LILIA MALIAR
SERGUEI MALIAR

The Neoclassical Growth Model with Heterogeneous Quasi-Geometric Consumers

This paper investigates how the assumption of quasi-geometric (hyperbolic) discounting affects the distributional implications of the standard one-sector neoclassical growth model with infinitely lived heterogeneous agents. The agents are subject to idiosyncratic shocks and face borrowing constraints. We confine attention to an interior Markov recursive equilibrium. The consequence of quasi-geometric discounting is that the effective discount factor of an agent is not a constant, but an endogenous variable which depends on the agent's current state. We show, both analytically and by simulation, that this new feature can significantly affect the distributional implications of the neoclassical growth model.

JEL codes: D91, E21, G11
Keywords: neoclassical growth model, time inconsistency, quasi-geometric discounting, hyperbolic discounting, idiosyncratic shocks, wealth inequality.

Quasi-geometric (hyperbolic) discounting is a form of time-inconsistency in preferences when the discount factor, applied between today and tomorrow, is different from the one employed for any other date further in the future. The first studies on quasi-geometric discounting date back to Strotz (1955–1956), Pollak (1968), and Phelps and Pollak (1968), although interest in this subject has recently been revived, e.g., Laibson (1997), Laibson, Repetto, and Tobacman (1998), Barro (1999), Harris and Laibson (2001), Angeletos et al. (2001), Krusell and Smith (2000, 2003), Krusell, Kurusçu, and Smith (2002), Luttmer and Mariotti (2002).

We are very grateful to two anonymous referees and editor Paul Evans for many useful comments and suggestions. Any remaining errors are the sole responsibility of the authors. This research was supported by the Instituto Valenciano de Investigaciones Económicas and the Ministerio de Ciencia y Tecnología de España, the Ramón y Cajal program and BEC 2001-0535.

Lilia Maliar *is from the Departamento de Fundamentos del Análisis Económico, Universidad de Alicante* (*E-mail:* maliarl@merlin.fae.ua.es). Serguei Maliar *is from the Departamento de Fundamentos del Análisis Económico, Universidad de Alicante* (*E-mail:* maliars@merlin.fae.ua.es).

Received March 17, 2003; and accepted in revised form July 2, 2004.

Journal of Money, Credit, and Banking, Vol. 38, No. 3 (April 2006)

This paper contributes to the literature by establishing how the assumption of quasi-geometric discounting affects the distributional implications of the standard one-sector neoclassical growth model. We consider an economy populated by a continuum of infinitely lived quasi-geometric agents, who are subject to idiosyncratic labor productivity shocks and who face borrowing constraints. We confine our attention to an interior Markov recursive solution to the individual utility-maximization problem. We solve for a stationary equilibrium in which the prices of capital and labor are consistent with the consumption-savings decisions of the agents (as in, e.g., Aiyagari, 1994, Huggett, 1997).

Two papers in the literature that are closely related to ours are Harris and Laibson (2001) and Krusell, Kurusçu, and Smith (2002). The former paper studies the behavior of a quasi-geometric consumer in a stochastic partial-equilibrium model, under the assumption of *iid* labor-income shocks. The latter paper analyzes the implications of a deterministic general-equilibrium model with homogeneous quasi-geometric consumers. As such, our study differs from the former paper in the sense that we use a general-equilibrium approach and introduce persistent shocks to labor income, and we differ from the latter paper in that we have heterogeneity of agents, due to idiosyncratic uncertainty.

With the assumption of quasi-geometric discounting, the effective discount factor of an agent is not a constant, but rather an endogenous variable that depends on the agent's current state. In particular, we show that if the consumption function is strictly concave, then the effective discount factor of the quasi-geometric short-run impatient agent is increasing in wealth. As a result, the rich are more patient than the poor, so that the model with quasi-geometric short-run impatient agents produces a larger dispersion of wealth than does the standard setup, where rich and poor are equally patient. This implication is of interest, given that the standard model with a constant discount factor dramatically underpredicts the size of wealth inequality, relative to the data (see, e.g., Quadrini and Ríos-Rull 1997).

In a calibrated version of the model, we find that the effects associated with the assumption of quasi-geometric discounting are quantitatively significant. For example, in our benchmark model with short-run impatient agents, the wealth holdings of the bottom 40% of the population decline by 28%, while those of the top 1% increase by 10%, and the Gini coefficient of the wealth distribution increases by 18% compared to the standard geometric-discounting setup. These improvements, however, are too small for the model to reproduce the size of the wealth inequality observed in the data. Furthermore, we find that in our general-equilibrium model, the size of precautionary savings is not substantially affected by the presence of quasi-geometric discounting. In fact, under some parameterizations, the precautionary savings of quasi-geometric consumers can be even greater than those of the standard geometric consumers. This is contrary to what Laibson, Repetto, and Tobacman (1998) have obtained in a partial-equilibrium setup.

The rest of the paper is organized as follows. Section 1 formulates the model, derives the optimality conditions, defines equilibrium and discusses some of the model's implications. Section 2 describes the methodology of the quantitative study and presents the results from simulations, and finally, Section 3 concludes.

1. THE MODEL

Time is discrete and the horizon is infinite, $t \in \{0, 1, 2, \ldots\}$. The economy is populated by a continuum of infinitely lived agents with names on a closed interval [0, 1]. The agents inelastically supply their total time endowment (equal to one) to the market. The labor productivities of the agents are subject to idiosyncratic shocks. The shocks follow a first-order Markov process and are uncorrelated across agents. All possible realizations of productivity shocks are in the set $\mathcal{S} = [s_{min}, s_{max}] \subset R_+$. At each point in time, the agents also differ in asset holdings, which somehow summarize information on past realizations of shocks. Assets are restricted to be in the set $\mathcal{A} = [-b, \infty) \subset R$. That is, agents may only borrow up to a certain amount b.

In every period t, an agent seeks to maximize the expected present value of the sum of one-period utilities from t forward by choosing an optimal path for consumption. The agent discounts the utility by using the quasi-geometric weights, 1, $\beta\delta$, $\beta\delta^2$, $\beta\delta^3$, …, where the discounting parameters β and δ are such that $\beta > 0$ and $0 < \delta < 1$. Consequently, on each date t, the agent solves the following problem

$$\max_{\{c_\tau, a_{\tau+1}\}_{\tau=t}^{\infty}} \left\{ u(c_t) + E_t \sum_{\tau=t}^{\infty} \beta\, \delta^{\tau+1-t}\, u(c_{\tau+1}) \right\} \tag{1}$$

subject to

$$c_\tau + a_{\tau+1} = ws_\tau + (1 + r)a_\tau \,, \tag{2}$$

$$a_{\tau+1} \geq -b \,, \tag{3}$$

where initial condition (a_t, s_t) is given. Here, c_τ, a_τ, and s_τ are consumption, asset holdings, and the labor productivity shock, respectively; r is the interest rate; w is the wage per unit of efficiency labor; E_τ is the expectation, conditional on all information about the agent's idiosyncratic shocks being available at τ. The momentary utility function $u(c)$ is continuously differentiable, strictly increasing, strictly concave, and satisfies $\lim_{c\to 0} u'(c) = \infty$.

As is argued in the literature (e.g., Laibson 1997) one can view a quasi-geometric consumer in different periods as a collection of temporal selves, who play an infinite-horizon game. Each self t has the preferences defined over the stream of consumption $\{c_\tau\}_{\tau=t}^{\infty}$ and solves the problem (1)–(3). We assume that self t has direct control only over the current consumption, c_t, i.e., the agent cannot commit herself to future actions.

