 simulations; each simulation consists of 115 periods. Numbers in parentheses are sample standard deviations of these statistics. All statistics are computed after first logging and then detrending the simulated time series using the Hodrick–Prescott filter.
[b]Source: Hansen (1985, Table 1).
[c]Source: Christiano and Eichenbaum (1992, Table 4).

represent the volatility of a logged variable x and correlation between logged variables x and y.

As we see from Table 4, in all of the cases considered, the values of X differ from one by less than 15%. In Section 3.1, we have argued that if the parameter X is not significantly different from one, then efficiency hours in the heterogeneous model behave in the same way as physical hours worked in the representative agent case. To verify this conjecture, we computed the solutions under $X = 1$ and compared the resulting statistics to those reported in the table. We find that the effect of this restriction on the model's predictions is fairly small (a few percent, at most). Therefore, the statistics σ_h, $\sigma_{y/h}$, $corr(y/h, h)$ and $corr(y/h, y)$ can be viewed as the volatilities of hours worked and productivity, and the correlations between productivity and hours worked and between productivity and output in the associated representative agent model.

Hansen (1985) and Christiano and Eichenbaum (1992) consider the representative agent version of the model under $\gamma = 1$ and $\sigma = 1$. Hansen (1985) shows that such a model underpredicts the volatility of hours worked and fails to account for the large fluctuations in hours compared to the relatively small fluctuations in productivity. Christiano and Eichenbaum (1992) point out another drawback of the model, precisely, its failure to produce the appropriate correlation between productivity and hours worked. Also, as follows from the table, the correlation between productivity and output in such a model is far from the one observed in the data. The results reported in Table 4 indicate that a variation in the values of γ and σ improves on statistics σ_h and $\sigma_h/\sigma_{y/h}$; however, it has only a minor effect on $corr(y/h, h)$ and $corr(y/h, y)$. In particular, our findings suggest that the representative agent version of the model cannot account for the weak correlation between productivity and hours worked under any reasonable values of γ and σ.

As is argued in Section 3.1, accounting for labor input bias, ξ, may bring the model into closer conformity with the data provided that $\xi < 1$. From Table 4, we can observe the following tendency. If $\gamma \simeq 1$, then $\xi \simeq 1$ and the effect of σ on ξ is ambiguous and small. If $\gamma > 1$, then $\xi > 1$ and it decreases with σ; finally, if $\gamma < 1$, then $\xi < 1$ and it increases with σ. In particular, under the elasticities $1/\gamma$ and $1/\sigma$, which are large enough, the value of ξ is substantially smaller than one, and, consequently, the effect of heterogeneity on aggregate dynamics is quantitatively significant.

Table 4 shows that under $\gamma = 0.6$ and $\sigma \in \{0.3, 0.2, 0.15\}$, the model is capable of accounting for the weak correlation between productivity and hours worked; it also generates the correlation between productivity and output which is close to that observed in the data. Finally, it reproduces the empirical observation discussed in Section 3 that the volatility of physical hours worked, σ_n, is larger than the volatility of efficiency hours, σ_h. Unfortunately, under these γ and σ, the model also has undesirable features. Specifically, the volatility of productivity is too low and the model's predictions are not robust to small changes in the parameters.

To understand the origin of the shortcomings, consider the correlation between productivity and hours worked

$$corr(y/n, n) = \frac{corr(y, n)\sigma_y - \sigma_n}{\sigma_{y/n}} = \frac{corr(y, n)\sigma_y - \sigma_n}{\sqrt{\sigma_y^2 + \sigma_n^2 - 2\,corr(y, n)\sigma_y\sigma_n}}. \quad (26)$$

If the model is to generate zero correlation between productivity and hours worked, it is necessary that $\sigma_y\, corr(y, n) \simeq \sigma_n$. Given that in our model $corr(y, n)$ is close to one (see Table 5), the preceding condition together with (26) implies both that $\sigma_{y/n}$ is small and that $corr(y/n, n)$ is highly sensitive to small changes in statistics σ_n and σ_y. It can be reasonably expected that any modification which reduces $corr(y, n)$ will improve the model's performance. Two examples of such

