Phase Retrieval for Structured Signals

Phase Retrieval

Recover $x_o \in \mathbb{C}^n$ from

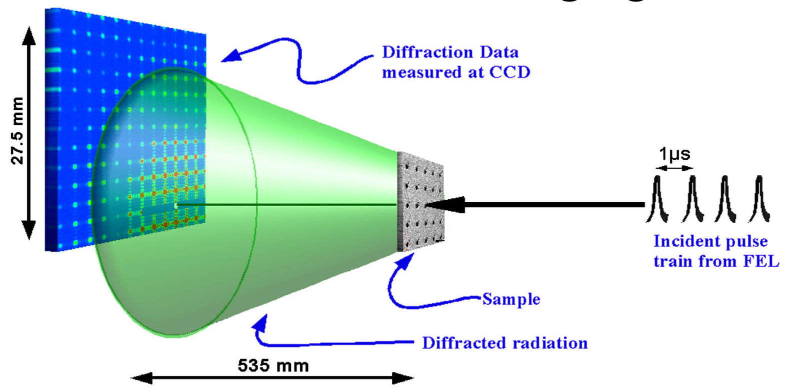
$$\left|a_i^t x_o\right|, \qquad 1 \le i \le m$$

- Gaussion $\quad y_i = a_i^t \overset{i.i.d.}{\sim} \mathcal{N}(0, \mathcal{I}_n)$
- DCT $\quad a_i^t = \left(\cos\frac{i\pi}{n}\left(k+\frac{1}{2}\right)\right)_{k=1}^{n}$
- Masks $\quad M_{k,l} \in \{-1, 0, 1\}$

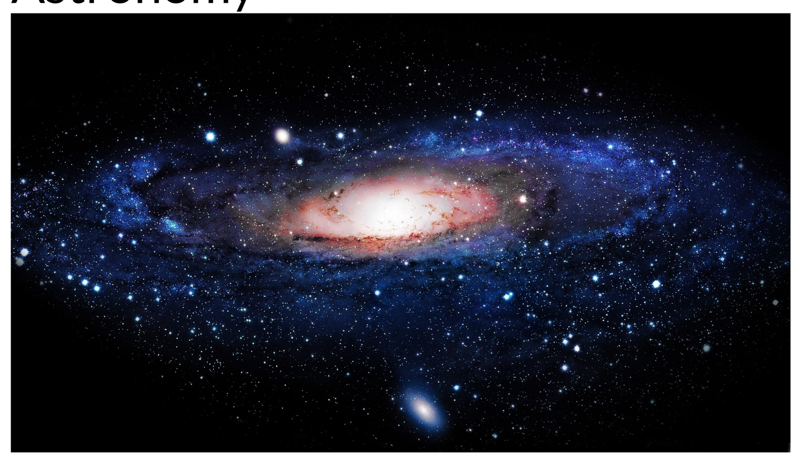
Applications

- Increasing Resolution

  Coherent Diffraction Imaging (CDI)



- Crystallography
- Astronomy



Wirtinger Flow

$$d(z) = \frac{1}{2m}\sum_{i=1}^{m}\left|y_i^2 - \left|a_i^t z\right|^2\right|^2$$

$$\text{dist}(z, x_o) = \min_{\phi\in\mathbb{R}}\left\|z - e^{i\phi}x\right\|$$

$$\text{dist}(z_o, x) \le \frac{1}{8}\|x\|$$

$$z_{\tau+1} = z_\tau - \mu_{\tau+1}\nabla d(z_\tau)$$

E. J. Candes, X. Li and M. Soltanolkotabi, "Phase Retrieval via Wirtinger Flow: Theory and Algorithms

- Compression family

$$x \in \mathcal{Q} \quad \text{compact}, \qquad \mathcal{E}_r : \mathcal{Q} \to \{0,1\}^r, \ \mathcal{D}_r : \{0,1\}^r \to \mathcal{Q}, \quad \|\mathcal{D}_r(\mathcal{E}_r(z)) - z\| \le \delta_r, \quad \forall\, z \in \mathcal{Q}$$

- α dimension

$$\mathcal{F} = \{\mathcal{E}_r, \mathcal{D}_r\}, \qquad \dim_\alpha(\mathcal{F}) = \lim_{r\to\infty}\frac{r}{\log\frac{1}{\delta_r}}$$

- COmpresive PhasE Retrieval (COPER)

$$\mathcal{C}_r = \mathcal{D}_r \circ \mathcal{E}_r(\mathcal{Q}), \quad d(z) = \frac{1}{2m}\sum_{i=1}^{m}\left|y_i^2 - \left|a_i^t z\right|^2\right|^2, \qquad \hat{x}_{\text{COPER}} = \arg\min_{c\in\mathcal{Q}} \mathbf{d}(c)$$

- $$\mathbb{P}\left(\inf_\theta\left\|e^{i\theta}x - \hat{x}\right\|^2 \le 32\sqrt{3}\delta_r^{\epsilon_\eta}\right) \ge 1 - 2^{-c_\eta r} - e^{-0.6m},$$

given $m \ge \eta \dim_\alpha(\mathcal{Q}), \quad \eta > 1.$

- error $\xrightarrow{\delta_r \to 0} 0$

- GD-COPER

$$\text{dist}(x, z_0) < \|z_0\|, \qquad z_{t+1} = \mathcal{D} \circ \mathcal{E}\left(z_t - \mu\nabla d(z_t)\right)$$

- $$\inf_{\theta\in\mathbb{R}}\left\|e^{i\theta}x - z_T\right\| \le (1-2\tau)(1-\tau)^T + \frac{3}{\tau}\delta_r$$ with high probability,

where $0 < \tau < 0.5$ depends on initial error.