We restrict our attention to the case when $\beta < 1$, so that the short-run discount factor, $\beta\delta$, is lower than the long-run discount factor, δ. Thus, the agent is short-run impatient: she plans to save much in the future, but as the future comes around, she changes her mind and saves less than she would have originally committed to if commitment had been available. This case is often referred to in the literature as hyperbolic discounting, because of a qualitative similarity with the case where the discount factor is given by an increasing-over-time generalized hyperbolic function (see, e.g., Laibson 1997). Following Krusell and Smith (2000), we refer to the case $\beta \neq 1$ as quasi-geometric discounting because except for the current date, the weights on momentary utility functions decline geometrically.

The production side of the economy consists of a representative firm. Given the factor prices, r and w, the firm rents capital, K_t, and hires labor, N_t, to maximize period-by-period profits. The technology is described by $F(K_t, N_t) + (1 - d)K_t$. The production function, F, has constant returns to scale, is strictly increasing, strictly concave, continuously differentiable and satisfies the appropriate Inada conditions. The depreciation rate of capital is $d \in (0, 1]$.

1.1 Recursive Formulation and the Euler Equation

As shown in Harris and Laibson (2001), the problem (1)–(3) can be written recursively. To be specific, let us assume that in all periods, the agent decides on consumption according to the same consumption function, $c_t = C(a_t, s_t)$. Without time subscripts, we have the following recursive formulation:

$$W(a, s) = \max_{c}\{u(c) + \beta\delta E[V(a',s')|s]\} , \tag{4}$$

where, given (a, s), the value function V solves the functional equation

$$V(a, s) = u[C(a,s)] + \delta E\{V[ws + (1 + r)a - C(a,s);s']|s\} \tag{5}$$

subject to the budget constraint

$$a' = \mathrm{ws} + (1 + r)a - c \tag{6}$$

and the borrowing constraint

$$a' \geq -b . \tag{7}$$

The problem (4)–(7) is to be solved for the current value function, $W(a, s)$, the continuation value function, $V(a, s)$, and the consumption function, $C(a, s)$. We assume that the above functions are continuous and differentiable. These assumptions will be in force throughout the remainder of the paper.

If the problem (4)–(7) has an interior solution, then such a solution satisfies the quasi-geometric Euler equation:

$$u'(c_t) \geq \delta E_t\,\{u'(c_{t+1})[1 + r - (1 - \beta)C_a(a_{t+1}, s_{t+1})]\} , \tag{8}$$

where u' is the derivative of the utility function u, and C_a is the first-order partial derivative of the consumption function, C, with respect to assets. The Euler equation holds with strict inequality if the borrowing limit is reached.

1.2 Equilibrium

Let x be a probability measure defined on $\mathcal{B}$, where $\mathcal{B}$ denotes the Borel subset of the set of all possible individual states $\mathcal{A} \times \mathcal{S}$. For all $B \in \mathcal{B}$, $x_t(B)$ is the mass of agents whose individual states lie in B at time t.

$P(a, s, B)$ denotes the conditional probability that an agent with state (a, s) will have an individual state lying in the set B in the next period. The function P is defined as

$$P(a,s,B) = \text{Prob}(\{s' \in \mathcal{S}:[A(a,s),s'] \in B\}|s) ,$$

where $A(a, s) \equiv a' = ws + (1 + r)a - C(a,s)$ is the decision function for assets (the asset function). The law of motion of x_t then is $x_{t+1}(B) = \int_{\mathcal{A}\times\mathcal{S}} P(a, s, B)dx_t$ for all $t \in [0,\infty)$ and all $B \in \mathcal{B}$.

Labor and capital inputs are given by $N_t = \int_{\mathcal{S}} s_t\, dx_t$ and $K_t = \int_{\mathcal{A}\times\mathcal{S}} a_t\, dx_t$, respectively. With a continuum of agents, N_t is a constant; for convenience, we normalize it to one, $N_t = 1$.

We only study such equilibria in which the period-$t + 1$ probability measure x_{t+1} is the same as the period-t probability measure x_t, for all $t \in [0,\infty)$. In this case, we say that the probability measure is stationary and denote it by x^*. The stationarity of x^* implies that the aggregate capital stock is constant, $K = \int_{\mathcal{A}\times\mathcal{S}} a_t dx^*$ for all $t \in [0,\infty)$ (even though the assets of each agent vary stochastically over time).

Definition: *A stationary equilibrium is defined as a stationary probability measure x^*, an optimal consumption function $C(a, s)$, and positive real numbers (K, r, w) such that*

1. *x^* satisfies $x^* = \int_{\mathcal{A}\times\mathcal{S}} P(a,s,B)dx^*$ for all $B \in \mathcal{B}$;*
2. *$C(a, s)$ solves the Euler equation (8) for a given pair of prices (r, w);*
3. *(r, w) are such that the firm's profit is maximum*

$$r = F_K(K,1) - d,\; w = F_N(K,1) ,$$

where F_K and F_N are the first-order partial derivatives of the production function, F, with respect to capital and labor inputs, respectively;

4. *K is the average of the agents' decisions: $K = \int_{\mathcal{A}\times\mathcal{S}} A(a, s)dx^*$.*

Thus, we focus exclusively on the interior solution to the individual problem (1)–(3). It has been shown in Krusell and Smith (2000, 2003) that the assumption of quasi-geometric discounting can lead to indeterminacy and multiplicity of equilibria. However, as is argued in Krusell, Kuruşçu, and Smith (2002), the solution to the Euler equation (the interior solution) is unique, as it is a unique limit of finite-horizon equilibria.[1] Focusing on the interior solution allows us to sidestep the indeterminacy and multiplicity problems pointed out in Krusell and Smith (2000, 2003).

1. Maliar and Maliar (2004) derive a closed-form solution to the problem of a quasi-geometric consumer (4)–(6) under the assumption of the exponential utility function. The solution obtained is interior and unique.

1.3 The Model's Implications

Under the assumption of standard geometric discounting, $\beta = 1$, the discount factor is a constant, equal to δ in all periods. However, if discounting is quasi-geometric, $\beta \neq 1$, the effective discount factor is an endogenous variable, which depends on the agent's current state. In this section, we illustrate some properties of such an endogenous discount factor. Let us rewrite the quasi-geometric Euler equation (8) as

$$u'(c_t) \geq \delta_{t+1} (1 + r) E_t [u'(c_{t+1})] , \tag{9}$$

where δ_{t+1} is the effective discount factor,

$$\delta_{t+1} \equiv \delta_{t+1} (a_{t+1}, s_{t+1}) = \delta \cdot \left[1 - \frac{1 - \beta}{1 + r} \cdot \frac{E_t [u'(c_{t+1}) C_a(a_{t+1}, s_{t+1})]}{E_t[u'(c_{t+1})]} \right] . \tag{10}$$

If $\beta = 1$, then $\delta_{t+1} = \delta$ for all t and Condition (9) reduces to the standard Euler equation. To characterize the properties of δ_{t+1} under $\beta \neq 1$, we first employ the simplifying assumption that the borrowing limit is never reached and then we discuss the effect of a binding borrowing constraint.[2]

In the absence of borrowing restrictions, the quasi-geometric Euler equation (9) holds with equality. We shall begin our analysis by establishing one useful result regarding the properties of the consumption function.

LEMMA 1: *$C(a, s)$ is strictly increasing in a for all $a \in \mathcal{A}$, $s \in \mathcal{S}$.*

PROOF: See Appendix A. ||

The proof of Lemma 1 relies on the assumption that a solution to the Euler equation (8) exists, that it is unique, and that the value function W is continuously differentiable. All of these properties were satisfied in our simulations, when β was sufficiently close to one. In general, the properties of the solution to the model studied here are not known.[3]

The implication of this result for the discount factor δ_{t+1} is as follows:

PROPOSITION 1: *If $\beta < 1$, then $\delta_{t+1} < \delta$ for all $a \in \mathcal{A}$, $s \in \mathcal{S}$.*

PROOF: Under the assumption that u is strictly concave and with the result of Lemma 1, the proof of Proposition 1 follows from Equation (10) directly. ||

Proposition 1 shows that a quasi-geometric short-run impatient agent has the discount factor $\delta_{t+1} < \delta$ and, therefore, is less patient than one with $\beta = 1$. Precisely what determines the patience of the agent? Two factors are relevant here. First, self t is impatient because she is impatient in the short-run, i.e., she has the short-run discount factor, $\beta\delta$, which is lower than the long-run discount factor, δ. Secondly,

2. At the point where the borrowing constraint begins to bind, the consumption function has a kink and, therefore, is not continuously differentiable.

