Slides Contents

Economics 136: Auctions & Market Design

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Topic #1: Introduction & Review

Market Design in Practice

- Spectrum auctions since 1994
  - FCC auctions
  - Worldwide innovations in auction design
- Other innovative auctions
  - Electricity
  - Carbon emissions
  - Timber
  - Asset sales
  - Procurement
- National Resident Matching Program since 1998
  - Matches 20,000 doctors to hospitals annually
- Other Innovative Matches
  - Psychology post-docs
  - High school placements
  - Course bidding
  - Kidney exchange

US Spectrum Allocation & Assignment

- Comparative hearings
  - public interest assessment
  - overwhelmed by cellular telephone applications
- Lotteries (Reagan era)
  - political compromise
  - “unjust enrichment” and administrative nightmares
- Auctions (Clinton era)
  - market determined “assignments”
  - regulation of “allocation” (uses)

The RCA Transponder Auction

- A sequential auction (1981)

Order	Winning Bidder	Price Obtained
1	TLC	14,400,000
2	Billy H. Batts	14,100,000
3	Warner Amex	13,700,000
4	RCTV	13,500,000
5	HBO	12,500,000
6	Inner City	10,700,000
7	UTV	11,200,000
Total		90,100,000

Australian Satellite TV Auction

A sealed bid auction with no withdrawal penalty (1991)

Initial Winning Bid	Final Transaction Price
212,000,000	117,000,000
177,000,000	77,000,000

New Zealand UHF License Auction

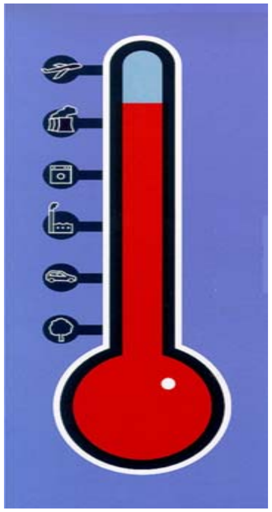
A simultaneous sealed-bid second price auction (1993)

Lot	Winner	High Bid	2nd Bid
1	Sky Network TV	2,371,000	401,000
2	Sky Network TV	2,273,000	401,000
3	Sky Network TV	2,273,000	401,000
4	BCL	255,124	200,000
5	Sky Network TV	1,121,000	401,000
6	Totalisator A.B	401,000	100,000
7	United Christian	685,200	401,000

Largest US Spectrum Auctions

Auction No.	Auction Name	Licenses Auctioned	Net High Bids (M)
1	Nationwide Narrowband PCS 7/25/1994-7/29/1994, Nationwide	10	$617.0
4	A & B Block PCS 12/5/1994-3/13/1995, MTA	99	$7,019.4
5	C Block PCS 12/18/1995-5/6/1996, BTA	493	$9,197.5
8	DBS (110 W) 1/24/1996-1/25/1996, Nationwide	1	$682.5
10	C Block PCS Reauction 7/3/1996-7/16/1996, BTA	18	$904.6
11	D, E, & F Block PCS 8/26/1996-1/14/1997, BTA	1479	$2,517.4
17	Local Multipoint Distribution Service (LMDS) 2/18/1998-3/25/1998, BTA	986	$578.7
33	Upper 700 MHz Guard Bands 9/6/2000-9/21/2000, MEA	104	$519.9
35	C & F Block PCS 12/12/2000-1/26/2001, BTA	422	$16,857.0

British CO_2 Auctions



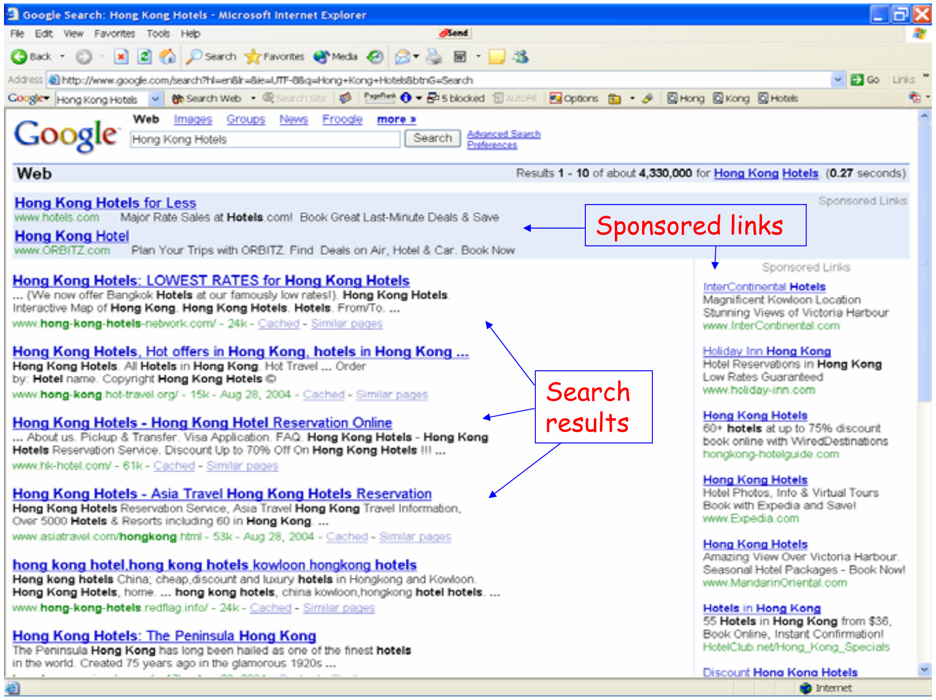
- Greenhouse Gas Emissions Trading Scheme Auction United Kingdom March 11-12, 2002
  - 38 bidders
  - 34 winners
  - 4 million metric tons of CO_2 emission reductions

EDF Generation Capacity Auction

EDF Electricité de France

PricewaterhouseCoopers

αγορα market design inc.



More Auction Mechanisms

- Dynamic Auctions
  - Dutch descending auctions
  - English ascending auctions
    - Clock vs open outcry auctions
- Static auctions
  - Sealed tenders
  - Second price auctions
  - Priority auctions
  - All-pay & two-pay auctions
  - "Package" auctions

Games & Mechanisms

- In normal form, a <u>game</u> is a triple Γ=(N,S,π) consisting of
  - A set of players
  - A strategy set for each player
  - A payoff function mapping strategy profiles to payoff vectors.
- In normal form, a <u>mechanism</u> is a triple Γ=(N,S,ω) consisting of
  - A set of players
  - A strategy set for each player
  - An outcome function mapping strategy profiles to outcomes
- Mechanism design theory:
  - Payoffs are jointly determined by outcomes and preferences
  - In principle, mechanisms can be designed.

"Mechanism Definitions"

- Extensive form: A complete mathematical description of all the detailed rules of the game, specifying:
  - Who are the players and
  - A (finite) labeled game tree describing:
    - An initial node, where the game begins
    - Which player moves at each node in the tree
    - What moves are available (specified by arcs)
    - Information sets to describe what the mover knows about preceding events
    - An outcome for each "terminal node" of the game tree
    - …but not payoffs for each outcome!
- Strategy: a complete specification of what the player does at every information set.
- Normal form: Lists of players and their strategies, and a function from strategy profiles to outcomes.

Background

- You are supposed to know about…
  - <u>Mechanisms</u>. Roughly, a normal-form mechanism (N,S,ω) is like a normal-form game (N,S,π) but with an outcome function ω instead of a payoff function π.
  - <u>Augmented mechanisms</u>. Roughly, an augmented mechanism is (N,S,ω,σ) is a mechanism plus a strategy profile that satisfies a specified solution concept.
- Some idiosyncratic language…
  - <u>Total performance</u>. The map from types to outcomes.
  - <u>Decision performance</u>. When an outcome consists of a decision and cash transfers, the map from types to the decision.

Dutch & Sealed-Tender Auctions

Initial formulation.

- In a "*Dutch Auction,*"
  - the auctioneer starts at a high price and reduces it continuously
  - the auction ends when some bidder shouts "Mine!" to claim the item at the current price
- Strategies
  - A *strategy* in the Dutch auction game specifies, for each price, whether to shout "Mine!"
  - A "*reduced strategy*" is a number specifying the highest price at which to shout "Mine!"

- In a "*first-price auction,*"
  - the object is assigned to the highest bidder
  - the price is the winning bid.
  - "Sealed tender" = "first-price"
- Strategies
  - A *strategy* in the sealed tender auction mechanism is a number, representing the amount bid.