Table 5
Selected statistics for U.S. and artificial economies

	RA model[a]	Heterogeneous model[b]						
	$\gamma = 1.0$ $\sigma = 1.0$	$\gamma = 1.5$ $\sigma = 1.0$	$\gamma = 1.0$ $\sigma = 1.0$	$\gamma = 1.0$ $\sigma = 0.2$	$\gamma = 0.6$ $\sigma = 1.0$	$\gamma = 0.6$ $\sigma = 0.2$	$\gamma = 0.6$ $\sigma = 0.15$	U.S. economy[a]
n_t	—	0.31	0.31	0.31	0.31	0.31	0.31	0.31[c]
		(0.00)	(0.00)	(0.01)	(0.01)	(0.01)	(0.01)	
h_t	—	0.29	0.31	0.31	0.34	0.36	0.36	—
		(0.00)	(0.00)	(0.01)	(0.01)	(0.01)	(0.01)	
σ_c	0.42	0.35	0.41	0.47	0.43	0.53	0.55	1.29
	(0.06)	(0.05)	(0.06)	(0.07)	(0.08)	(0.09)	(0.10)	
σ_k	0.36	0.33	0.36	0.43	0.39	0.50	0.53	0.63
	(0.07)	(0.07)	(0.08)	(0.09)	(0.09)	(0.10)	(0.12)	
σ_i	4.24	3.78	4.07	4.99	4.51	5.84	6.18	8.60
	(0.51)	(0.45)	(0.53)	(0.61)	(0.59)	(0.72)	(0.80)	
σ_y	1.35	1.27	1.37	1.65	1.45	1.88	1.95	1.76
	(0.16)	(0.15)	(0.18)	(0.20)	(0.19)	(0.23)	(0.25)	
$corr(c, y)$	0.89	0.94	0.90	0.89	0.79	0.79	0.77	0.85
	(0.03)	(0.01)	(0.02)	(0.02)	(0.03)	(0.03)	(0.03)	
$corr(n, y)$	0.98	0.97	0.98	0.99	0.99	0.99	0.98	0.76
	(0.01)	(0.01)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	
$corr(h, y)$	—	0.97	0.98	0.99	0.99	0.99	0.99	—
		(0.01)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	
$corr(i, y)$	0.99	1.00	0.99	0.99	0.99	0.99	0.99	0.92
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	

[a]Source: Except for [c], Hansen (1985, Table 1).
[b]The standard deviations and correlations are sample averages of statistics computed for each of 400 simulations; each simulation consists of 115 periods. Numbers in parentheses are sample standard deviations of these statistics. All statistics are computed after first logging and then detrending the simulated time series using the Hodrick–Prescott filter.
[c]Juster and Stafford (1991) is the source.

modifications are incorporating home production and allowing for variation in labor input along both the hours-per-worker margin and the employment margin.[3]

In Table 5, we report the remaining statistics for the U.S. and artificial economies. For comparison, we also include the predictions of the representative agent version of the model under $\gamma = 1$, $\sigma = 1$. As we see from the table, in our model all the statistics are similar to those in the representative agent model.

[3] Introducing home production and the two margins in the representative agent model reduces $corr(y, n)$ from 0.99 to 0.91 and 0.75 respectively (Source: Kydland (1995, Table 5.2)).

Observe that under γ and σ which are lower than one, the volatilities of consumption, capital, investment and output are closer to the corresponding volatilities in the data.

Summarizing, on the one hand, the heterogeneous agents model improves on some labor market statistics which the representative agent model seriously fails to predict. On the other hand, the model with heterogeneity preserves the positive features of the representative agent setup.

6. Conclusion

This paper incorporates into the neoclassical model with labor–leisure choice heterogeneous agents who differ with respect to endowments and non-acquired skills. We show that the qualitative and quantitative analysis of the equilibrium in the model can be simplified substantially by using the results from aggregation. We demonstrate that under the standard for the macroeconomic literature assumptions on preferences, the model cannot account for cross-sectional observations. Specifically, we find that if agents' preferences are of the Cobb–Douglas type, the model fails to reproduce the simple empirical regularity that high-productivity agents work more compared to the low productivity. We reach the same conclusion for the case of addilog utility with the standard (smaller than one) intertemporal elasticity of substitution for consumption. We show, however, that if such elasticity in the addilog utility is set to a value which is large enough, then the model is capable of generating the appropriate pattern for hours worked. Moreover, such model can account for several time-series labor market facts which cannot be reconciled within a similar representative agent setup.