3. For a similar utility-maximization problem with *iid* shocks, Harris and Laibson (1999, 2001) prove the existence of equilibrium and provide sufficient conditions for continuity and differentiability of the value and policy functions.

self t is impatient because the subsequent self $t + 1$ is impatient in the short-run. To see the point, consider the first-order condition of the problem (4)–(7) with respect to consumption. Self $t + 1$'s choice of c_{t+1} is determined by the short-run discount factor, $\beta\delta$,

$$u'(c_{t+1}) = \beta\delta E_{t+1} V_a(a_{t+2}, s_{t+2}) ,$$

where V_a is the first-order partial derivative of V with respect to assets. For self t, however, the discount factor between periods $t + 1$ and $t + 2$ is the long-run one, δ, and thus, the values of consumption and assets for self t at period $t + 1$ are related as

$$u'(c_{t+1}) < \delta E_{t+1} V_a(a_{t+2}, s_{t+2}) .$$

Given that the marginal utility of consumption is decreasing, from the perspective of self t, self $t + 1$ overconsumes. The fact that a part of savings is misused by self $t + 1$, makes self t save less, i.e., act impatiently.

The assumption of quasi-geometric discounting has another important implication: the effective discount factor, δ_{t+1}, depends on the agent's wealth. By finding a partial derivative of δ_{t+1} with respect to a_{t+1} from Equation (10) and omitting the arguments of the functions for the sake of compactness, we can write

$$\frac{\partial\delta_{t+1}}{\partial a_{t+1}} = -\frac{\delta(1-\beta)}{1+r} \cdot \frac{E_t[u''C_a^2 + u'C_{aa}]E_t[u'] - E_t[u'C_a]E_t[u''C_a]}{(E_t[u'])^2} , \tag{11}$$

where $C_{aa} \equiv C_{aa}(a_{t+1}, s_{t+1})$ is the second-order partial derivative of C with respect to assets. Consider a nonstochastic steady state of Expressions (9) and (10) such that $s_t = \bar{s}$, $c_t = \bar{c}$ and $a_t = \bar{a}$ for all t. By evaluating Equation (11) in the steady state, we get

$$\frac{\partial\delta(\bar{a}, \bar{s})}{\partial a} = -\frac{\delta(1-\beta)}{1+r} \cdot C_{aa}(\bar{a}, \bar{s}) . \tag{12}$$

Hence, if the consumption function is strictly concave, $C_{aa} < 0$, then the effective discount factor of short-run impatient consumers is strictly increasing in wealth, at least near the steady state.[4]

Let us now analyze what happens to δ_{t+1} when the borrowing limit is reached. As implied by the budget constraint (6), the marginal propensity to consume out of assets of a liquidity-constrained agent is $C_a(a_{t+1}, s_{t+1}) = 1 + r$ (except at the point where the borrowing limit is just reached and where there is a kink). Thus, according to Equation (10), the effective discount factor is $\delta_{t+1} = \beta\delta$, i.e., $\delta_{t+1} < \delta$. Furthermore, by evaluating Equation (11) in the presence of binding borrowing restrictions, we obtain

4. Carroll and Kimball (1996) show analytically that introducing labor income uncertainty into a similar finite-horizon problem with standard geometric discounting, $\beta = 1$, and with no restrictions on borrowing induces a concave consumption function. Carroll and Kimball (2001) demonstrate that the concavity of the consumption function is preserved even in the presence of borrowing restrictions. The proof of a parallel result for our setup is beyond the scope of this paper. In our simulations, the consumption function was concave under all parameterizations considered.

$$\frac{\partial \delta_{t+1}}{\partial a_{t+1}} = -\frac{\delta(1-\beta)}{1+\mathrm{r}} \cdot \frac{E_t[u' C_{aa}]}{E_t[u']} . \tag{13}$$

Again, given a strict concavity of the consumption function, $C_{aa} < 0$, we have that the effective discount factor of short-run impatient consumers is strictly increasing in wealth. Consequently, the properties of δ_{t+1} are the same here as they were in the absence of borrowing restrictions.

The fact that the effective discount factor of agents depends on wealth can play a potentially important role in the model's distributional implications. We should recall that under standard geometric discounting, $\beta = 1$, the model severely underpredicts the wealth of rich agents and overpredicts the wealth of poor agents (see, e.g., Aiyagari, 1994, Quadrini and Ríos-Rull, 1997). Note that the assumption of quasi-geometric discounting can help us improve on the above shortcoming. Specifically, if $\beta < 1$, the rich act more patiently (have a higher discount factor) than do the poor. As a result, the difference between the rich and the poor will be greater in an economy with $\beta < 1$ than in one with $\beta = 1$, where rich and poor are equally patient. In the remainder of the paper, we shall evaluate the effects associated with the assumption of quasi-geometric discounting by using numerical methods.

2. QUANTITATIVE ANALYSIS

In this section, we describe the calibration and solution procedures and discuss the numerical results.

2.1 Calibration and Solution Procedures

The model's period is 1 year. The long-run discount factor is set at $\delta = 0.96$. We assume that the production function is Cobb–Douglas, $F(K, N) = K^{\alpha} N^{1-\alpha}$, with a capital share set at $\alpha = 0.36$. The depreciation rate of capital, d, is equal to 0.08. The debt limit is set at zero, $b = 0$.

We assume that the momentary utility function is of the Constant Relative Risk Aversion (CRRA) type,

$$u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}, \quad \gamma > 0 , \tag{14}$$

where γ is a coefficient of relative risk aversion. As in Aiyagari (1994), we assume that idiosyncratic shocks follow an $AR(1)$ process given by

$$\log s_{t+1} = \rho \log s_t + \sigma(1-\rho^2)^{1/2} \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim N(0,1) ,$$

where $\rho \in [0, 1]$ is the autocorrelation coefficient, and $\sigma \geq 0$ is the unconditional standard deviation of the variable $\log s_t$.

We consider four alternative sets of values of

$$(\gamma, \rho, \sigma) \in \{(1.0, 0.6, 0.2), (1.0, 0.9, 0.2), (3.0, 0.9, 0.2), (1.0, 0.9, 0.4)\} .$$

We assume two alternative values of $\beta = \{0.8, 1.0\}$, which correspond to the cases of quasi-geometric short-run impatient consumers and the standard geometric consumers, respectively.

As we argued in Section 1.3, the presence of quasi-geometric discounting in the model has two effects. First, if $\beta < 1$, the effective discount factor, δ_{t+1}, is lower than one in the standard case, $\beta = 1$, which raises the equilibrium interest rate. Secondly, the effective discount factor, δ_{t+1}, is not a constant, as in the standard case, but rather is a function of the individual state (a_{t+1}, s_{t+1}). To distinguish between the two effects and to isolate the role of the short-run discount factor in the equilibrium, we also solve the quasi-geometric-discounting model by setting the interest rate, r (the aggregate capital stock, K), at the value obtained in the standard geometric-discounting model and by adjusting the long-run discount factor, δ, correspondingly.