"Dutch ≡ First Price"

- *Theorem. Suppose outcomes are identified by the assignment of the item and the price paid for it. Then, in reduced normal form, the Dutch auction and the first-price auction are equivalent mechanisms.*
- Proof. In each, a (reduced) strategy is described by a single number, the item is assigned to the bidder who names the largest number, and the price paid is that number. *QED*

English & "2nd-price" Auctions

- A "simplified *English auction*" is a mechanism in which
  - the auctioneer raises the price continuously
  - *bidders observe only the current price*
  - at each price, bidders decide whether to become (permanently) inactive.
  - the auction ends when only one bidder is active
  - the last active bidder gets the item for the final price.

- A "*second price auction*" is a sealed bid auction in which
  - the item is awarded to the highest bidder, but
  - the price is set equal to the highest bid among the remaining bidders.
- A "*reduced strategy*" in this simplified English auction specifies the price at which to become inactive.

"Simplified English ≡ 2nd Price"

- *Theorem. Suppose the outcome is identified as the assignment of the item and the price paid for it. Then, in reduced normal form, the* "simplified" *English auction and the second-price auction are equivalent mechanisms.*
- Proof. In each, a (reduced) strategy is described by a single number, the item is assigned to the bidder who names the largest number, and the price paid is the "second highest" number. *QED*

English Auctions Generally

- Strategy sets are larger in the English auction than in the 2nd price auction.
- That matters because…
  - Enforcement of collusive agreements
  - Attacking a competitor's bid budget
  - Value inferences
- Value assumptions
  - Private value models
  - Interdependent value models

Old Methods: Formulation

- First price auctions, the old way.
- Model
  - N bidders
  - Each bidder has a value v_n for the item that is drawn according to distribution F with positive density f.
  - Values are independently distributed.
  - Sealed tender rules: each bidder, knowing its value, places a bid. High bidder wins. Price is high bid.
  - **Question: What is a strategy in this context?**

Old Methods: Formulation

- First price auctions, the old way.
- Model
  - N bidders, minimum bid is zero.
  - Each bidder has a value v_n for the item that is drawn according to distribution F on [0,V] with positive density f.
  - Values are independently distributed.
  - Sealed tender rules: each bidder, knowing its value, places a bid. High bidder wins. Price is high bid.
  - A strategy is a function $\beta : [0,V] \to \mathbb{R}_+$
  - Guess that the function is increasing. Then,

$$\beta(v) \in \arg\max_b (v-b)F^{N-1}(\beta^{-1}(b))$$

Analysis Strategy

- Use minimum bid of zero to determine a boundary condition.
- Derive the first-order condition.
- Substitute the equilibrium conditions that $b=\beta(v)$ and the inverse function condition that $d\beta^{-1}/db = 1/(d\beta/dv)$.

Old Methods: Analysis

- First order condition is:
  $0=-F^{N-1}(\beta^{-1}(b))+(N-1)(v-b)f(\beta^{-1}(b))(F^{N-2}(\beta^{-1}(b))/\beta'(\beta^{-1}(b))$
- At the solution, b=β(v) and $\beta^{-1}(b)=v$, so the FOC becomes:
  $0=-F^{N-1}(v)+(N-1)(v-\beta(v))F^{N-2}(v)f(v)/\beta'(v)$
- Rearranging, we get the ("envelope") formula:
  $\frac{d}{dv}\left((v-\beta(v))F^{N-1}(v)\right)=F^{N-1}(v)$. Integrating and rearranging: $\beta(v)=v-\frac{K+\int_0^v F^{N-1}(s)ds}{F^{N-1}(v)}$, with $K=0$ because $\beta(0)=0$. (Why?)

Bidding to Buy or Sell?

- In auctions where parties *bid to buy* with value *v* and bid *b*, the winner's payoff is *v - b* and the losers get zero.
- In auctions where parties *bid to sell* goods or services at cost c, the winner is paid its bid amount. So, the winner gets *b – c* and losers get zero.
  - An equivalent formulation is that the parties are bidding to buy with a "value" of *–c* and a bid of *–b*.
  - As in any "bid to buy" auction, the winner's payoff is its value minus its bid, or *–c*-(-*B*)=*B*-*c*.
- Conclusion: our results about *bidding-to-buy* also apply to the case of *bidding-to-sell*.

Homework #1

- Problem #1:
  - Obtain another formula for the equilibrium bid function of the first-price auction which can be interpreted as: β(v) = E[highest value among the N-1 other values | my value of v is highest].
  - Explain how your formula merits this interpretation.
- Problem #2: Show that there can be other equilibria if there is no minimum bid at all.
- Problem #3:
  - Use this "old" method to find the symmetric equilibrium of an auction with two bidders in which both bidders pay their own bids but only the highest bidder wins the object.
  - Be sure to use the same model and notation, adjusting only for the difference in rules.

Topic #2: Vickrey-Clarke-Groves ("VCG") Mechanisms

Motivation

- The problem studied in this section is how to implement efficient allocations when those allocations depend on participants' preferences, which only they know.
- Participants may misrepresent their preferences.
  - A seller might exaggerate its costs, hoping to get a higher price.
  - A homeowner may claim that she doesn't benefit from certain public services, hoping that other will pay.
- This section describes a class of mechanisms that can implement efficient allocations and yet make it in every individual's interest to report her preferences truthfully.

Notation

- N = set of participants, including the designer, participant 0. $S \subset N$. $j \in N$. Type vector $\vec{t}$.
- Outcome is (x,p) where x is a decision and p is a vector of payments by participants.
- Payoffs: $u^i\left((x,p),\vec{t}\right) \equiv v^i(x,t^i) - p^i$
- Additional notation:

$$V(X,S,\vec{t}) = \max_{x \in X} \sum_{j \in S} v^j(x,t^j)$$

$$\hat{x}(X,S,\vec{t}) \in \arg\max_{x \in X} \sum_{j \in S} v^j(x,t^j)$$

Remark on the Formulation

- In some respects, the scope of the preceding formulation is very wide
  - The decision x can be anything:
    - » the allocation of a good or goods in an auction
    - » a public goods decision
    - » the design of a new product
  - Transfers ("payments") can be anything
  - Values can be "anything" provided participants know their values
- Restrictions
  - Participants must know their values!
  - Money must enter the payoffs linearly.
  - Decisions must be capable of money compensation.

VCG Mechanism

- **Defining characteristic**: Value-maximizing outcomes. Player *i's* report does not affect the total payoff to others, including transfers.
- Suppose *i* reports being indifferent among all decisions. Then,
  - *i* pays some amount $h^i(t^{-i})$.
  - The optimal decision and total payoff to others are:

$$\hat{x}(X,N-i,t^{-i}) \in \arg\max_{x \in X} \sum_{j \in N-i} v^j(x,t^j)$$

$$Total\ \ Payoff_{-i} = V(X,N-i,t^{-i}) + h^i(t^{-i})$$

  - Note that payments besides *i*'s are irrelevant for this calculation. (Why?)

Payment Formula

- If i makes a different report, the decision and values for the others are given by:
  - $\hat{x}(X,N,\vec{t}) \in \arg\max_{x\in X} \sum_{j\in N} v^j(x,t^j)$
  - $\sum_{j\in N-i} v^j(\hat{x}(X,N,\vec{t}),t^j) + p^i(\vec{t}) = V(X,N-i,t^{-i}) + h^i(t^{-i})$
- So, the VCG payment formula must satisfy:
  - $p^i(\vec{t}) = V(X,N-i,t^{-i}) - \sum_{j\in N-i} v^j(\hat{x}(X,N,\vec{t}),t^j) + h^i(t^{-i})$
- The VCG mechanism with $h^i(t^{-i})=0$ for all i is called the "pivot mechanism."

Dominant strategies

- We say that all reports are "potentially pivotal" if for all i, and any two types t^i & $\tilde{t}^i$, there exists t^{-i} such that:

$$\sum_{j\in N} v^j(\hat{x}(X,N,\tilde{t}^i,t^{-i}),t^j) < V(X,N,\vec{t})$$

- Theorem. In any VCG mechanism, truthful reporting is always a best reply. If all reports are potentially pivotal, then truthful reporting is a dominant strategy.
  - Here, we prove the first part only.

Proof of "Always Best Reply"

- We compare i's payoff from truth-telling versus reporting falsely, using the definition of the function $\hat{x}$.

$$\begin{aligned}
& v^i(\hat{x}(X,N,\tilde{t}^i,t^{-i}),t^i) - p^i(\tilde{t}^i,t^{-i}) \\
&= v^i(\hat{x}(X,N,\tilde{t}^i,t^{-i}),t^i) - \left(V(X,N-i,t^{-i}) - \sum_{j\in N-i} v^j(\hat{x}(X,N,\tilde{t}^i,t^{-i}),t^j) + h^i(t^{-i})\right) \\
&= \sum_{j\in N} v^j(\hat{x}(X,N,\tilde{t}^i,t^{-i}),t^j) - \left(V(X,N-i,t^{-i}) + h^i(t^{-i})\right) \\
&\le \sum_{j\in N} v^j(\hat{x}(X,N,t^i,t^{-i}),t^j) - \left(V(X,N-i,t^{-i}) + h^i(t^{-i})\right) \\
&= v^i(\hat{x}(X,N,t^i,t^{-i}),t^i) - \left(V(X,N-i,t^{-i}) - \sum_{j\in N-i} v^j(\hat{x}(X,N,t^i,t^{-i}),t^j) + h^i(t^{-i})\right) \\
&= v^i(\hat{x}(X,N,t^i,t^{-i}),t^i) - p^i(t^i,t^{-i})
\end{aligned}$$

- The conclusion is that the payoff from truth-telling is higher.