Appendix A

With an interior solution, the first-order conditions (FOCs) of agent's $s \in S$ utility maximization problem (1), (2) with respect to insurance holdings, capital holdings and the transversality conditions are as follows:

$$u_1(c_t^s, n_t^s)p_t(\theta) = \delta u_1(c_{t+1}^s(\theta), n_{t+1}^s(\theta)) \Pr\{\theta_{t+1} = \theta \mid \theta_t = \theta'\}_{\theta,\theta' \in \Theta}, \tag{A.1}$$

$$u_1(c_t^s, n_t^s) = \delta \mathrm{E}_t[u_1(c_{t+1}^s, n_{t+1}^s)(1 - d + r_{t+1})], \tag{A.2}$$

$$\lim_{t \to \infty} \mathrm{E}_0\left[\delta^t u_1(c_t^s, n_t^s)\left(k_{t+1}^s + \int_\Theta p_t(\theta) m_{t+1}^s(\theta)\,\mathrm{d}\theta\right)\right] = 0, \tag{A.3}$$

where u_1, u_2 are the first-order partial derivatives of the utility u with respect to consumption and labor and $c_{t+1}^s(\theta)$, $n_{t+1}^s(\theta)$ are equilibrium consumption and working hours as functions of the realization of the aggregate shock.

FOC (27) implies

$$\mathrm{E}_{t-1}\left[\delta\frac{u_1(c_t^s,n_t^s)}{u_1(c_{t-1}^s,n_{t-1}^s)}\,m_t^s(\theta_t)\right]=\int_{\Theta}m_t^s(\theta)p_{t-1}(\theta)\,\mathrm{d}\theta. \tag{A.4}$$

Further, FOC (28) together with the fact that k_t^s is known at $t-1$ yields

$$\mathrm{E}_{t-1}\left[\delta\frac{u_1(c_t^s,n_t^s)}{u_1(c_{t-1}^s,n_{t-1}^s)}\,k_t^s(1-d+r_t)\right]=k_t^s. \tag{A.5}$$

Multiplying each term of (2) by

$$\frac{\delta u_1(c_t^s,n_t^s)}{u_1(c_{t-1}^s,n_{t-1}^s)},$$

taking the expectation E_{t-1} on both sides and using Eqs. (A.5) and (A.4), one can show that for all $t>0$ the following condition holds:

$$\begin{aligned}
&k_t^s+\int_{\Theta}m_t^s(\theta)p_{t-1}(\theta)\,\mathrm{d}\theta\\
&\quad=\mathrm{E}_{t-1}\left[\delta\frac{u_1(c_t^s,n_t^s)}{u_1(c_{t-1}^s,n_{t-1}^s)}(c_t^s-n_t^se^sw_t)\right]\\
&\qquad+\ \mathrm{E}_{t-1}\left[\delta\frac{u_1(c_t^s,n_t^s)}{u_1(c_{t-1}^s,n_{t-1}^s)}\left(k_{t+1}^s+\int_{\Theta}m_{t+1}^s(\theta)p_t(\theta)\,\mathrm{d}\theta\right)\right].
\end{aligned}$$

Applying forward recursion, using the law of iterative expectations and imposing transversality condition (A.3), we get

$$\begin{aligned}
&(1-d+r_0)k_0^s+m_0^s(\theta_0)\\
&\quad=c_0^s-n_0^se^sw_0+k_1^s+\int_{\Theta}m_1^s(\theta)p_0(\theta)\,\mathrm{d}\theta\\
&\quad=\mathrm{E}_0\left[\sum_{\tau=0}^{1}\delta^t\frac{u_1(c_\tau^s,n_\tau^s)}{u_1(c_0^s,n_0^s)}(c_\tau^s-n_\tau^se^sw_\tau)\right]\\
&\qquad+\mathrm{E}_0\left[\delta\frac{u_1(c_1^s,n_1^s)}{u_1(c_0^s,n_0^s)}\left(k_2^s+\int_{\Theta}m_2^s(\theta)p_1(\theta)\,\mathrm{d}\theta\right)\right]\\
&\quad=\cdots=\mathrm{E}_0\left[\sum_{\tau=0}^{\infty}\delta^\tau\frac{u_1(c_\tau^s,n_\tau^s)}{u_1(c_0^s,n_0^s)}(c_\tau^s-n_\tau^se^sw_\tau)\right]\\
&\qquad+\lim_{t\to\infty}\mathrm{E}_0\left[\delta^t\frac{u_1(c_t^s,n_t^s)}{u_1(c_0^s,n_0^s)}\left(k_{t+1}^s+\int_{\Theta}m_{t+1}^s(\theta)p_t(\theta)\,\mathrm{d}\theta\right)\right]\\
&\quad=\mathrm{E}_0\left[\sum_{\tau=0}^{\infty}\delta^\tau\frac{u_1(c_\tau^s,n_\tau^s)}{u_1(c_0^s,n_0^s)}(c_\tau^s-n_\tau^se^sw_\tau)\right].
\end{aligned}$$

The last result corresponds to expected lifetime budget constraint (6) used in the main text.