We also analyze the robustness of our results to the introduction of two types of consumers who differ in the degrees of their short-run patience. We specifically consider an economy in which a fraction λ of the population is quasi-geometric short-run impatient, $\beta = 0.8$, and a fraction $(1 - \lambda)$ is standard geometric, $\beta = 1$, with $\lambda \in \{1/3, 2/3\}$. In fact, we would have the same distributional implications if everyone was short-run impatient, $\beta = 0.8$, but a fraction $(1 - \lambda)$ of the population could commit while the rest could not. Indeed, if a short-run impatient agent committed, starting from the second period, her behavior would be the same as one of the standard geometric agent, except that it would be in the first period. However, given that we solve for stationary distributions, the first-period decisions play no role in the distributional implications of the model.

Thus, for each parameterization (γ, ρ, σ), we report seven computational experiments: one experiment under the standard geometric-discounting, $\beta = 1$, and $\delta = 0.96$; three experiments under $\beta = 0.8$ and $\lambda \in \{1/3, 2/3, 1\}$ holding the long-run discount factor fixed; and three experiments under $\beta = 0.8$ and $\lambda \in \{1/3, 2/3, 1\}$ holding the interest rate fixed.

To solve the model, we use an algorithm iterating on the Euler equation. The description of the algorithm is provided in Appendix B.[5] In the standard geometric-discounting case, $\beta = 1$, the algorithm had no difficulty in computing the solution. Under quasi-geometric discounting, $\beta < 1$, however, the convergence was more costly to achieve. In several experiments, it was necessary to make a good initial guess at the interest rate, r, and then to slowly update the decision rules. Furthermore, the algorithm typically failed to converge when β was lower than 0.8. The computational problems described, however, do not appear to be specific to our solution method.[6]

5. Maliar and Maliar (2005) study the convergence properties of this Euler equation method in the context of models with quasi-geometric discounting. The method used in the present paper is shown to yield the same solutions as those obtained by the perturbation method proposed by Krusell, Kuruşçu, and Smith (2002).

6. The difficulties in finding numerical solutions have been reported in other papers on quasi-geometric discounting. Laibson, Repetto, and Tobacman (1998) study a finite-horizon model similar to ours and also find that solution can be computed only if β is not too low (they use $\beta = 0.85$). In the context of a deterministic version of the neoclassical growth model with quasi-geometric discounting, Krusell and Smith (2000) argue that numerical problems are related to the fact that in addition to a smooth interior solution, the model has an infinite number of discontinuous solutions.

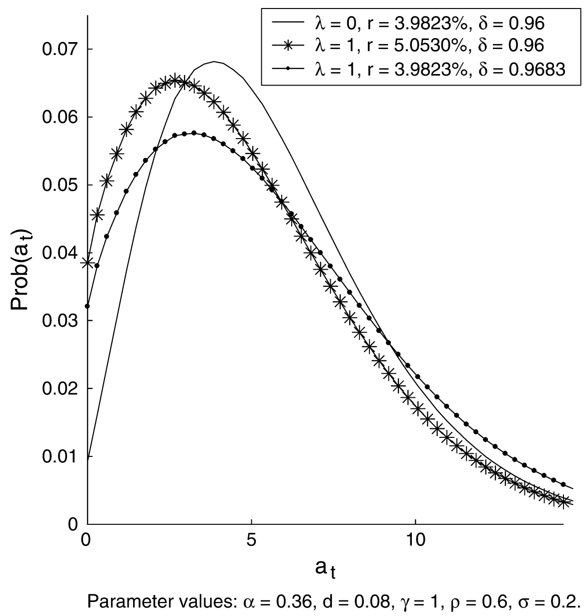


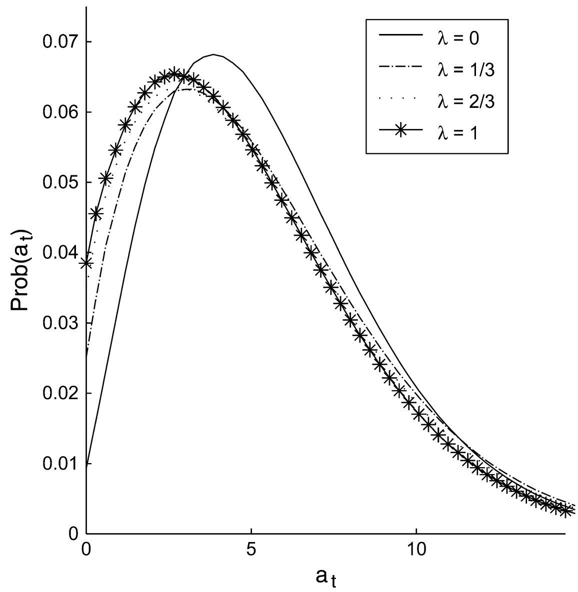
FIG. 1. The Asset Distribution: The Benchmark Cases

2.2 Results

Figures 1–3 plot the stationary probability distribution of assets (wealth) in the model under $\gamma = 1.0$, $\rho = 0.6$, and $\sigma = 0.2$. A comparison of the cases $\lambda = 1$ and $\lambda = 0$ in Figure 1 reveals the following tendencies: if δ is fixed, the unconditional mean of the wealth distribution increases in β and the fraction of the liquidity-constrained population decreases in β. The latter tendency is still observed if δ is adjusted to hold the same interest rate, r (the same mean of the wealth distribution), in the economies with $\lambda = 1$ and $\lambda = 0$. Figures 2 and 3 plot the wealth distribution obtained in the sensitivity experiments with respect to λ holding δ and r fixed, respectively. As is seen, independently of whether δ or r is fixed, an increase in the fraction of short-run impatient agents in the economy raises the fraction of the liquidity-constrained population and, hence, increases the dispersion of wealth.

Table 1 summarizes the statistics on the wealth distribution generated by the model economies. We report two measures of wealth inequality: the Gini coefficient and the percentages of wealth held by different groups of the population. For the sake of comparison, we also provide the corresponding statistics on the U.S. economy.

We must first note that the model with standard geometric discounting, $\beta = 1$, cannot generate the realistic relative degrees of wealth inequality. To be more specific, the poor agents are not so poor and the rich agents are not so rich in the model as they are in the data. For instance, in the model, under $\gamma = 1.0$, $\rho = 0.6$, and $\sigma = 0.2$ (the first panel in the table), the bottom 40% of the population



Parameter values: α = 0.36, d = 0.08, γ = 1, ρ = 0.6, σ = 0.2.

FIG. 2. The Asset Distribution under $\delta = 0.96$

holds 17.1% of total wealth and the upper 1% of the population holds 3.1% of total wealth, whereas in the U.S. economy, these numbers are 2.2% and 28.2%, respectively. The Gini coefficient reflects the same tendency: it is much lower in the model (0.32) than in the data (0.76). Variations in the parameters γ, ρ, and σ (the remaining three panels in the table) can help generate a higher concentration of wealth in the model; however, the improvements are not sufficient to account for the data.

We now analyze the case of quasi-geometric discounting. As we mentioned before, the consumption function proved to be concave in our simulations, which implies that under the assumption of short-run impatience, $\beta < 1$, wealth inequality increases in comparison to the standard geometric-discounting case, $\beta = 1$. The results in Table 1 make it possible to appreciate the quantitative expressions of this effect. First, consider the model with short-run impatient agents, $\lambda = 1$, when δ is fixed at 0.96. For example, under $\gamma = 1.0$, $\rho = 0.6$, and $\sigma = 0.2$, we have that the wealth holdings of the poorest 40% of the population are 12.3% and those of the richest 1% of the population are 3.4% (i.e., decline by 28% and increase by 10%, respectively, compared to the corresponding statistics in the geometric-discounting case $\beta = 1.0$); similarly, the Gini coefficient rises to 0.38 (i.e., increases by 19%). The same regularities are observed under the other parameterizations of (γ, ρ, σ).

