1 Introduction

In this note, we briefly discuss the algebraic properties of the sub-Weibull distributions. This class of
distributions is an extension of commonly used sub-Gaussian and sub-Exponential classes that includes
distributions with heavier tails. As the class name suggests, the tail of a sub-Weibull distribution is upper-
bounded by the tail of a Weibull distribution. All moments of such distribution are finite. However, the
Moment Generating Function (MGF) may be infinite at all positive points.

Sub-Gaussian distributions have been extensively discussed in [6]. We encourage the reader to get familiar
with the concepts explained in [6], particularly those from Chapter 2, before proceeding with this note. This
note extends the properties of the sub-Gaussian class to sub-Weibull distributions. Such properties help one
to recognize a sub-Weibull distribution. Several practical tools, such as concentration inequalities and large
deviation limits, have been developed to analyze statistics constructed by such heavy-tailed components
(see [1-5]).

2 Sub-Weibull distributions

Recall the below Proposion from [6]:

Proposition (2.5.2 of [6]). A random variable $X$ is called sub-Gaussian if it satisfies any of the following
equivalent properties.

(i) $\Pr(|X| > t) \leq 2 \exp \left( -\frac{t^2}{K_1^2} \right) , \ \forall t \geq 0$ \quad (1)

(ii) $\|X\|_p \leq K_2 \sqrt{p} , \ \forall p \geq 1$ \quad (2)

(iii) $\mathbb{E} \left[ \exp \left( \lambda X^2 \right) \right] \leq \exp \left( K_3^2 \lambda^2 \right) , \ \forall \lambda \text{ s.t. } |\lambda| \leq \frac{1}{K_3}$ \quad (3)

(iv) $\exists K_4 \text{ s.t. } , \ \mathbb{E} \left[ \exp \left( \frac{X^2}{K_4^2} \right) \right] \leq 2$ \quad (4)

To extend the above proposition, we define tail capturing function below.

Definition 1. Suppose $X$ is a random variable. We say function $I : \mathbb{R}^\geq 0 \to \mathbb{R}^\geq 0$ captures the tail of $X$ if

$$\Pr(|X| > t) \leq \exp \left( -I(t) \right) , \ \forall t \geq T$$ \quad (5)

where $T > 0$ is fixed. Moreover, $I(t)$ is said to be asymptotically tight if

$$\lim_{t \to \infty} -\log \frac{\Pr(|X| > t)}{I(t)} = 1.$$ \quad (6)
Therefore, based on (1), \(X\) is a sub-Gaussian variable if it has a quadratic tail capturing function. Sub-Weibull distributions have tail capturing functions with possibly slower growth.

**Definition 2.**

1. A random variable \(X\) belongs to the sub-Weibull class (SW) if \(I(t) = c \sqrt{t}\) captures its tail for some \(c, \alpha > 0\).

2. \(\alpha\) is called the **order** of the tail capturing function \(I(t)\). \(SW(\alpha)\) is the class of all random variables that have a tail capturing function of order \(\alpha\).

3. The **order** of the tail of \(X\) is defined as the minimum \(\alpha\) for which \(X \in SW(\alpha)\). In other words,

\[
\text{Ord}(X) \equiv \min \{\alpha \mid X \in SW(\alpha)\}.
\]

Note that if \(0 < \alpha \leq \beta\) then \(SW(\alpha) \subseteq SW(\beta)\), and \(SW = \bigcup_{\alpha > 0} SW(\alpha)\). Moreover, if \(I(t)\) is an asymptotically tight tail capturing function for \(X\), then orders of \(I(t)\) and \(X\) are the same.

**Remark 1.** Note that sub-Gaussian class is \(SW(\frac{1}{2})\). Additionally, \(SW(1)\) is the class of sub-Exponential distributions.

**Lemma 1.** For random variable \(X\), the below conditions are equivalent

1. \(I(t) = c_\alpha \sqrt{t}\) captures tail of \(X\), i.e. \(P(\|X| > t) = O\left(\exp\left(-c_\alpha t^{\frac{1}{2}}\right)\right)\),

2. \(\|X\|_p = \left(\mathbb{E}[|X|^p]\right)^{\frac{1}{p}} = O(p^\alpha)\), where \(c_\alpha \in (0, \infty)\).