Second price auctions

- The good is worth t^i to bidder i.
- Pivot mechanism:
  - Item is awarded to bidder with highest value.
  - Losing bidders pay 0.
  - Winning bidder pays the second highest value.

$$\begin{aligned}
p^i(\vec{t}) &= V(X,N-i,t^{-i}) - \sum_{j\in N-i} v^j(\hat{x}(X,N,\vec{t}),t^j) \\
&= V(X,N-i,t^{-i}) - 0
\end{aligned}$$

- The second price auction is sometimes called a "Vickrey auction."

Budget balance

- There does not generally exist any VCG mechanism to balance the budget. Adding up the payments yields a restriction:

$$\sum_{i\in N} p^i(\vec{t}) = \sum_{i\in N} f^i(t^{-i}) - (|N|-1)V(X,N,\vec{t}) = 0$$

where $f^i(t^{-i}) = V(X,N-i,t^{-i}) + h^i(t^{-i})$

$$V(X,N,\vec{t}) = \sum_{i\in N} f^i(t^{-i})/(|N|-1)$$

Proof by Example

- Single good to be allocated to bidder 1 or 2, where 1's values are in {1,3} and 2's values are in {2,4}.
  - Total payment in value profile (1,2) plus those in value profile (3,4) is $4+h^2(1)+h^1(2)+h^2(3)+h^1(4)$.
  - Total payment in value profile (1,4) plus those in value profile (3,2) is $3+h^2(1)+h^1(2)+h^2(3)+h^1(4)$.
  - Not possible that both sums are zero.

Vickrey Package Auctions

- Advantages
  - Wide scope of mechanism
  - Efficient outcomes
  - Dominant strategies
    - Predictions robust to details of specification
    - Transaction costs are reduced
- Disadvantages
  - Computational complexity/cognitive burden
  - Information revealed
  - Unlimited budgets required
  - Several **monotonicity-related problems:**
    - Illustrated below
  - Investment-merger incentives

Low & Non-Monotonic Revenues

- Two spectrum licenses, three potential bidders
  - Bidder 1 is a new entrant who needs two licenses for efficient scale operation and will pay $1 billion for the pair
  - Bidders 2 and 3 are incumbents who seek to expand capacity. Each needs just one license and will pay $1 billion.
- Auction outcomes:
  - If just bidders 1 and 2 compete, revenue is $1 billion.
  - If all three bidders compete, prices and revenues are $0.
  - Conclusion: outcome is not in the core ("low revenues") and revenue is not monotonic in participation or bidder values.

Losing Bidders Can Collude to Win

- Two spectrum licenses, three bidders
  - Bidder 1 is a new entrant who needs two licenses for efficient scale operation and will pay $1 billion for the pair
  - Bidders 2 and 3 are incumbents who seek to expand capacity. Each needs just one license and will pay $250 million.
- Auction outcomes
  - If the incumbents bid honestly, they lose.
  - If the incumbents each bid $1 billion, they win at a total price of zero.

Profitable Use of Shills

- Two spectrum licenses, two bidders
  - Bidders 1 and 2 are both new entrants who needs two licenses for efficient scale operation.
  - Bidder 1 will pay up to $1 billion for the pair
  - Bidder 2 will pay up to $900 million for the pair.
- Auction outcomes
  - If bidder 2 bids honestly, it loses.
  - If bidder 2 enters the auction as 2A and 2B, each of which bids $1 billion for a single license, it wins both licenses at a total price of zero.

Mergers and Investments

- Two spectrum licenses, three bidders
  - Bidder 1 is a new entrant who needs two licenses for efficient scale operation and will pay up to $1 billion for the pair
  - Bidders 2 and 3 are incumbents who seek to expand capacity. Each needs just one license and will pay up to $1 billion.
  - If bidders 2 & 3 merge their operations, total value increases by 25% to $2.5 billion.
- Auction Outcomes
  - Unmerged bidders pay $0; net profit is $2 billion.
  - Merged bidder pays $1 billion, net profit is $1.5B.
  - Value-enhancing merger is deterred.

Two Lessons

- The real game is always bigger than you first think
  - Mergers
  - Investments
  - License designs
  - Auction rules
  - Shills
- The Vickrey auction has significant drawbacks
  - Complexity and privacy issues
  - “Monotonicity problems”
  - Merger-investment disadvantage

Topic #3: Envelope Theorem

Optimization in Economics

- Maximization/equilibrium models are widespread.
  - Traditional
    - Consumer theory
    - Producer theory
  - Newer
    - Auction theory
    - Game and incentive theories
- The new applications require more general theorems about optimization than the producer and consumer theory applications.
- Two kinds of results are particularly useful
  - Envelope theorems (treated today)
  - "Robust" comparative statics/sensitivity analysis theorems (treated another day)

Envelope Formula Example

- Consider the problem $\max_{x\in[0,1]} f(x,t)$ where $f(x,t)=(3t-1)x-\frac{1}{2}x^2$.
- The solution is described by the maximizer and the maximum value:

$$x(t)\equiv \text{"}\arg\max f(x,t)\text{"}=\begin{cases}0 & \text{if } t\le \frac{1}{3}\\ 3t-1 & \text{if } \frac{1}{3}<t<\frac{2}{3}\\ 1 & \text{if } t\ge\frac{2}{3}\end{cases}$$

$$V(t)\equiv \max_{x\in X} f(x,t)=\begin{cases}0 & \text{if } t\le \frac{1}{3}\\ \frac{1}{2}(3t-1)^2 & \text{if } \frac{1}{3}<t<\frac{2}{3}\\ 3t-\frac{1}{2} & \text{if } t\ge\frac{2}{3}\end{cases}$$

Example, continued

- Observe that $x(t)$ is not differentiable; it has kinks at t = 1/3 and 2/3.
- However, V is continuously differentiable and satisfies:

$$V'(t)=f_2(x(t),t)=\begin{cases}0 & \text{if } t< \frac{1}{3}\\ 3t-1 & \text{if } \frac{1}{3}\le t\le\frac{2}{3}\\ 1 & \text{if } t>\frac{2}{3}\end{cases}$$

where $f_2(x,t)\equiv \frac{\partial f}{\partial t}$.

- The "envelope formula" ($V'=f_2$) reappears in many guises. Important special cases in economic theory are Hotelling's lemma and Shepard's lemma.

Producer Theory

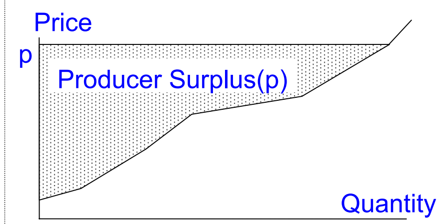
- With production set X in $\mathbf{R}^L$, a price-taking firm's indirect profit function expresses its maximum profits as a function of the prevailing prices:

$$\pi(p) = \max_{x \in X} p \bullet x$$

- Lemma (Hotelling): If $\pi(\cdot)$ is differentiable at p, then $x_j^*(p) = \partial \pi / \partial p_j$ for all j. Moreover, if $\pi(\cdot)$ is absolutely continuous, then

$$\pi(p) = \pi(0, p_{-1}) + \int_0^{p_1} \pi_1(s, p_{-1})ds$$
$$= \pi(0, p_{-1}) + \int_0^{p_1} x_1^*(s, p_{-1})ds$$

Producer Surplus



The shaded area between a firm's supply curve and the vertical is the firm's producer surplus.

- Producer surplus can be expressed as an integral in two ways, one of which is Hotelling's lemma.

Envelope Theorems

- Generally, envelope theorems deal with the properties of the value function:

$$V(t) \equiv \max_{x \in X} f(x, t)$$

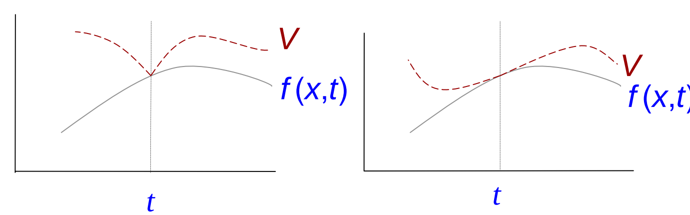
- Usual textbook intuition.
  - If "everything" is differentiable, then using the chain rule,

$$V(t) = f(x(t), t)$$
$$V'(t) = f_1(x(t), t)x'(t) + f_2(x(t), t)$$
$$= 0 \cdot x'(t) + f_2(x(t), t)$$
$$= f_2(x(t), t)$$

  - In the special case where f(x,p)=p·x, this becomes Hotelling's lemma.
- But what if $f(x,t)$ is not differentiable in x?