Furthermore, as is seen from Table 1, the predictions of the model with short-run impatient agents, $\lambda = 1$, do not significantly change if instead of δ, the interest

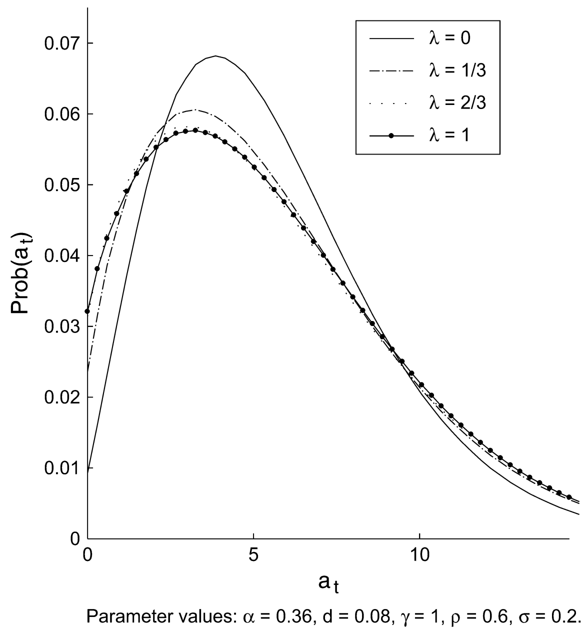


FIG. 3. The Asset Distribution under $r = 3.9823\%$

rate, r, is fixed. We therefore conclude that the effect of quasi-geometric discounting on the degrees of wealth inequality in the model comes mostly from the endogenous dependence of the individual effective discount factor on the individual state and not from the implied differences in the equilibrium interest rate (the aggregate capital stock).

Finally, the sensitivity experiments $\lambda \in \{1/3, 2/3\}$ show that in order to increase wealth inequality, it is not necessary that the entire population be short-run impatient, but just a fraction of it. In fact, the dispersion of wealth in the economy populated by both quasi-geometric short-run impatient and standard geometric agents can be even larger than in the economy, where all agents are short-run impatient.

In Table 2, we include the same statistics on the income distribution, as we previously did for the wealth distribution. As one can see, in the U.S. economy, there is much less dispersion across individuals in income compared with wealth. All model economies are capable of reproducing this regularity, but, again, they dramatically underpredict the degrees of income inequality. The main point to note from the table is that the role of quasi-geometric discounting in the income distribution is quite modest.

The results in Table 3 allow us to appreciate the effect of quasi-geometric discounting on aggregate capital stock. The comparison of the models with $\lambda \in \{0, 1/3, 2/3, 1\}$ under fixed δ shows that an increase in the fraction of the short-run

TABLE 1

Selected Statistics of the Wealth Distribution in the U.S. and Artificial Economies

λ	r, %	δ	Gini	Wealth Groups 0%–40%	80%–100%	90%–95%	95%–99%	99%–100%
$\gamma = 1.0$, $\rho = 0.6$, $\sigma = 0.2$								
1	5.0530	0.9600	0.3832	12.2587	40.4248	10.4202	10.3289	3.3584
2/3	4.7012	0.9600	0.3814	14.0402	41.1009	10.4438	10.3862	3.4389
1/3	4.3502	0.9600	0.3664	14.0346	41.0514	10.2353	10.0030	3.3818
1	3.9823	0.9683	0.3777	12.8730	40.2449	10.3571	10.0481	3.3418
2/3	3.9823	0.9659	0.3805	12.9006	40.4797	10.4906	10.3743	3.4520
1/3	3.9823	0.9633	0.3666	15.4237	39.8945	10.2369	9.9384	3.3441
0	3.9823	0.9600	0.3221	17.1209	38.0764	9.4314	9.3635	3.0890
$\gamma = 1.0$, $\rho = 0.9$, $\sigma = 0.2$								
1	4.9723	0.9600	0.4427	9.1699	45.7881	11.2498	11.4023	3.9160
2/3	4.6250	0.9600	0.4430	10.7607	44.6939	11.4690	11.7306	3.9854
1/3	4.2757	0.9600	0.4371	10.4274	44.8827	11.4269	11.5875	3.9171
1	3.9067	0.9682	0.4362	9.9773	45.2729	11.4580	11.5234	3.8455
2/3	3.9067	0.9659	0.4400	9.8947	45.5243	11.4198	11.5996	3.9590
1/3	3.9067	0.9632	0.4358	9.9829	45.2272	11.3716	11.5473	3.9446
0	3.9067	0.9600	0.4127	12.5869	42.5223	10.8946	11.0387	3.7187
$\gamma = 3.0$, $\rho = 0.9$, $\sigma = 0.2$								
1	4.2086	0.9600	0.4197	10.8272	43.7405	11.1473	10.9888	3.6610
2/3	3.8858	0.9600	0.4117	12.5302	43.8980	10.8683	10.9044	3.6102
1/3	3.5874	0.9600	0.4014	12.3919	42.3923	10.7028	10.5768	3.5583
1	3.3076	0.9677	0.4161	11.4832	43.6710	10.9409	11.0198	3.7318
2/3	3.3076	0.9653	0.4088	11.7127	43.3011	10.9243	10.9010	3.6331
1/3	3.3076	0.9628	0.3996	11.9999	42.8423	10.7349	10.5379	3.5299
0	3.3076	0.9600	0.3890	14.1303	42.3206	10.4659	10.5094	3.5029
$\gamma = 1.0$, $\rho = 0.9$, $\sigma = 0.4$								
1	4.4031	0.9600	0.4640	9.8054	46.7830	12.0835	12.5520	4.3022
2/3	4.0689	0.9600	0.4632	9.3941	47.1823	12.0195	12.4920	4.3228
1/3	3.7162	0.9600	0.4573	9.0731	47.0350	11.8894	12.2560	4.2151
1	3.3252	0.9689	0.4604	10.2048	47.5991	12.2376	12.5323	4.3149
2/3	3.3252	0.9666	0.4602	10.1668	47.4830	12.1909	12.4856	4.3100
1/3	3.3252	0.9637	0.4550	10.2394	47.0156	11.9712	12.0935	4.1977
0	3.3252	0.9600	0.4437	10.4973	46.2484	11.6965	11.9547	4.0663
U.S. Economy[a]			0.76	2.2	77.1	12.6	23.1	28.2

[a] *Source: Quadrini and Ríos-Rule (1997).*

impatient population drives down the aggregate capital stock. This effect is quite sizable: for example, under $\gamma = 1.0$, $\rho = 0.6$, and $\sigma = 0.2$, the aggregate capital stock in the economy with quasi-geometric short-run impatient agents, $\lambda = 1$, is 12.5% lower than in the economy with geometric agents, $\lambda = 0$. In Table 3, we also report the amount of precautionary savings, *PS*, %, which are defined as the percentage difference between the capital stocks in the stochastic economy, K, and in the associated deterministic economy, K^{ss}. The main finding here is that the difference in precautionary savings across the models in each panel is relatively small. In fact, precautionary savings in the economy with quasi-geometric consumers can be even larger than those in the standard geometric-discounting case (see the panel $\gamma = 3.0$, $\rho = 0.9$, and $\sigma = 0.2$ in Table 3). This is in contrast to the result of Laibson, Repetto, and Tobacman (1998) where under low values of β (specifically, they use