Appendix B

The FOCs of planner's problem (4), (5) with respect to c_t^s, n_t^s, k_t are

$$\lambda_s u_1(c_t^s, 1 - n_t^s) = \zeta_t, \tag{B.1}$$

$$\lambda_s u_2(c_t^s, 1 - n_t^s) = \zeta_t e^s w_t, \tag{B.2}$$

$$\zeta_t = \delta \mathrm{E}_t[\zeta_{t+1}(1 - d + r_{t+1})], \tag{B.3}$$

where ζ_t is the Lagrange multiplier associated with the economy's resource constraint, $w_t \equiv \theta_t f_2(k_t, h_t)$ is marginal product of labor input.

Proof of Proposition 2. Under the Cobb–Douglas utility, solving for c_t^s and $(1 - n_t^s)$ from FOCs (B.1), (B.2), we get

$$c_t^s = \left[\frac{\mu}{\zeta_t}\left(\frac{1-\mu}{\mu w_t}\right)^{(1-\mu)(1-\eta)}\right]^{1/\eta} (\lambda^s)^{1/\eta}(e^s)^{-(1-\mu)(1-\eta)/\eta}, \tag{B.4}$$

$$1 - n_t^s = \left[\frac{1-\mu}{\zeta_t}\left(\frac{1-\mu}{\mu w_t}\right)^{-\mu(1-\eta)}\right]^{1/\eta} (\lambda^s)^{1/\eta}(e^s)^{(\mu(1-\eta)-1)/\eta}. \tag{B.5}$$

Integration of (B.4) and (B.5) over the set of agents (both sides of the latter are previously premultiplied by e_s) yields

$$c_t = \left[\frac{\mu}{\zeta_t}\left(\frac{1-\mu}{\mu w_t}\right)^{(1-\mu)(1-\eta)}\right]^{1/\eta} \int_S (\lambda^s)^{1/\eta}(e^s)^{-(1-\mu)(1-\eta)/\eta}\, \mathrm{d}\omega^s, \tag{B.6}$$

$$1 - h_t = \left[\frac{1-\mu}{\zeta_t}\left(\frac{1-\mu}{\mu w_t}\right)^{-\mu(1-\eta)}\right]^{1/\eta} \int_S (\lambda^s)^{1/\eta}(e^s)^{-(1-\mu)(1-\eta)/\eta}\, \mathrm{d}\omega^s. \tag{B.7}$$

Dividing (B.4) by (B.6) and (B.5) by (B.7) and introducing f^s, we obtain (9) in the main text.

Rearranging the terms of Eqs. (B.6) and (B.7), we get

$$u_1(c_t, 1 - h_t) = \zeta_t \left(\int_S (\lambda^s)^{1/\eta}(e^s)^{-(1-\mu)(1-\eta)/\eta}\, \mathrm{d}\omega^s\right)^{-\eta}, \tag{B.8}$$

$$u_2(c_t, 1 - h_t) = \zeta_t w_t \left(\int_S (\lambda^s)^{1/\eta}(e^s)^{-(1-\mu)(1-\eta)/\eta}\, \mathrm{d}\omega^s\right)^{-\eta}, \tag{B.9}$$

where $u(c_t, 1-h_t) = [(c_t^{\mu}(1-h_t)^{1-\mu})^{1-\eta} - 1]/(1-\eta)$. Combining (B.8), (B.9) yields the FOC of (8) with respect to labor; substituting (B.8) into (B.3) gives us the FOC with respect to capital. These are, respectively,

$$u_2(c_t, 1-h_t) = w_t u_1(c_t, 1-h_t),$$

$$u_1(c_t, 1-h_t) = \delta \mathrm{E}_t[u_1(c_{t+1}, 1-h_{t+1})(1-d+r_{t+1})].$$

This proves that aggregate dynamics of the heterogeneous economy is described by single-agent utility maximization problem (8). Finally, condition (10) follows after substituting into expected lifetime budget constraint (6) both conditions given in (9). □