The Derivative Formula

- Let $V(t) \equiv \max_{x \in X} f(x, t)$, $X^*(t) = \arg\max_{x \in X} f(x, t)$
- Theorem 1. Take $t \in [0,1]$ and $x \in X^*(t)$, and suppose that $f_2(x,t)$ exists.
  - If t<1 and $V'(t+)$ exists, then $V'(t+) \geq f_2(x,t)$.
  - If t>0 and $V'(t-)$ exists, then $V'(t-) \leq f_2(x,t)$.
  - If $t \in (0,1)$ and $V'(t)$ exists, then $V'(t) = f_2(x,t)$.
- Proof:



The Integral Formula

- Theorem 2(A). Let X be the choice set and [0,1] the parameter set. Suppose
  - For all t, $X^*(t) \neq \varnothing$
  - for all (x,t), $f_2(x,t)$ exists
  - $V(t)$ is absolutely continuous.

  Then for any selection $x(t)$ from $X^*(t)$,

$$V(t) = V(0) + \int_0^t f_2(x(s),s)ds.$$

- Note similarity to the derivative formula:

$$V'(t) = f_2(x(t),t)$$

Proof of Theorem 2(A)

1. Since V is absolutely continuous, it is differentiable almost everywhere.
2. By Theorem 1, where the derivative exists, it satisfies $V'(t) = f_2(x(t),t)$.
3. By the Fundamental Theorem of Calculus, an absolutely continuous function is the integral of its derivative.

   QED

Applications

- The first application of the envelope theorem is to derive Hotelling's lemma.
  - Let $t=p_1$. Let $f(x,t)=tx_1+p_2x_2\ldots+p_Nx_N$. Write $\pi(p)$ for $V(t)$.
  - Then, $f_2(x,t) = x_1$.
  - So, the envelope formula is

$$\begin{aligned} V(t) = \pi(t,p_{-1}) &= \pi(0,p_{-1}) + \int_0^t \pi_1(s,p_{-1})ds \\ &= \pi(0,p_{-1}) + \int_0^t x_1^*(s,p_{-1})ds \end{aligned}$$

- In the coming slides, we will apply the theorem to more complicated looking objective functions *f*.
  - For understanding, the trick will be to look past the enormous quantities of ink used in writing the objective functions and just apply the envelope formula.

Multi-Dimensional Parameters

- The same theorem can be applied to paths through a multidimensional parameter space.
- Sample objective: $f(x,t)$
  - Let $t(\cdot)$ be a smooth path through $[0,1]^N$.
  - Define $g(x,s)=f(x,t(s))$, where $s\in[0,1]$.
  - Applying the envelope theorem and the chain rule,

$$\begin{aligned} V_f(t(s)) &= V_g(s) \\ &= V_g(0) + \int_0^s g_2(x(t(r)),r)dr \\ &= V_f(0) + \int_0^s f_2(x(t(r)),t(r))\cdot t'(r)dr. \end{aligned}$$

Absolute Continuity

- A math topic, not included on the course examination
  - Same comment applies also to the next two slides
- <u>Theorem 2(B)</u>. Suppose that
  - $f(x,\cdot)$ is absolutely continuous for all $x \in X$.
  - there exists an integrable function $b(t)$ such that $|f_2(x,\cdot)| \le b(t)$ for all $x \in X$ and almost all $t \in [0,1]$.

  Then V is absolutely continuous and $|V'(t)| \le b(t)$ for almost all $t \in [0,1]$.

Proof of Theorem 2(B)

- Define

$$B(t) = \int_0^t b(s)ds$$

- Then:

$$|V(t'') - V(t')| \le \sup_{x \in X} |f(x,t'') - f(x,t')|$$

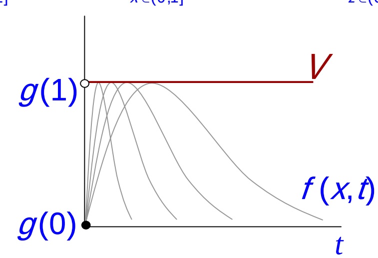
$$= \sup_{x \in X} \left| \int_{t'}^{t''} f_t(x,t)dt \right| \le \int_{t'}^{t''} \sup_{x \in X} |f_t(x,t)| dt$$

$$\le \int_{t'}^{t''} b(t)dt = B(t'') - B(t')$$

The Integrable Bound

- The bounding function b is indispensable
- Let $X=(0,1]$ and $f(x,t)=g(t/x)$, where g is smooth and single-peaked with unique maximum at 1.
  - $V(0)=g(0)$, $V(t)=g(1)$: V is discontinuous at 0.
  - The example has no integrable bound:

$$\sup_{x \in (0,1]} |f_2(x,t)| = \sup_{x \in (0,1]} \left| \tfrac{1}{t}\left(\tfrac{t}{x} g'(\tfrac{t}{x})\right) \right| = \tfrac{1}{t} \sup_{z \in (0,\infty)} |zg'(z)|$$



Necessity of the VCG Mechanisms

Goals of this section

- We have seen in the previous section that the VCG mechanisms are dominant strategy mechanisms that implement efficient outcomes.
- In this section, we show a converse, that the VCG mechanisms are the only mechanisms with those properties.
  - This conclusion requires an assumption that the set of possible preferences is sufficiently rich.
  - Method: use the envelope theorem.

Dominant Strategies

- Letting S^i denote a strategy set.
- Fixing the strategies played by others:

$$V^i(t^i,\sigma^{-i}) = \max_{\sigma_i \in S^i} u^i(x(\sigma^i,\sigma^{-i}),t^i)$$

$$= V^i(0,\sigma^{-i}) + \int_0^{t^i} u_2^i\left(x(\sigma^{*i}(s),\sigma^{-i}),s\right)ds$$

- The dominant strategy property is reflected in the fact that the maximizing strategy σ^{*i} depends only on i's type and not on the other players' strategies.

Holmstrom's lemma

- An outcome is a pair (z,p) where z is a decision from some finite set and p is a vector of cash payments.
- Suppose payoffs can be written in the quasi-linear form:

$$u^i(z,p,t^i) = v^i(z,t^i) - p^i = v^i(t^i)\cdot z - p^i$$

- Applying the envelope theorem leads to:
- <u>Holmstrom's lemma</u>: In this problem,

$$V^i(t^i,\sigma^{-i}) \equiv v^i(t^i)\cdot z(\sigma^{*i}(t^i),\sigma^{-i}) - p^i(\sigma^{*i}(t^i),\sigma^{-i})$$

$$= V^i(0,\sigma^{-i}) + \int_0^{t^i} v^{i\prime}(s)\cdot z(\sigma^{*i}(s),\sigma^{-i})ds$$

Green-Laffont-Holmstrom Theorem

- Under certain conditions, the VCG mechanism is the <u>only</u> way to implement efficient outcomes in dominant strategies.
- <u>Theorem</u>. Suppose that for each bidder i, the type space is compact and smoothly path-connected and $v^i(\cdot)$ is continuously differentiable. Then, any mechanism that implements the efficient outcome in dominant strategies entails the same payments as some Vickrey-Clarke-Groves mechanism.

Proof Idea

- By Holmstrom's lemma, once the outcome function is fixed, a player's payoffs as a function of its type are fixed up to a constant that depends on the others' types.
- Once the outcome and a player's payoffs are fixed, its payment is fixed.
- So, the set of payment functions for player i in an efficient dominant strategy mechanisms is a family of functions that varies up to a constant that depends on the others' types.
- The set also includes all the VCG payment mechanisms, but that leaves no room for anything else.