**Proof.** We claim if \(X \in SW(\alpha)\), then \(Y = |X|^{\frac{1}{\alpha}}\) is a sub-Gaussian random variable. To see this note that

\[
P(\|Y| > t) = P(\|X| > t^{2\alpha}) \leq K \exp\left(-ct^2\right),
\]

which yields (1) of Proposition 2. Moreover, note that \(\|X\|_p = \left(\|Y\|_{2\alpha}\right)^{\frac{2}{2\alpha}}\), thus

\[
\|X\|_p = O(p^\alpha) \iff \|Y\|_{2\alpha} = O(\sqrt{p}).
\]

Hence, the claimed equivalence in the Lemma 1 is inherited from the equivalence of (1) and (2) of sub-Gaussian family.

**2.1 Algebra of sub-Weibull distributions**

The order of the tail has nice relation with the algebraic operators over random variables. Below Lemma declares this fact.

**Lemma 2.** Assume \(X \in SW(\alpha), Y \in SW(\beta)\) are two random variables. Then

1. \(XY \in SW(\alpha + \beta)\).

2. \(X + Y \in SW(\max(\alpha, \beta))\)

**Proof.** By Cauchy-Schwartz inequality we obtain

\[
\|XY\|_p \leq \|X\|_p\|Y\|_p^p \leq k_2k'_2(2p)^\alpha(2p)^\beta = (2^{\alpha+\beta}k_2k'_2)^p = O(p^{\alpha+\beta}).
\]

Hence, Lemma 1 yields \(XY \in SW(\alpha + \beta)\).

For the second part, one only needs to note that \(\|X + Y\|_p \leq \|X\|_p + \|Y\|_p = O(\max(\alpha, \beta))\).
Example 1. One can use Lemma 2 to determine the order of powers of a random variable. For instance, if for $X \sim \mathcal{N}(0, 1)$ we have

$$\text{Ord}(X) = \frac{1}{2}, \quad \text{Ord}(X^2 \sim \chi^2(1)) = 1, \quad \text{Ord}(X^n) = \frac{n}{2}.$$  

Moreover, one can verify $\text{Ord}(\chi^2(k)) = 1$ for an arbitrary degrees of freedom $k$.

Remark 2. While Lemma 2 says the class of sub-Weibull variables is closed under summation and multiplication, $\text{SW}$ is not closed under division. To see this, consider independent $X, Y \sim \mathcal{N}(0, 1)$. Then $\frac{X}{Y} \sim \text{Cauchy}(0, 1)$ which has a polynomial tail, hence not in $\text{SW}$.

2.2 Relation to MGF

The order of the tail of a random variable $X \in \text{SW}$ determines in what range the moment generating function of its absolute value, $mgf(\lambda) = \mathbb{E} \left[ \exp \left( \lambda |X| \right) \right]$, exists and is finite. Below Lemma explains this relation:

Lemma 3. Let $X$ be a random variable in $\text{SW}$.

1. If $X \in \text{SW}^{(\frac{1}{2})}$, then $mgf(\lambda)$ is finite for all $\lambda \in \mathbb{R}$.

   In fact, $\exists K_5, K_6$ s.t. $mgf(\lambda) \leq K_6 \exp \left( K_5 \lambda^2 \right)$.

2. If $X \in \text{SW}(1)$, then $\exists k_3 > 0$ such that $mgf(\lambda)$ is finite for $|\lambda| \leq k_3$.

3. If $X \notin \text{SW}(1)$, then $mgf(\lambda) = \infty$ for all $\lambda > 0$.

Proof. For parts 1 and 2 refer to Sections 2.5 and 2.7 from [6]. For the last part, assume for a positive $\lambda$ we have $mgf(\lambda) = K < \infty$. Then,

$$\mathbb{P} \left( |X| > t \right) \leq \exp \left( -\lambda t \right) \mathbb{E} \left[ \exp \left( \lambda |X| \right) \right] = K \exp \left( -\lambda t \right).$$

Hence, $K \exp \left( -\lambda t \right)$ is a tail capturing function for $X$ which means $X \in \text{SW}(1)$. The contradiction denotes no such $\lambda$ exists.

References