Proof of Proposition 3. Under the addilog utility, conditions (B.1), (B.2) become

$$c_t^s = \zeta_t^{-1/\gamma}(\lambda^s)^{1/\gamma}, \tag{B.10}$$

$$1 - n_t^s = (\zeta_t w_t)^{-1/\sigma} B^{1/\sigma} (\lambda^s)^{1/\sigma} (e^s)^{-1/\sigma}. \tag{B.11}$$

Integrating them over the set of agents (first, premultiplying the latter by e^s) yields

$$c_t = \zeta_t^{-1/\gamma} \int_S (\lambda^s)^{1/\gamma} \, \mathrm{d}\omega^s, \tag{B.12}$$

$$1 - h_t = (\zeta_t w_t)^{-1/\sigma} B^{1/\sigma} \int_S (\lambda^s)^{1/\sigma} (e^s)^{1-1/\sigma} \, \mathrm{d}\omega^s. \tag{B.13}$$

Taking the ratios of (B.10) to (B.12) and (B.11) to (B.13), introducing f^s and defining the parameter X as it is in (13) of the main text, we get the two conditions in (14).

Rearranging the terms in (B.12) and (B.13), we have

$$u_1(c_t, 1-h_t) = \zeta_t \left(\int_S (\lambda^s)^{1/\gamma} \, \mathrm{d}\omega^s \right)^{-\gamma}, \tag{B.14}$$

$$u_2(c_t, 1-h_t) = \zeta_t w_t \left(\int_S (\lambda^s)^{1/\sigma} (e^s)^{1-1/\sigma} \, \mathrm{d}\omega^s \right)^{-\sigma}, \tag{B.15}$$

where $u(c_t, 1-h_t) = (c_t^{1-\gamma} - 1)/(1-\gamma) + B((1-h_t)^{1-\sigma} - 1)/(1-\sigma)$. Eqs. (B.14), (B.15) combined together yield the FOC of single-agent problem (12) with respect to h_t. Further, substituting (B.14) in (B.3) we get the corresponding intertemporal condition of (12). They are, respectively,

$$X u_2(c_t, 1-h_t) = w_t u_1(c_t, 1-h_t),$$

$$u_1(c_t, 1-h_t) = \delta \mathrm{E}_t[u_1(c_{t+1}, 1-h_{t+1})(1-d+r_{t+1})].$$

This verifies that at the aggregate level the heterogeneous economy behaves as single-agent economy (12). Finally, after substituting (14) into expected lifetime budget constraint (6) we get (15). □

Appendix C

We introduce growth as it is usually done in the RBC literature. In the economy with growth, the problem of an individual $s \in S$ is

$$\max_{\{c_t^s,\, n_t^s,\, k_{t+1}^s,\, m_{t+1}^s(\theta)\}_{\theta\in\Theta,\, t\in T}} \mathrm{E}_0 \sum_{t=0}^{\infty} \delta^t \left\{ \frac{(c_t^s)^{1-\gamma}-1}{1-\gamma} + B\frac{(1-n_t^s)^{1-\sigma}-1}{1-\sigma} g^{t(1-\gamma)} \right\}$$

s.t.

$$c_t^s + k_{t+1}^s + \int_\Theta p_t(\theta) m_{t+1}^s(\theta)\,\mathrm{d}\theta = (1-d+r_t)k_t^s + n_t^s e^s w_t g^t + m_t^s(\theta_t),$$

where the parameter g is the rate of labor-augmenting technological progress. Finding the FOCs of the above problem, introducing new variables $\tilde{c}_t^s = c_t^s g^{-t}$, $\tilde{k}_t^s = k_t^s g^{-t}$ and $\tilde{m}_t^s = m_t^s g^{-t}$ and following the procedure described in Appendix B, we obtain the conditions that identify aggregate dynamics:

$$g\tilde{c}_t^{-\gamma} = \tilde{\delta}\mathrm{E}_t[\tilde{c}_{t+1}^{-\gamma}(1-d+\tilde{r}_{t+1})], \tag{C.1}$$

$$\tilde{c}_t^{-\gamma}\tilde{w}_t = BX(1-h_t)^{-\sigma}, \tag{C.2}$$

$$\tilde{c}_t + \tilde{k}_{t+1}g = \tilde{k}_t(1-d) + \theta_t f(\tilde{k}_t, h_t), \tag{C.3}$$