Proof, 1

- Let p be the payment rule of some mechanism that implements the efficient decision performance and let V be its value function.
- Let p^* be a VCG payment function with value function V^* and with $h^i(t^{-i})$ chosen so that

$$V^{*i}(0,t^{-i}) \equiv V^i(0,\sigma^{-i}(t^{-i})).$$

- Applying Holmstrom's lemma twice,

Proof, 2

$$p^i(\sigma^{*i}(t^i),\sigma^{-i}(t^{-i})) = -V^i(0,\sigma^{-i}(t^{-i})) + v^i(t^i)\cdot z(\sigma^{*i}(t^i),\sigma^{-i}(t^{-i})) - \int_0^{t^i} v^{i\prime}(s)\cdot z(\sigma^{*i}(s),\sigma^{-i}(t^{-i}))ds$$

and

$$p^{*i}(t^i,t^{-i}) = -V^{*i}(0,t^{-i}) + v^i(t^i)\cdot z(t^i,t^{-i}) - \int_0^{t^i} v^{i\prime}(s)\cdot z(s,t^{-i})ds$$

- By inspection, $p^i(\sigma^{*i}(t^i),\sigma^{-i}(t^{-i})) \equiv p^{*i}(t^i,t^{-i})$

QED

Topic #4: "Revenue Equivalence" and Related Results

Motivation

- The most famous theorem in auction theory is the revenue equivalence theorem.
- History
  - Vickrey (1961, 1962) introduced a now widely used auction model, studied a variety of auction rules, computed equilibrium strategies, calculated the seller's expected revenues, and always found the same answer.
  - This was a puzzle until about 1981, when papers by Myerson and by Riley & Samuelson lent deep insight into the reasons for the conclusion.
  - We give a modern treatment, in which the envelope theorem is the key ingredient.
- To study this theory, we need first to review the theory of Bayesian games and mechanisms.

Bayesian Mechanisms

- Players are 1,…,N
- "Actions" (strategies?) available to player i are S^i, i=1,…,N.
- Outcome function, $\omega : S^1 \times \ldots \times S^N \to \Omega$.
- … and, for each , i=1,…,N,
  - "Types" $t^i \in T^i$
  - Payoffs: $u^i(\omega(s^1,\ldots,s^N),t^1,\ldots,t^N)$
  - Beliefs: $\pi^i(t^{-i}|t^i)$
  - "Strategies" $\sigma^i : T^i \to S^i$

Private Values Assumption

- Private values assumption
  - $u^i(\omega(s^1,\ldots,s^N),t^1,\ldots,t^N) = u^i(\omega(s^1,\ldots,s^N),t^i)$.
- Discussion
  - "I know my preferences."
  - But what if I may someday want to resell this good?
  - What if you have information about its authenticity?
  - General case is called "interdependent values."

Bayes-Nash Equilibrium

- Definition. A strategy profile σ is a *Bayes-Nash equilibrium* of Γ if for all types t^i,

$$\sigma^i(t^i) \in \arg\max_{\tilde{\sigma}^i \in S^i} E^i\left[u^i\left(\omega\left(\tilde{\sigma}^i, \sigma^{-i}(t^{-i})\right), \vec{t}\right) \middle| t^i\right]$$

$$= \arg\max_{\tilde{\sigma}^i \in S^i} \int u^i\left(\omega\left(\tilde{\sigma}^i, \sigma^{-i}(t^{-i})\right), \vec{t}\right) d\pi^i(t^{-i} \mid t^i).$$

- Notation.

$$\sigma^i = i\text{'s strategy}$$
$$\omega = \text{outcome function}$$
$$u^i(\omega, \vec{t}) = i\text{'s payoff}$$
$$\pi^i = i\text{'s beliefs}$$

An Aside:

"Purely Technical" Assumptions

- Let us call an assumption *purely technical* when either
  1. The formulation can always be modified to make the assumption true, or
  2. The assumption can never be refuted by a finite number of "simple" empirical observations.
- We use such assumptions freely to simplify formulations and distinguish them from *restrictive* assumptions.
- Example:
  - Consider any function *f(x)* and the finite set of simple observations $(x_1,f(x_1)),\ldots,(x_n,f(x_n))$.
  - No finite set of simple observations can refute the assumption that *f* is continuous or continuously differentiable.
  - The assumptions that *f* is positive or increasing, however, can be refuted and hence are restrictive.

Assumptions

- Restrictive payoff assumptions
  - Quasi-linear payoffs.
  - Risk neutrality.
- Restrictive belief assumptions
  - Identical beliefs.
  - Types independently distributed.
- Purely "technical" assumptions
  - Each v^i is continuously differentiable (so the envelope theorem applies) and non-decreasing.
  - Types distributed <u>uniformly</u> on [0,1]. If v^i is increasing, then $\Pr\{v^i(t^i)\le\gamma\}=(v^i)^{-1}(\gamma)$.

Myerson's lemma

- For any player *i*:

$$V^i(t^i;\sigma^{-i})=\max_{\tilde\sigma^i} E^i\left[z\left(\tilde\sigma^i,\sigma^{-i}(t^{-i})\right)\bullet v^i(t^i)-p^i\left(\tilde\sigma^i,\sigma^{-i}(t^{-i})\right)\right]$$

- <u>Myerson's lemma</u>: Let $z(\vec t)\equiv z\left(\sigma^i(t^i),\sigma^{-i}(t^{-i})\right)$ Then, at equilibrium, the expected payoffs satisfy:

$$V^i(\tau;\sigma^{-i})=V^i(0;\sigma^{-i})+\int_0^\tau E^i[z(\vec t)\mid t^i=s]\cdot\frac{dv^i}{ds}ds$$

- <u>Proof Idea</u>: Envelope theorem applied to Bayesian maximization problem.

Proof Details

- Identify the objective function *f* and apply the envelope theorem, as follows:

$$V^i(t^i;\sigma^{-i})=\max_{\tilde\sigma^i} E^i\left[z\left(\tilde\sigma^i,\sigma^{-i}(t^{-i})\right)\bullet v^i(t^i)-p^i\left(\tilde\sigma^i,\sigma^{-i}(t^{-i})\right)\right]$$

$$\equiv\max_{\tilde\sigma^i} f\left(\tilde\sigma^i,t^i;\sigma^{-i}\right)$$

$$f_2\left(\tilde\sigma^i,t^i;\sigma^{-i}\right)=E^i\left[z\left(\tilde\sigma^i,\sigma^{-i}(t^{-i})\right)\cdot v^{i\prime}(t^i)\right]$$

$$V^i(\tau;\sigma^{-i})=V^i(0;\sigma^{-i})+\int_0^\tau f_2(\sigma^{*i}(s),s)ds$$

$$=V^i(0;\sigma^{-i})+\int_0^\tau E^i[z(\vec t)\mid t^i=s]\cdot\frac{dv^i}{ds}ds$$

"Revenue Equivalence"

- Theorem. Consider
  - the standard symmetric auction model with a M indivisible goods for sale and identical, independent atomless type distributions, and each bidder able to buy just one item.
  - a mechanism for which the outcome is always efficient and the lowest type bidder always pays zero.
- For every such mechanism,
  - every type of every bidder has same conditional expected payoff, given its type, as in the "highest rejected bid" auction (in which price is the M+1st highest bid).
  - the seller's expected revenue is M times the expectation of the M+1st highest buyer value.
- Proof Idea. Apply Myerson's lemma. Notation: $t^{(1)}$, $t^{(2)}$,… are order statistics.

Proof Details

- Let z(·) be the efficient decision. In any auction satisfying the hypotheses, the expected payoff of a type zero bidder is zero. So, by Myerson's lemma, the expected payoff of a bidder of type τ is:

$$\int_0^\tau E[z(\vec{t}\,)\,|\,t^i = s]\cdot\frac{dv^i}{ds}ds$$

- Also, the expected total payoff to all parties, including the seller, is:

$$\int_0^1 \ldots \int_0^1 \left(v(s^{(1)}) + \ldots + v(s^{(M)})\right) ds^n \ldots ds^1$$

- So, the seller's expected payoff must be the same as at the highest rejected bid auction:

$$M\int_0^1 \ldots \int_0^1 v(s^{(M+1)}) ds^n \ldots ds^1$$

Vickrey's Examples

- M items for sale. N bidders. Each bidder wants only one item.
- Auction designs:
  - Each of the M highest bidders pays the M+1st highest bid (the "pivot mechanism").
  - Each of the M highest bidders pays the amount of its own bid (a "sealed tender").
  - Each of the M highest bidders pays the lowest winning bid (T-bill mechanism).
- Vickrey's surprise: all lead to the same average price!

Example 1: Pivot Mechanism

- Consider a bidder whose value is v and bids b.
- If the M^{th} highest opposing bid is B, then the bidder's payoff is
  - v-B if b>B
  - 0 if b<B
- Payoff is always maximized by bidding b=v; no other bid is always optimal.
- If all play their dominant strategies, seller's revenue is equal to M times the M+1th highest value.