TABLE 2

SELECTED STATISTICS OF THE INCOME DISTRIBUTION IN THE U.S. AND ARTIFICIAL ECONOMIES

				Income Groups				
λ	r, %	δ	Gini	0%–40%	80%–100%	90%–95%	95%–99%	99%–100%
$\gamma = 1.0, \rho = 0.6, \sigma = 0.2$								
1	5.0530	0.9600	0.1294	31.2398	26.9603	6.7298	6.0367	1.7627
2/3	4.7012	0.9600	0.1284	31.2080	26.9331	6.7347	6.0199	1.7557
1/3	4.3502	0.9600	0.1265	31.1930	26.8051	6.6625	5.9683	1.7374
1	3.9823	0.9683	0.1258	31.2656	26.7582	6.6732	5.9707	1.7463
2/3	3.9823	0.9659	0.1262	31.5868	26.7520	6.6672	5.9678	1.7466
1/3	3.9823	0.9633	0.1255	31.6263	26.6791	6.6596	5.9683	1.7427
0	3.9823	0.9600	0.1227	31.6205	26.4795	6.5753	5.9169	1.7163
$\gamma = 1.0, \rho = 0.9, \sigma = 0.2$								
1	4.9723	0.9600	0.1368	30.6160	27.4283	6.9277	6.1128	1.7538
2/3	4.6250	0.9600	0.1353	30.7182	27.3726	6.9232	6.0964	1.7480
1/3	4.2757	0.9600	0.1337	30.9087	27.2600	6.8864	6.0561	1.7313
1	3.9067	0.9682	0.1312	31.1122	27.1301	6.8631	6.0117	1.7094
2/3	3.9067	0.9659	0.1316	31.0789	27.1367	6.8757	6.0217	1.7120
1/3	3.9067	0.9632	0.1319	31.0409	27.1085	6.8604	6.0116	1.7107
0	3.9067	0.9600	0.1314	31.0490	27.0449	6.8599	5.9979	1.6984
$\gamma = 3.0, \rho = 0.9, \sigma = 0.2$								
1	4.2086	0.9600	0.1329	30.9465	27.2226	6.8689	6.0240	1.7102
2/3	3.8858	0.9600	0.1312	31.0517	27.0234	6.8564	5.9990	1.6954
1/3	3.5874	0.9600	0.1294	31.1683	26.9785	6.8371	5.9728	1.6772
1	3.3076	0.9677	0.1286	31.2709	26.9275	6.8110	5.9650	1.6777
2/3	3.3076	0.9653	0.1284	31.2674	26.9075	6.8092	5.9601	1.6727
1/3	3.3076	0.9628	0.1281	31.2681	26.8842	6.8071	5.9482	1.6616
0	3.3076	0.9600	0.1276	31.2731	26.8595	6.8018	5.9433	1.6581
$\gamma = 1.0, \rho = 0.9, \sigma = 0.4$								
1	4.4031	0.9600	0.2292	24.8330	33.4053	8.4104	8.1016	2.3548
2/3	4.0689	0.9600	0.2294	24.8016	33.4565	8.4075	8.0989	2.3463
1/3	3.7162	0.9600	0.2296	24.8084	33.4828	8.3660	8.0940	2.3324
1	3.3252	0.9689	0.2270	25.0795	33.2413	8.3370	8.1027	2.3239
2/3	3.3252	0.9666	0.2279	24.9973	33.2493	8.3335	8.1186	2.3329
1/3	3.3252	0.9637	0.2288	24.8921	33.4464	8.3706	8.1222	2.3290
0	3.3252	0.9600	0.2294	24.7770	33.4579	8.3686	8.1350	2.3225
U.S. Economy [(a)]			0.51	10.3	53.6	10.7	13.5	14.1

[(a)] *Source: Quadrini and Ríos-Rule (1997).*

$\beta = 0.85$), the presence of quasi-geometric discounting leads to the missing precautionary savings effect.[7] The discrepancy between the results of Laibson, Repetto, and Tobacman (1998) and ours is explained by the fact that in their model, the interest rate is given exogenously, whereas, in our model, it is determined endogenously. In a general-equilibrium setup like ours, the agents' willingness to save more (less) drives the interest rate down (up), which, in turn, decreases (increases) the incentive to save. This is precisely what mitigates the effect of quasi-geometric discounting on precautionary savings.

7. The empirical findings about the importance of a precautionary savings motive are mixed. For example, Carroll (1994), Carroll and Samwick (1997) find strong evidence of precautionary savings, while Dynan (1993), Guiso, Jappelli, and Terlizzese (1992) report the missing precautionary savings effect.

TABLE 3

THE AGGREGATE CAPITAL STOCK AND PRECAUTIONARY SAVINGS IN THE ARTIFICIAL ECONOMIES

λ	r (%)	δ	K	ΔK (%)	K^{ss}	PS (%)
$\gamma = 1.0, \rho = 0.6, \sigma = 0.2$						
1	5.0530	0.9600	4.8801	−12.5176	4.7907	1.8657
2/3	4.7012	0.9600	5.0929	−8.7027	4.9956	1.9480
1/3	4.3502	0.9600	5.3209	−4.6158	5.2140	2.0492
1	3.9823	0.9683	5.5783	0	5.4960	1.4986
2/3	3.9823	0.9659	5.5783	0	5.4827	1.7435
1/3	3.9823	0.9633	5.5783	0	5.4722	1.9396
0	3.9823	0.9600	5.5783	0	5.4468	2.4148
$\gamma = 1.0, \rho = 0.9, \sigma = 0.2$						
1	4.9723	0.9600	4.9276	−12.5339	4.7907	2.8580
2/3	4.6250	0.9600	5.1410	−8.7459	4.9956	2.9116
1/3	4.2757	0.9600	5.3714	−4.6567	5.2140	3.0183
1	3.9067	0.9682	5.6337	0	5.4886	2.6440
2/3	3.9067	0.9659	5.6337	0	5.4801	2.8046
1/3	3.9067	0.9632	5.6337	0	5.4717	2.9617
0	3.9067	0.9600	5.6337	0	5.4468	3.4317
$\gamma = 3.0, \rho = 0.9, \sigma = 0.2$						
1	4.2086	0.9600	5.4176	−11.2889	4.7907	13.0866
2/3	3.8858	0.9600	5.6492	−7.4966	4.9956	13.0835
1/3	3.5874	0.9600	5.8782	−3.7475	5.2140	12.7388
1	3.3076	0.9677	6.1070	0	5.4421	12.2188
2/3	3.3076	0.9653	6.1070	0	5.4508	12.0398
1/3	3.3076	0.9628	6.1070	0	5.4524	12.0063
0	3.3076	0.9600	6.1070	0	5.4468	12.1214
$\gamma = 1.0, \rho = 0.9, \sigma = 0.4$						
1	4.4031	0.9600	5.2855	−13.2418	4.7907	10.3282
2/3	4.0689	0.9600	5.5159	−9.4594	4.9956	10.4161
1/3	3.7162	0.9600	5.7775	−5.1647	5.2140	10.8074
1	3.3252	0.9689	6.0922	0	5.5574	9.6226
2/3	3.3252	0.9666	6.0922	0	5.5401	9.9660
1/3	3.3252	0.9637	6.0922	0	5.5081	10.6040
0	3.3252	0.9600	6.0922	0	5.4468	11.8488

NOTES: Statistic ΔK is the percentage difference between the capital stock in a given row, K, and the one in the row $\lambda = 0$ of the same panel, $K(\lambda = 0)$, i.e., $\Delta K = (K - K(\lambda = 0))/K(\lambda = 0) \times 100\%$. Precautionary savings, PS (%), are the percentage difference between the capital stock in the stochastic model, K, and the one in the deterministic model, K^{ss}, i.e., $PS = (K - K^{ss})/K^{ss} \times 100\%$.

3. CONCLUDING REMARKS

The standard one-sector growth model, with a large number of agents who are subject to uninsured idiosyncratic shocks, predicts substantially less wealth inequality than what is observed in the data. One way of generating more skewness in the distribution of wealth is to assume that agents differ in patience (discount factors) (e.g., Krusell and Smith, 1995, 1998, Carroll, 2000). In the paper, we argue that the introduction of quasi-geometric discounting can have the same effect on the equilibrium as postulating heterogeneity in the discount factors. This is because the effective discount factor becomes an endogenous state-dependent variable. In particular, if agents are short-run impatient, then the effective discount factor increases in wealth, which accentuates the differences between the saving rates of rich and poor agents. The consequence is that the model with quasi-geometric short-run impatient agents produces a larger dispersion of wealth than does the standard geometric-discounting setup. We evaluate the effects associated with quasi-geometric discounting in a calibrated version of the model. We find that such effects are quantitatively significant but not sufficiently so for the model to be able to reproduce the true degree of inequality in wealth or income as observed in the data.