- Remarks

Theoretical

- m needs to be as large as $\dim_\alpha(\mathcal{Q})$
- Mild initial condition

General

- It can employ ANY structure
- Having the compression method is enough

Practical

- Stable to the initialization
- Fast and efficient

Full Paper

My Website

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*: Columbia University **: Nokia Bell-labs

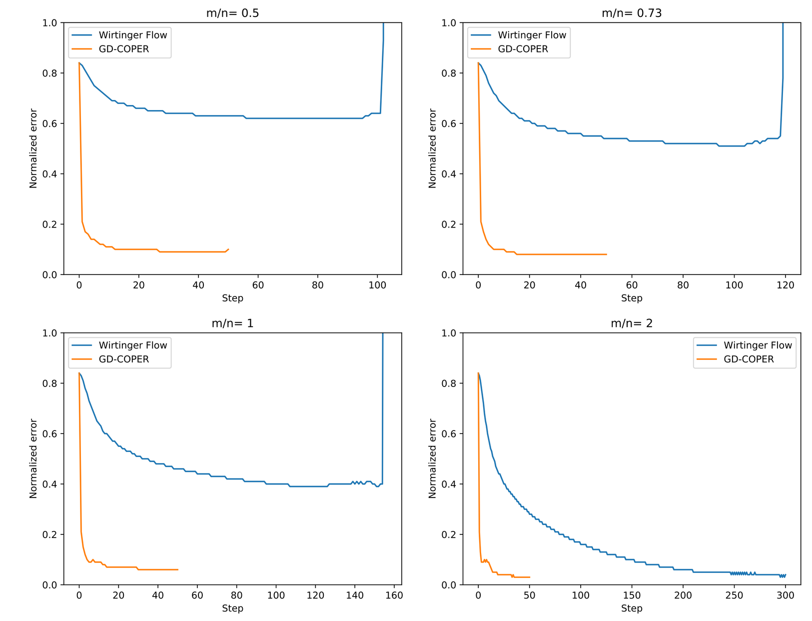
Results

- Peak Signal to Noise Ratio

$$\text{PSNR}(z) = 20\log_{10}\frac{255}{\sqrt{\text{MSE}}}$$

$$\text{MSE}(z) = \frac{1}{n}\|z - x\|^2$$

- Convergence

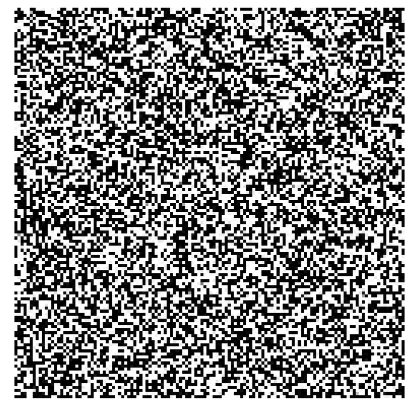


GD-COPER

Gaussion, $m/n = .73$

PSNR = 27.4

Wirtinger Flow



Gaussion, $m/n = .73$

DVG

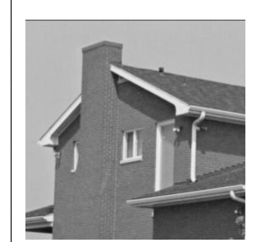
DCT, $m/n = 10$

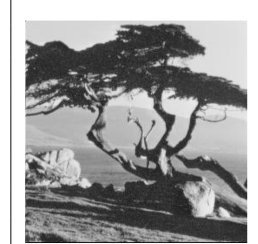
PSNR = 49.9

DCT, $m/n = 10$

PSNR = 37.6

Initialization

Target	n-init-error	λ	$\frac{m}{n}=1$		$\frac{m}{n}=2$		$\frac{m}{n}=3$	
			GD-C	WF	GD-C	WF	GD-C	WF
	0.0	0.0	27.84	inf	31.55	inf	35.11	inf
	0.09	0.1	28.04	DVG	31.5	DVG	35.19	DVG
	0.17	0.2	27.44	DVG	31.24	DVG	35.12	DVG
	0.26	0.3	26.99	DVG	31.47	DVG	35.26	DVG
	0.35	0.4	26.68	DVG	31.23	DVG	35.02	DVG
	0.43	0.5	26.89	DVG	31.62	DVG	34.66	19.12
	0.52	0.6	26.5	DVG	32.18	DVG	33.89	18.97
	0.61	0.7	26.69	DVG	32.4	DVG	33.54	17.94
	0.7	0.8	26.56	DVG	31.97	13.86	33.71	17.13
	0.78	0.9	26.26	DVG	31.74	12.92	34.16	16.12
	0.87	1.0	26.71	DVG	32.0	12.11	34.6	15.21

Target	$\frac{m}{n}$	All-white			Spectral		
		n-init-err	PSNR	Run time	n-init-err	PSNR	Run time
	1	0.98	DVG	2.6	1.39	DVG	5.0
	2	0.98	DVG	2.8	1.39	DVG	7.2
	3	0.98	14.0	9.6	1.39	DVG	8.9
	4	0.98	17.0	11.9	1.4	DVG	10.6
	5	0.98	20.0	15.9	1.38	DVG	12.6
	6	0.98	23.2	17.7	1.21	DVG	15.0
	7	0.98	26.1	21.8	1.31	DVG	17.0
	8	0.98	29.0	24.1	1.39	DVG	17.9
	9	0.98	32.2	26.2	0.65	20.4	30.8
	10	0.98	34.7	13.6	0.6	21.3	30.9
	15	0.98	57.1	21.9	0.48	21.2	55.3