Example 2: Sealed Tender

- Model
  - N bidders, minimum bid is zero.
  - Each bidder has a value v_n for the item that is drawn according to distribution F on [0,V] with positive density f.
  - Values are independently distributed.
  - Sealed tender rules: each bidder, knowing its value, places a bid. High bidder wins. Each winning bidder's price is his bid.
  - A strategy is a function $\beta:[0,V]\to\mathbb{R}_+$
  - Guess that the function is increasing. Then,

$$\beta(v)\in\arg\max_b (v-b)\sum_{k=0}^{M-1}\frac{(N-1)!}{k!(N-1-k)!}\left(1-F(\beta^{-1}(b))\right)^k F^{N-1-k}(\beta^{-1}(b))$$

Sealed Tender Analysis

- First order condition is:

$$0=-\sum_{k=0}^{M-1}\frac{(N-1)!}{k!(N-1-k)!}\left(1-F(\beta^{-1}(b))\right)^k F^{N-1-k}(\beta^{-1}(b))+(v-b)\ldots$$

- At the solution, b=β(v) and $\beta^{-1}(b)=v$, so the FOC becomes:

$$0=-\sum_{k=0}^{M-1}\frac{(N-1)!}{k!(N-1-k)!}\left(1-F(v)\right)^k F^{N-1-k}(v)+(v-b)\ldots$$

- Rearranging, we get the ("envelope") formula:

$$\frac{d}{dv}\left((v-\beta(v))\sum_{k=0}^{M-1}\frac{(N-1)!}{k!(N-1-k)!}\left(1-F(v)\right)^k F^{N-1-k}(v)\right)=$$
$$\sum_{k=0}^{M-1}\frac{(N-1)!}{k!(N-1-k)!}\left(1-F(v)\right)^k F^{N-1-k}(v).$$

Continuation

- Integrating and rearranging terms:

$$\beta(v)=v-\frac{K+\int_0^v\sum_{k=0}^{M-1}\frac{(N-1)!}{k!(N-1-k)!}\left(1-F(s)\right)^k F^{N-1-k}(s)ds}{\sum_{k=0}^{M-1}\frac{(N-1)!}{k!(N-1-k)!}\left(1-F(v)\right)^k F^{N-1-k}(v)}$$

- With the constant of integration K equal to zero.
  - Look past all the ink!
  - Notice that a bidder's expected profits satisfy the envelope formula.

Guessing the Equilibrium

- In this case, we could have guessed the equilibrium strategy by equating our two formula for the winner's expected profits:

$$(v-\beta(v))\sum_{k=0}^{M-1}\frac{(N-1)!}{k!(N-1-k)!}\left(1-F(v)\right)^k F^{N-1-k}(v)$$
$$=\int_0^v\sum_{k=0}^{M-1}\frac{(N-1)!}{k!(N-1-k)!}\left(1-F(s)\right)^k F^{N-1-k}(s)ds$$

Two Person Bargaining, 1

- Background:
  - The outcome of bargaining may depend on the bargaining protocol, that is, the *mechanism* used for bargaining
  - The VCG pivot mechanism is one mechanism that supports efficient outcomes, but it may require a subsidy to run
- Questions:
  - How large, on average, is the subsidy required by the pivot mechanism?
  - What other mechanisms that lead to efficient outcomes at a Bayesian-Nash equilibrium require smaller subsidies than the pivot mechanism, at least on average?

Two-Person Bargaining, 2

- A seller has a value s and a buyer has value b, both distributed on [0,1].
- In the VCG "pivot" mechanism, they trade if and only if b>s.
  - Total surplus is max(b-s,0).
  - If trade takes place,
    - the seller then receives price b.
    - the buyer then pays price s.
  - In every event, at VCG equilibrium, both buyer and seller have payoff equal to max (b-s,0).
- Each player's expected payoff is E[max(b-s,0)].
  - So, on average, the VCG mechanism incurs a loss equal to E[max(b-s,0)].

Myerson-Satterthwaite

- Theorem. Any mechanism that
  - (1) results in efficient trade in the two-person bargaining problem at Bayes-Nash equilibrium, and
  - (2) entails no payments when there is no trade

  incurs an expected loss for the mechanism operator equal to E[max(0,b-s)], which is also the total expected gains from trade.
- Proof.
  - By Myerson's lemma, *any mechanism* that implements efficient trade with $V_b(0)=0$ and $V_s(1)=0$ has the same expected payoffs for each type of the buyer and for each type of the seller.
  - Since expected total surplus is E[max(0,b-s)] and each player expects to gain that amount, the result follows.

FCC Auction Application

- FCC sells the licenses
  - Efficient outcomes are theoretically *implementable* in the independent, private values environments
  - Theory: Vickrey auction theorem
- FCC uses a lottery among bidders
  - Initial misallocation may be uncorrectable by any incentive compatible mechanism in an independent private values environment
  - Contrary to Coase theorem
  - Theory: Myerson-Satterthwaite theorem
- Conclusion
  - Getting the initial allocation right can matter, as US and European experience confirms

Topic #5a: Optimal Auctions

The Problem

- A natural question that sellers designing an auction may ask is: "what auction leads to the highest expected revenue, or selling price?"
- The revenue equivalence theorem suggests that the key lies not in the payment rules but in the allocation that is induced.
- Can we even formulate the problem of maximizing over all possible mechanisms?
  - Yes, we can.

Auction Revenues

- Suppose that
  - value of good to bidder i is $v^i(t^i)$ where each v^i is increasing and differentiable
  - types distributed independently, uniformly on [0,1]
- Definitions.
  - An *augmented mechanism* (mechanism $(S,\omega)\equiv(S,x,p)$ plus equilibrium strategies σ) is voluntary if the maximal payoff $V^i(t^i)$ is non-negative everywhere.
  - The expected revenue from an augmented mechanism is the expected sum of payments:

$$R(S,\omega,\sigma)=E\left[\sum_{i=1}^{N}p^i(\sigma^1(t^1),\ldots,\sigma^N(t^N))\right]$$

Revenue Max Problem

- Types uniform on [0,1].
  - Values distributed according to $(v^i)^{-1}$.
- Problem $\max_{S,\omega,\sigma} R(S,\omega,\sigma)$
- Generalize to allow randomized mechanisms, so x^i is the *probability* of outcome i.
  - Then, we have these constraints on feasible mechanisms:

$$x^i(\vec{t}) \geq 0 \text{ for all } i \neq 0$$

$$\sum_{i\neq 0} x^i(\vec{t}) \leq 1$$

Total & Marginal Revenue

- Temporarily assume a single bidder with value $v(t)$, where v is increasing and t is uniformly distributed on [0,1].
- If the seller fixes a price $v(s)$, it sells when the buyer's value is higher, which happens with probability 1-s.
  - Then, 1-s is like the "quantity" sold.
  - Expected <u>total revenue</u> is $(1-s)v(s)$.
  - <u>Marginal revenue</u> is the derivative of total revenue with respect to quantity 1-s.

$$m(s)=\frac{d(1-s)v(s)}{d(1-s)}=-\frac{d(1-s)v(s)}{ds}=v(s)-(1-s)v'(s)$$

Revenue Characterization

- <u>Theorem</u>. Suppose that, at a Nash equilibrium of some mechanism, the probability that the good is allocated to a bidder j when the type profile is t is $x_j(t)$ and that the expected payoff of type 0 is $V_j(0)$. Then, the expected auction revenues are:

$$\int_0^1 \ldots \int_0^1 \sum_{i=1}^N x^i(s^1,\ldots,s^N)m^i(s^i)ds^1\ldots ds^N-\sum_{i=1}^N V^i(0)$$

  - Notice that this characterization involves x but not the expected payment function p.
  - Notice, too, that this expression is <u>linear</u> in the various $x(t)$'s.

Proof Outline: Calculate!

- There is lots of ink on the pages to follow, so keep track of what is going on!
  1. Use envelope theorem to express bidder profits for each type as a <u>linear</u> function of the allocation probabilities x.
  2. Express expected profits as a linear function of x, gathering coefficients of the x terms. (This is the tricky part; it entails reversing an order of integration).
  3. The total value of the allocation is also a linear function of x.
  4. Seller revenue is the expected total value minus bidder expected profits, still a linear function of x.

Proof

- Bidder 1's maximal payoff satisfies:

$$V^1(\tau)-V^1(0)=\int_0^\tau E[x^1(s^1,t^{-1})\,|\,t^1=s^1]\frac{dv^1}{ds^1}ds^1$$

$$=\int_0^\tau\int_0^1\ldots\int_0^1\frac{dv^1}{ds^1}x^1(s^1,\ldots,s^N)ds^2\ldots ds^N ds^1$$

- So, the bidder's <u>expected</u> payoff is:

$$E[V^1(t^1)]-V^1(0)=E[V^1(t^1)-V^1(0)]$$

$$=\int_0^1\int_0^\tau\int_0^1\ldots\int_0^1\frac{dv^1}{ds^1}x^1(s^1,\ldots,s^N)ds^2\ldots ds^N ds^1 d\tau$$

Proof Continued

- Reverse the order of integration and re-express…

$$E[V^1(t^1)] - V^1(0) = \int_0^1 \int_0^\tau \int_0^1 \ldots \int_0^1 \frac{dv^1}{ds^1} x^1(s^1,\ldots,s^N) ds^2 \ldots ds^N ds^1 d\tau$$

$$= \int_0^1 \ldots \int_0^1 \int_{s^1}^1 d\tau \frac{dv^1}{ds^1} x^1(s^1,\ldots,s^N) ds^1 \ldots ds^N$$

$$= \int_0^1 \ldots \int_0^1 (1-s^1) \frac{dv^1}{ds^1} x^1(s^1,\ldots,s^N) ds^1 \ldots ds^N$$

Proof Completed.