APPENDICES

Appendix A contains the proof to Lemma 1. Appendix B presents a description of the computational algorithm.

APPENDIX A

PROOF TO LEMMA 1: Denote $u(c_t) = u((1 + r)a_t + ws_t - a_{t+1}) \equiv u(a_t,a_{t+1})$.

We first prove that the asset function, $A(a_t, s_t)$, is strictly increasing in a_t. For any two levels of current wealth a_t^1 and a_t^2 and the corresponding next period's wealth $a_{t+1}^1 = A(a_t^1, s_t)$ and $a_{t+1}^2 = A(a_t^2, s_t)$, we have

$$u(a_t^1,a_{t+1}^1) + \beta\delta E_t[V(a_{t+1}^1,s_{t+1})] > u(a_t^1,a_{t+1}^2) + \beta\delta E_t[V(a_{t+1}^2,s_{t+1})]$$

$$u(a_t^2,a_{t+1}^2) + \beta\delta E_t[V(a_{t+1}^2,s_{t+1})] > u(a_t^2,a_{t+1}^1) + \beta\delta E_t[V(a_{t+1}^1,s_{t+1})] .$$

On adding up these equations and rearranging the terms, we obtain

$$u(a_t^1,a_{t+1}^1) - u(a_t^1,a_{t+1}^2) > u(a_t^2,a_{t+1}^1) - u(a_t^2,a_{t+1}^2) .$$

The strict concavity of the utility function implies that if $a_t^1 > a_t^2$, then $a_{t+1}^1 > a_{t+1}^2$, i.e., that $A(a_t, s_t)$ is strictly increasing in a_t.

In order to prove that the consumption function, $C(a_t, s_t)$, is strictly increasing in a_t, we use the results that the optimal value function $W(a_t, s_t)$ is strictly increasing and strictly concave in a_t.

The fact that W is strictly increasing in a_t follows from the assumption of the strictly increasing utility function, u, and by the definition of

$$W(a_t,s_t) = \max_{a_{t+1}}\{u((1 + r)a_t + ws_t - a_{t+1}) + \beta\delta E_t[V(a_{t+1}, s_{t+1})]\} .$$

The strict concavity of W can be shown as follows: fix a sequence of realizations for shocks $(s_t, s_{t+1},\dots) \in \mathcal{S}$. Consider a_t^1 and a_t^2 such that $a_t^1 > a_t^2$. By using the asset function, $A(a_t, s_t)$, iteratively, we find the corresponding optimal sequences for assets $(a_t^1, a_{t+1}^1,\dots) \in \mathcal{A}$ and $(a_t^2,\ a_{t+1}^2,\ \dots) \in \mathcal{A}$. The fact that $A(a_t, s_t)$ is strictly increasing in a_t implies that $a_\tau^1 > a_\tau^2$ for all $\tau \geq t$.

Consider the sequence $(va_t^1 + (1 - v)a_t^2, va_{t+1}^1 + (1 - v)a_{t+1}^2,\dots) \in \mathcal{A}$, where $v \in [0, 1]$. The strict concavity of W follows from the strict concavity of the utility function, u

$$
\begin{aligned}
W(va_t^1 + (1-v)a_t^2,s_t) &= u(va_t^1 + (1-v)a_t^2,va_{t+1}^1 + (1-v)a_{t+1}^2) \\
&+ E_t \sum_{\tau = t}^{\infty} \beta\delta^{\tau+1-t} u(va_{\tau+1}^1 + (1-v)a_{\tau+1}^2,va_{\tau+2}^1 \\
&+ (1-v)a_{\tau+2}^2) > \\
&> v\left[u(a_t^1,a_{t+1}^1) + E_t \sum_{\tau = t}^{\infty} \beta\delta^{\tau+1-t} u(a_{\tau+1}^1,a_{\tau+2}^1)\right] \\
&+ (1-v)\left[u(a_t^2,a_{t+1}^2) + E_t \sum_{\tau = t}^{\infty} \beta\delta^{\tau+1-t} u(a_{\tau+1}^2,a_{\tau+2}^2)\right] \\
&= vW(a_t^1,s_t) + (1-v)W(a_t^2,s_t)\ .
\end{aligned}
$$

To complete the proof, we find the derivative of W with respect to assets from Equations (4)–(6):

$$W_a(a_t,s_t) = u'(C(a_t,s_t))(1 + r)\ .$$

The fact that W is strictly increasing and strictly concave in a_t implies that $C(a_t, s_t)$ is strictly increasing in a_t. ||

APPENDIX B

In our economy, each agent solves the problem (4)–(7), which is, in effect, a variant of the problem with an occasionally binding inequality constraint studied in Christiano and Fisher (2000). Let us rewrite the Euler equation (8) as

$$u'(C(a,s)) - h(a,s) = \delta E\{u'(C(a',s'))[1 + r - (1-\beta)C_a(a',s')]\}\ , \tag{B1}$$

where $h(a, s)$ is the Lagrange multiplier associated with the borrowing constraint (7). The corresponding set of Kuhn-Tucker conditions is given by

$$h(a, s) \geq 0\ , \tag{B2}$$

$$A(a, s) - b \geq 0, \quad h(a, s)\,(A(a,s) - b) = 0\ . \tag{B3}$$

The solution to the individual problem is defined as a set of time-invariant functions $C(a, s)$, $A(a, s)$, and $h(a, s)$ satisfying the Euler equation (B1), the budget constraint (6) and the Kuhn-Tucker conditions (B2) and (B3).

Our solution method is similar to the parameterized expectations algorithm used in Den Haan and Marcet (1990) and Christiano and Fisher (2000), however, unlike those papers, we parameterize the asset function and not the expectation term in the Euler equation. We compute the solution on a grid of prespecified points. We approximate the autoregressive process for the shocks by a seven-state Markov chain, as in Aiyagari (1994). For each state $s \in \{s_1, \ldots, s_7\}$, we parametrize the asset demand by a function of the agent's current asset holdings.[8] The grid for asset holdings consists of 100 equally spaced points in the range $[a_{\min}, a_{\max}]$, where $a_{min} \equiv b = 0$ and $a_{\max}$ is the maximum sustainable capital stock (i.e., the solution to $F(a, 1) = da$). Our choice of the value of $a_{\max}$ ensures that the upper bound is never reached in equilibrium. To evaluate the asset function outside the grid, we use cubic polynomial interpolation.

Under the assumption of the CRRA momentary utility function (14), the Euler equation (B1) and the budget constraint (6), combined together, yield

$$a' = (1 + r)a + ws - \tag{B4}$$

$$-\left\{h(a,s) + \delta \sum_{s \in \{s_1,\ldots,s_7\}} \frac{(1 + r - (1 - \beta)C_a(A(a,s),s'))\mathrm{Prob}(s'|s)}{(A(a,s)(1 + r) + ws' - A(A(a,s),s'))^{\gamma}}\right\}^{-1/\gamma}.$$

Consequently, we implement the following iterative procedure:

- *Step* 1. Fix some asset function, $A(a, s)$, on the grid and compute the corresponding consumption function, $C(a, s)$, from the budget constraint (6).
- *Step* 2. Use the decision rules to calculate the right side of Equation (B4) in each point on the grid by setting the Lagrange multiplier equal to zero, $h(a, s) = 0$ for all a, s. The left side of Equation (B4) will be the new asset function, $\tilde{A}(a, s)$. For each point, such that $\tilde{A}(a, s)$ does not belong to $[a_{\min}, a_{\max}]$, set $\tilde{A}(a, s)$ at the corresponding boundary value.
- *Step* 3. Compute the asset function for next iteration $\tilde{\tilde{A}}(a, s)$ by using the updating:

  $$\tilde{\tilde{A}}(a,s) = \eta\tilde{A}(a,s) + (1 - \eta)A(a,s), \quad \eta \in (0,1].$$

- Iterate on *Steps* 1–3 until $\tilde{\tilde{A}}(a, s) = A(a, s)$ with a given precision.