where $\tilde{r}_t \equiv \theta_t \partial f(\tilde{k}_t, h_t)/\partial \tilde{k}_t$, $\tilde{w}_t \equiv \theta_t \partial f(\tilde{k}_t, h_t)/\partial h_t$, the parameter X is defined by (C.7), and $\tilde{\delta} \equiv \delta g^{1-\gamma}$ is the discount rate adjusted to growth. The expected lifetime budget constraint takes the form

$$\mathrm{E}_0 \sum_{\tau=0}^{\infty} \tilde{\delta}^\tau \frac{\tilde{c}_\tau^{-\gamma}}{\tilde{c}_0^{-\gamma}} [\tilde{c}_\tau f^s - \tilde{w}_\tau e^s(1-(1-h_\tau)X^{-1/\sigma}(e^s)^{-1/\sigma}(f^s)^{\gamma/\sigma})] = \kappa_0^s. \tag{C.4}$$

Eqs. (14), (16), (18) do not change after introducing growth except that the appropriate variables in (14) are $\tilde{c}_t^s$ and $\tilde{c}_t$.

Optimality conditions (C.1)–(C.3) provide a basis for calibrating the parameters $\tilde{\delta}$, d and B. Evaluating (C.1) and (C.3) in the steady state yields

$$\tilde{\delta} = \frac{\pi_k g}{\pi_k g + \pi_c + \alpha - 1}, \qquad d = \frac{1-\pi_c}{\pi_k} + 1 - g.$$

Condition (C.2) in the steady state can be written as

$$BX = (1-\alpha)\pi_k^{(1-\gamma)\alpha/(1-\alpha)}\pi_c^{-\gamma}(1-h)^\sigma h^{-\gamma}, \tag{C.5}$$

where h denotes steady-state efficiency labor. The parameter B is calibrated to the same value as in the representative agent case. Setting $X = 1$ and $\xi = 1$, we get

$$B = (1-\alpha)\pi_k^{(1-\gamma)\alpha/(1-\alpha)}\pi_c^{-\gamma}(1-n)^{\sigma}n^{-\gamma}. \tag{C.6}$$

Dividing (C.5) by (C.6) and substituting the resulting condition into (16) evaluated in the steady state, we obtain

$$X = \xi^{\sigma}n^{\gamma}(1-(1-n)\xi)^{-\gamma}. \tag{C.7}$$

This formula relates the parameters X and ξ.

Appendix D

To compute the solution, we use the following iterative algorithm:

Step 1. Fix ξ to some level and compute X according to (C.7).
Step 2. Use (C.1)–(C.3) to solve for aggregate equilibrium quantities.
Step 3. Recover f^s's from (C.4) and recompute ξ according to (18).

Iterate on these steps until the fixed point value of ξ is found.

Once the equilibrium law of motion for the aggregate quantities and the corresponding set of the agent-specific parameters $\{f^s\}^{s\in S}$ is known, the remaining variables can be restored by direct calculations.

To complete Step 2 of the solution algorithm, we use parametrized expectations algorithm, see, e.g., Marcet (1989). Under this algorithm, the conditional expectation in (C.1) is parameterized by a function of the state variables and, subsequently, iterations on the parameters of this function are performed until the equilibrium law of motion for the marginal utility is found. As a function parametrizing (C.1), we use second-order degree exponentiated polynomial; the length of simulations is 10,000.

To compute the expectations in expected lifetime budget constraints (C.4), we follow the approach described in GMV (1995). Under this approach, the expected infinite sum is approximated by the average of N simulations of length T

$$\mathrm{E}_0 \sum_{\tau=0}^{\infty} \left(\frac{\tilde{c}_\tau^{-\gamma}}{\tilde{c}_0^{-\gamma}}\right)\tilde{\delta}^{\tau} z_\tau \simeq \frac{1}{N}\sum_{n=1}^{N}\sum_{\tau=0}^{T}\left(\frac{\tilde{c}_\tau^{-\gamma}}{\tilde{c}_0^{-\gamma}}\right)\tilde{\delta}^{\tau} z_\tau, \tag{D.1}$$

where $z_\tau \in \{\tilde{c}_\tau, \tilde{w}_\tau, \tilde{w}_\tau(1-h_\tau)\}$. To obtain a precise estimate, both N and T have to be large. Since it is computationally costly to set both N and T large, the right-hand side of (D.1) is subdivided in two parts, the head and the tail. Subsequently, the head is computed as average of N short draws of length T'