- Total payoff is $x(\vec{t}) \cdot v(\vec{t})$
- Total expected revenue $R(S,\omega,\sigma)$ is therefore:

$$= E[x(\vec{t}) \cdot v(\vec{t})] - \sum_{i=1}^N E[V^i(t^i)]$$

$$= \int_0^1 \ldots \int_0^1 \sum_{i=1}^N x^i(s^1,\ldots,s^N) v^i(s^i) ds^1 \ldots ds^N - \sum_{i=1}^N E[V^i(t^i)]$$

$$= \int_0^1 \ldots \int_0^1 \sum_{i=1}^N x^i(s^1,\ldots,s^N) \left(v^i(s^i) - (1-s^i) \frac{dv^i}{ds^i} \right) ds^1 \ldots ds^N - \sum_{i=1}^N V^i(0)$$

$$= \int_0^1 \ldots \int_0^1 \sum_{i=1}^N x^i(s^1,\ldots,s^N) m^i(s^i) ds^1 \ldots ds^N - \sum_{i=1}^N V^i(0)$$

Revenue Max Theorem

- Define the "marginal revenue" functions:

$$m^i(s^i) \equiv v^i(s^i) - (1-s^i) dv^i / ds^i$$

- Theorem. Suppose that the marginal revenue functions are non-decreasing. Then, an augmented mechanism is expected revenue maximizing if (i) each $V^i(0)=0$ ("no subsidies") and (ii) the good is allocated to bidder i exactly when $m^i(t^i) > \max\left(0, \max_{j \neq i} m^j(t^j)\right)$. Furthermore, at least one such augmented mechanism exists. The maximum expected revenue is:

$$E\left[\max\left(0, m^1(t^1), \ldots, m^N(t^N)\right)\right].$$

Proof.

- By the previous theorem, the revenue function is:

$$R(S,\omega,\sigma) = \int_0^1 \ldots \int_0^1 \sum_{i=1}^N x^i(s^1,\ldots,s^N) m^i(s^i) ds^1 \ldots ds^N - \sum_{i=1}^N V^i(0)$$

$$\leq \int_0^1 \ldots \int_0^1 \max(0, \max m^i(s^i)) ds^1 \ldots ds^N$$

  - Aside: the 0 in the expression $\max(0,m^1\ldots)$ corresponds to a "reserve," that is, a condition under which the item is not sold to any bidder (and so "reserved for the seller").
- This revenue bound is achievable by a dominant strategy mechanism, as follows:

Proof Completed.

- Ask bidders to report their types. Assign the item to the bidder with the highest marginal revenue, provided that exceeds zero.
- Payments are made as follows:

$$p^i(\vec{t}) = p^i(t^{-i}) = \begin{cases} v^i\left((m^i)^{-1}\left(\max(0,\max_{j\neq i} m^j(t^j)\right)\right) & \text{if } x^i(t)=1 \\ 0 & \text{otherwise} \end{cases}$$

  - Note that $V^j(0)=0$ for all j.
  - It is straightforward to verify that truthful reporting is a dominant strategy.

Relation to Monopoly Theory

- Bulow-Roberts "interpretation":
  - The problem is analogous to a multi-market monopoly.
  - Each bidder is analogous to a separate "market" where the good can be sold.
  - The monopolist can price discriminate, setting different prices in different markets.
  - Quantity is analogous to probability of winning.
  - Deciding to whom to allocate the item is analogous to allocating quantities across separated markets.

Optimal Reserve Prices

- Corollary. Suppose that the valuation functions are identical ($v^1=\ldots=v^N\equiv v$) and that v and the corresponding marginal revenue function $m(s)=v(s)-(1-s)v'(s)$ are increasing. Then an auction maximizes expected revenue if (and only if) at equilibrium, it assigns the item to the bidder with the highest value provided that value exceeds $v(r)$, where $m(r)=0$.

More Optimal Auctions

- Corollary. Suppose that the valuation functions are identical ($v^1=\ldots=v^N\equiv v$) and that v and the corresponding marginal revenue function $m(s)=v(s)-(1-s)v'(s)$ are increasing. Then the following auctions all maximize expected revenue at their symmetric Bayes-Nash equilibria:
  - A second-price auction with minimum bid $v(r)$.
  - A first-price auction with minimum bid $v(r)$.
  - An all-pay auction with minimum bid $v(r)r^{N-1}$.

Aside

- Notice that the equilibrium strategies are for the first-price ("1"), second-price ("2") and all-pay-own-bid ("A") auctions are as follows:

$$\beta_2(t) = \begin{cases} 0 \text{ if } t < r \\ v(t) \text{ otherwise} \end{cases}$$

$$\beta_1(t) = \begin{cases} 0 \text{ if } t < r \\ E\left[\max\left(v(r), \beta_2(t^2), \ldots, \beta_2(t^N)\right) | \max(t^2, \ldots, t^N) < t\right] \text{ otherwise} \end{cases}$$

$$\beta_A(t) = \begin{cases} 0 \text{ if } t < r \\ \beta_1(t) F^{N-1}(t) \text{ otherwise} \end{cases}$$

Proof.

- The symmetric equilibrium strategies in each case are increasing functions for $t>r$.
- At the equilibrium of all of the preceding auctions, the highest type wins provided the type exceeds r. By Myerson's lemma, all such auctions lead to the same expected payoffs for every type of every bidder (and for the seller).

Bulow-Klemperer Theorem

- Jeremy Bulow and Paul Klemperer: "Auctions versus Negotiations"
- **Theorem**: Suppose that the marginal revenue function is increasing and that $v^i(0)=0$ for all i. Then, adding a single buyer to an "otherwise optimal auction but with zero reserve" yields more expected revenue than setting the reserve optimally, that is,

$$E\left[\max\left(m^1(t^1), \ldots, m^N(t^N), m^{N+1}(t^{N+1})\right)\right]$$
$$\geq E\left[\max\left(m^1(t^1), \ldots, m^N(t^N), 0\right)\right]$$

Math Observation, 1

- Given any random variable X and any real number α, $E[\max(X,\alpha)] \geq \max(E[X],\alpha)$.
- Proof: Suppose $E[X] \geq \alpha$. Then,

$$E\left[\max(X,\alpha)\right] = \int \max(x,\alpha) f(x) dx \geq \int x f(x) dx$$
$$= E[X] = \max(E[X],\alpha)$$

Next, suppose $E[X] \leq \alpha$. Then,

$$E\left[\max(X,\alpha)\right] = \int \max(x,\alpha) f(x) dx \geq \int \alpha f(x) dx$$
$$\geq \alpha = \max(E[X],\alpha)$$

  - Aside: This is a special case of "Jensen's inequality."

Math Observation, 2

- $E[m^i(t^i)]=v^i(0)$.
- Proof. Recall that:

$$TR^i(s) \equiv (1-s)v^i(s)$$

$$m^i(s) = \frac{d}{d(1-s)}\left(TR^i(s)\right) = -\frac{d}{ds}\left(TR^i(s)\right)$$

$$\therefore E[m^i(t^i)] = \int_0^1 m^i(s)ds = -\left(TR^i(1) - TR^i(0)\right)$$

$$= -\left(0 - v^i(0)\right) = v^i(0)$$

Proof: Bulow-Klemperer Theorem

- Applying the two math observations (with $X=m^{N+1}(t^{N+1})$):

E[Auction Revenue, N+1 bidders & no reserve]

$$= E\left[\max\left(m^1(t^1),\ldots,m^N(t^N),m^{N+1}(t^{N+1})\right)\right]$$

$$= \int\ldots\int \max\left(m^1(s^1),\ldots,m^N(s^N),m^{N+1}(s^{N+1})\right)ds^{N+1}\ldots ds^1$$

$$\geq \int\ldots\int \max\left(m^1(s^1),\ldots,m^N(s^N),\int m^{N+1}(s^{N+1})ds^{N+1}\right)ds^N\ldots ds^1$$

$$= \int\ldots\int \max\left(m^1(s^1),\ldots,m^N(s^N),0\right)ds^N\ldots ds^1$$

$$= E\left[\max\left(m^1(t^1),\ldots,m^N(t^N),0\right)\right]$$

$= E$[Auction Revenue, N bidders, optimal auction]. ■

Topic #5b: Related Applications

Additional Applications

- Weak Cartels
- Interdependent values
- The auction martingale theorem

The Cartel Problem

- In auctions, bidders sometimes form “rings” or “cartels” which meet secretly to collude in the bidding.
- In “strong” cartels, bidders may make payments among themselves to divide their ill-gotten gains. Such payments are illegal and leave a cash trail, so cartels prefer to avoid that.
  - Alternatively, other favors may be exchanged, but those, usually, either leave a trail or are a poor substitute for cash. .
- In “weak” cartels, the bidders cannot make payments but can talk before the auction to agree which of them will bid.
  - Questions: What is the most profitable strategy for a weak cartel, when preferences are private information? How much profit does it earn?