Note that by construction, the obtained solution satisfies the Euler equation (B1), the budget constraint (6), and the Kuhn-Tucker conditions in Equation (B3). We are left to check that our solution satisfies the remaining Kuhn–Tucker condition (B2), i.e., that the Lagrange multiplier is nonnegative whenever the borrowing constraint (7) binds. Notice that under $\gamma > 0$, the term $\{h(a, s) + \ldots\}^{-1/\gamma}$ in Equation (B4) is decreasing in the value of $h(a, s)$. Thus, when the unconstrained solution, which we obtained under $h(a, s) = 0$, violates the borrowing constraint (7), so that we set the asset holdings in the left side of Equation (B4) at the borrowing limit, we should increase the Lagrange multiplier in the right side of Equation (B4) in order to

8. The borrowing restriction on assets used in our paper, $a' \geq 0$, is not equivalent to the one in Aiyagari (1994). In the latter paper, the restriction is imposed on total resources. These are restricted to being no lower than the wage corresponding to an interest rate equal to the time preference rate (the highest possible interest rate under $\beta = 1$). Such restriction on the total resources would not be appropriate if discounting is quasi-geometric, $\beta \neq 1$, as the equilibrium interest rate can be either higher and or lower than the time preference rate.

preserve the equality sign. Hence, our method ensures that the Lagrange multiplier is always nonnegative.

In the stochastic version of the model, we compute the interest rate corresponding to a given asset function, $A(a, s)$, by calculating the stationary probability distribution of shocks and assets, as described in Ríos-Rull (1999):

$$\text{Prob}(a',s') = \sum_{s \in \{s_1,\ldots,s_7\}} \text{Prob}(A^{-1}(a', s),s) \cdot \text{Prob}(s'|s) \,,$$

where $A^{-1}\,(a', s) = \{a, a' = A(a, s)\}$ is the inverse of the asset function. In the deterministic case, we compute the interest rate corresponding to a given asset function, $A(a, 1)$, by solving for a capital stock satisfying a fixed-point property $A(a^*, 1) = a^*$.

Finally, in order to solve for the equilibrium interest rate, r, in the model with fixed discount factor, and to solve for the equilibrium discount factor, δ, in the model with fixed interest rate, we use the bisection method, described in Aiyagari (1994).

LITERATURE CITED

Aiyagari, Rao S. (1994). "Uninsured Idiosyncratic Risk and Aggregate Saving." *Quarterly Journal of Economics* 109, 659–684.

Angeletos, Marios, David Laibson, Andrea Repetto, Jeremy Tobacman, and Stephen Weinberg (2001). "The Hyperbolic Consumption Model: Calibration, Simulation, and Empirical Evaluation." *Journal of Economic Perspectives* 15, 47–68.

Barro, Robert J. (1999). "Ramsey Meets Laibson in the Neoclassical Growth Model." *Quarterly Journal of Economics* 114, 1125–1152.

Carroll, Christopher D. (1994). "How Does Future Income Affect Current Consumption?" *Quarterly Journal of Economics* 109, 111–148.

Carroll, Christopher D. (2000). "Requiem for the Representative Consumer? Aggregate Implications of Microeconomic Consumption Behavior." *American Economic Review* 90, 110–115.

Carroll, Christopher D., and Miles S. Kimball (1996). "On the Concavity of the Consumption Function." *Econometrica* 64, 981–992.

Carroll, Christopher D., and Miles S. Kimball (2001). "Liquidity Constraints." NBER Working Paper No. 8496.

Carroll, Christopher D., and A. Samwick (1998). "How Important is Precautionary Saving?" *Review of Economics and Statistics* 80, 410–419.

Christiano, Lawrence J., and Jonas D.M. Fisher (2000). "Algorithms for Solving Dynamic Models with Occasionally Binding Constraints." *Journal of Economic Dynamics and Control* 24, 1179–1232.

Den Haan, Wouter J., and Albert Marcet (1990). "Solving the Stochastic Growth Model by Parametrizing Expectations." *Journal of Business and Economic Statistics* 8, 31–34.

Dynan, Karen E. (1993). "How Prudent are Consumers?" *Journal of Political Economy* 101, 1104–1113.

Harris, Christopher, and David Laibson (1999). "Dynamic Choices of Hyperbolic Consumers." Harvard University Discussion Paper.

Harris, Christopher, and David Laibson (2001). "Dynamic Choices of Hyperbolic Consumers." *Econometrica* 69, 935–959.

Huggett, Mark (1997). "The One-Sector Growth Model with Idiosyncratic Shocks: Steady States and Dynamics." *Journal of Monetary Economics* 39, 385–403.

Guiso, Luigi, Tullio Jappelli, and Daniele Telizzese (1992). "Earning Uncertainty and Precautionary Saving." *Journal of Monetary Economics* 30, 307–337.

Krusell, Per, and Anthony A. Smith (1995). "Income and Wealth Heterogeneity in the Macroeconomy." Working Paper No 399. Rochester, NY: Rochester Center for Economic Research.

Krusell, Per, and Anthony A. Smith (1998). "Income and Wealth Heterogeneity in the Macroeconomy." *Journal of Political Economy* 106, 868–896.

Krusell, Per, and Anthony A. Smith (2000). "Consumption-Savings Decisions with Quasi-Geometric Discounting." CEPR Discussion Paper No. 2651.

Krusell, Per, and Anthony A. Smith (2003). "Consumption-Savings Decisions with Quasi-Geometric Discounting." *Econometrica* 71, 365–375.

Krusell, Per, Burhanettin Kuruşçu, and Anthony A. Smith. (2002) "Equilibrium Welfare and Government Policy with Quasi-Geometric Discounting." *Journal of Economic Theory*, 105, 42–72.

Laibson, David (1997). "Golden Eggs and Hyperbolic Discounting." *Quarterly Journal of Economics* 112, 443–477.

Laibson, David, Andrea Repetto, and Jeremy Tobacman (1998). "Self-Control and Saving for Retirement." *Brookings Papers on Economic Activity* 1, 91–172.

Luttmer, Erzo G.J., and Thomas Mariotti (2002). "Subjective Discounting in an Exchange Economy." *Journal of Political Economy*, forthcoming.

Lilia, Maliar, and Serguei Maliar (2005). "Solving the Neoclassical Growth Model with Quasi-Geometric Discounting: Non-Linear Euler-Equation Methods." *Computational Economics* 26, 163–172.

Lilia, Maliar, and Serguei Maliar (2004). "Quasi-Geometric Discounting: A Closed-Form Solution under the Exponential Utility Function." *Bulletin of Economic Research* 56, 201–206.

Phelps, Edmund S., and Robert A. Pollak (1968). "On Second-Best National Saving and Game-Equilibrium Growth." *Review of Economic Studies* 35, 185–199.

Pollak, Robert A. (1968). "Consistent Planning." *Review of Economic Studies* 35, 201–208.

Quadrini, Vincenzo, and Victor J. Ríos-Rull (1997). "Understanding the US Distribution of Wealth." *Federal Reserve Bank of Minneapolis Quarterly Review* 21, 22–36.

Ríos-Rull, Victor J. (1999). "Computation of Equilibria in Heterogeneous Agent Models." In *Computational Methods for Study of Dynamic Economies*, edited by R. Marimon and A. Scott, pp. 238–273. New York: Oxford University Press.

Strotz, Robert H. (1955–1956). "Myopia and Inconsistency in Dynamic Utility Maximization." *Review of Economic Studies* 23, 165–180.