and the tail is approximated by a single long draw of the length T''

$$\frac{1}{N}\sum_{n=1}^{N}\sum_{\tau=0}^{T}\left(\frac{\tilde{c}_\tau^{-\gamma}}{\tilde{c}_0^{-\gamma}}\right)\tilde{\delta}^\tau z_\tau \simeq \frac{1}{N}\sum_{n=1}^{N}\sum_{\tau=0}^{T'}\left(\frac{\tilde{c}_\tau^{-\gamma}}{\tilde{c}_0^{-\gamma}}\right)\tilde{\delta}^\tau z_\tau + \tilde{\delta}^{T'}\sum_{\tau=0}^{T''}\left(\frac{\tilde{c}_\tau^{-\gamma}}{\tilde{c}_0^{-\gamma}}\right)\tilde{\delta}^\tau z_\tau.$$

We choose $N = 400$, $T' = 115$, $T'' = 10{,}000$. As is argued in GMV (1995), a similar choice guarantees substantial accuracy of simulated solutions.

Finally, we show that a solution $\{f^s\}^{s\in S}$ to expected lifetime budget constraints (C.4) computed on Step 3 exists, is unique and such that $f^s > 0$ for $\forall s \in S$. Indeed, let $\{\tilde{c}_\tau, \tilde{w}_\tau, h_\tau\}_{\tau\in T}$ be such that $0 < \tilde{c}_\tau$, $\tilde{w}_\tau < \infty$ and $0 < h_\tau < 1$ for any $\tau \in T$. Consider functions $\psi^s(f^s)$ and $\phi^s(f^s)$ such that

$$\phi^s(f^s) \equiv X^{-1/\sigma}(e^s)^{1-1/\sigma}(f^s)^{\gamma/\sigma}\mathrm{E}_0\sum_{\tau=0}^{\infty}\tilde{\delta}^\tau\frac{\tilde{c}_\tau^{-\gamma}}{\tilde{c}_0^{-\gamma}}\tilde{w}_\tau(1-h_\tau),$$

$$\psi^s(f^s) \equiv \kappa_0^s + e^s\mathrm{E}_0\sum_{\tau=0}^{\infty}\tilde{\delta}^\tau\frac{\tilde{c}_\tau^{-\gamma}}{\tilde{c}_0^{-\gamma}}\tilde{w}_\tau - f^s\mathrm{E}_0\sum_{\tau=0}^{\infty}\tilde{\delta}^\tau\frac{\tilde{c}_\tau^{-\gamma}}{\tilde{c}_0^{-\gamma}}\tilde{c}_\tau.$$

In terms of the above functions, the agent's expected lifetime budget constraint (C.4) can be expressed as $\phi^s(f^s) = \psi^s(f^s)$. For any $X \in R_+$, we have $\phi^s(0) = 0$ and $(\phi^s)' > 0$. Further, given that $\kappa_0^s > 0$ for $\forall s \in S$, we have $\psi^s(0) > 0$ and $(\psi^s)' < 0$. Finally, the functions $\phi^s(f^s)$ and $\psi^s(f^s)$ are continuous on R_+. Thus, there exists a unique value f^s which satisfy the expected lifetime budget constraint of each agent $s \in S$. A solution $\{f^s\}^{s\in S}$ and formula (18) determine uniquely the corresponding value of the parameter ξ. Therefore, if the equilibrium exists, is interior and unique, then the value of ξ which is consistent with the equilibrium is also unique.

General results about the existence of the equilibrium are hard to achieve. Whether the equilibrium in our economy exists will depend on a particular choice of the model's parameters. To see the point, consider, for example, the case when all consumers have the preferences of Cobb–Douglas type and assume that there are some consumers whose endowment to skills ratio is very high. According to formulas (9), (10) such consumers will choose to work a negative amount of hours, which implies that there is no interior equilibrium in the model. However, in all the numerical experiments which we reported, the equilibrium exists and is interior; also, the iterative procedure had no difficulties to converge to a fixed point value of the parameter ξ.

7. For further reading

Constantinides, 1982; Cooley and Ogaki, 1996; Danthine and Donaldson, 1995; Huang, 1987; King et al., 1988; Mas-Colell et al., 1995; Negishi, 1960; Ogaki, 1997; Ríos-Rull, 1995; Rosen, 1968; Stokey and Lucas, 1989.

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