Weak Cartels (McAfee-McMillan)

- Given a mechanism, the *corresponding random allocation* ignores the types and assigns the good to bidder i with probability $\bar{x}^i = E[x^i(t^i)]$.
- <u>Theorem</u>. Suppose that for each *i*, $(1-t^i)dv^i/dt^i$ is decreasing and each $v^i(0)=0$. Then, any mechanism for a weak cartel (hence with $V^i(0)=0$ for all *i*) is ex ante Pareto dominated (for cartel members) by its corresponding random allocation.

Majorization Inequalities

- If $f,g:\mathbb{R}\to\mathbb{R}$ are non-decreasing functions, and $E[f(X)]$ and $E[g(X)]$ exist, then $E[f(X)g(X)] \geq E[f(X)]\bullet E[g(X)]$.
- If $f,g:\mathbb{R}\to\mathbb{R}$ with f non-decreasing and g non-increasing and $E[f(X)]$ and $E[g(X)]$ exist, then $E[f(X)g(X)] \leq E[f(X)]\bullet E[g(X)]$.

Proof

- A bidder’s expected profit is:

$$E[V^i(t^i)] = \int_0^1 V^i(\tau)d\tau = \int_0^1\int_0^\tau \frac{dv^i}{ds}x^i(s)dsd\tau$$
$$= \int_0^1\int_s^1 d\tau \frac{dv^i}{ds}x^i(s)ds = \int_0^1 (1-s)\frac{dv^i}{ds}x^i(s)ds.$$

- Similarly, $E[\bar{V}^i(t^i)] = \int_0^1 (1-s)\frac{dv^i}{ds}\bar{x}^i ds.$
- Since $x(\cdot)$ is non-decreasing, the majorization inequality implies:

$$\int_0^1 (1-s)\frac{dv^i}{ds}x^i(s)ds \leq \int_0^1 (1-s)\frac{dv^i}{ds}ds\int_0^1 x^i(s)ds$$
$$= \int_0^1 (1-s)\frac{dv^i}{ds}\bar{x}^i ds.$$

Interdependent Values

- We have seen that the VCG mechanism implements efficient outcomes in dominant strategies for the private values case.
- Question: Can we drop the private values assumption?
  - Example: suppose one bidder knows whether a painting is fake or authentic. Can the bidder be induced to reveal that?
- Problem: Solution concepts
  - The dominant strategy solution concept is hard to achieve without private values. A useful extension of the concept is the ex post Nash equilibrium.
  - If we further weaken the concept to Bayesian-Nash equilibrium, can more be achieved?

Ex Post Nash

- A strategy profile σ is an ex post Nash equilibrium if for all type profiles, the strategy profile $\sigma(t)$ is a Nash equilibrium:

$$\sigma^i(t^i) \in \arg\max_{\tilde{\sigma}^i} u^i(\tilde{\sigma}^i, \sigma^{-i}(t^{-i}), \vec{t})$$

- Note well that i's strategy in this definition depends only on *i*'s type, on not on the other types.
- If
  - σ^{-i} maps onto the set of opposing strategy profiles and
  - the private values assumption applies, that is,

$$u^i(\tilde{\sigma}^i, \sigma^{-i}(t^{-i}), \vec{t}) \equiv u^i(\tilde{\sigma}^i, \sigma^{-i}(t^{-i}), t^i)$$

  then, an ex post the condition is "nearly" a dominant strategy equilibrium.
- Sometimes argued to be an "appropriate" extension of dominant strategies to implementation environments with general payoffs.

Math Review

- Lemma. Let f,g:[0,1]→[0,1] satisfy

$$\left(\forall t \in (0,1)\right) F(t) \equiv \int_0^t f(s)ds = \int_0^t g(s)ds$$

  Then, $f=g$ almost everywhere.

- Simple case for intuition: If f and g are continuous functions, then F is continuously differentiable and f and g are both equal to the derivative F'.

Interdependent Values

- Suppose any bidder's value for a good may depend on what others know.
  - Its type is $t^i = (t_1^i, \ldots, t_N^i)$
  - Its value is $t_i^i + v^i(t_i^{-i})$
- Can *i*'s information about *j*'s value be used to improve the allocation at *ex post* equilibrium?

$$V^i(t_i^i, \cancel{t_{-i}^i}, t^{-i}) = \max_{\sigma^i} z^i(\sigma^i, \sigma^{-i}(t^{-i})) \cdot (t_i^i + v^i(t_i^{-i})) + p^i(\sigma^i, \sigma^{-i}(t^{-i}))$$

$$= V^i(0, \cancel{t_{-i}^i}, t^{-i}) + \int_0^{t_i^i} z^i(\sigma^{*i}(s, t_{-i}^i), \sigma^{-i}(t^{-i}))ds$$

- So, by the lemma, the integrand does not depend on t_{-i}^i (except possibly on a set of t_i^i of zero measure.)

Impossibility (Ex Post)

- Theorem (Jehiel-Moldovanu, 1999). In the model described above, no player's probability of acquiring an item can depend, at ex post equilibrium, on its knowledge about the other players' values.
- Corollary. Efficient outcomes cannot "generally" be implemented in dominant strategies in this environment.
  - "Counterexample": 1 always has the lowest value and knows the values of players 2 and 3.
  - Idea: you can't give me incentives for truthful reporting unless I'm already indifferent.

Impossibility (Bayesian)

- With private values, moving from the VCG dominant strategy implementation to Bayesian implementation does not help auction revenues, bargaining outcomes, etc.
  - Is the same true for interdependent values?
  - Or can Bayesian mechanisms make use of i's information about others (t^i_{-i})?
- Theorem (Jehiel-Moldovanu). Consider any decision performance $\hat{z}^i(\vec{t})$ such that $E\left[\hat{z}^i(\vec{t}) \mid t^i\right]$ depends non-trivially on t^i_{-i}. Then that decision performance is not Bayesian implementable by any mechanism.

Proof

- Observe that the function

$$V^i(t^i_i, \cancel{t^i_{-i}}) = \max_{\hat{\sigma}^i} E\left[z^i\left(\hat{\sigma}^i, \sigma^{-i}(t^{-i})\right)\left(t^i_i + v^i(t^{-i}_i)\right) - p^i\left(\hat{\sigma}^i, \sigma^{-i}(t^{-i})\right) \mid t^i\right]$$

  cannot depend on t^i_{-i}, because the RHS does not.
- Also, by the envelope theorem,

$$V^i(t^i_i, \cancel{t^i_{-i}}) - V^i(0, \cancel{t^i_{-i}}) = \int_0^{t^i_i} E\left[z^i\left(\sigma^i(s, t^i_{-i}), \sigma^{-i}(t^{-i})\right) \mid t^i\right] ds$$

- As one varies t^i_{-i}, the integrand appears to change but the integral does not. So, by the lemma, the integrand can depend only on t^i_{-i} only on a null set.

Optional Exercise

- Extended Revenue Equivalence Theorem
  - Suppose that payoffs are quasi-linear and the value to a bidder of winning the item is $v^i(t^i, t^{-i})$.
  - Show that if v^i is continuously differentiable, then any two mechanisms such that
    - (i) the maximum value satisfies $V^i(0)=0$ and
    - (ii) at equilibrium, the highest type bidder always wins

    must have the same average revenue for the seller.

Sequences of Auctions

- Sometimes, auction houses sell a sequence of similar goods, one after the other.
  - The RCA transponder sale from lecture #1 is an example.
- What does auction theory predict about the pattern of prices from a sequence of sales?
  - Do we expect prices to trend up or down over time?
  - Are prices a martingale (no trend on average), a submartingale (trending upwards) or a supermartingale (trending downwards)?

Calculation

- Suppose there are three bidders whose values are uniformly distributed on (0,1).
- Calculate the equilibrium bids in a sequence of two first-price auctions, using the revenue equivalence theorem.
  - A bidder in the second auction with type *t* who sees the winning bid of $b>\beta_1(t)$ at the first round imagines that the remaining types are drawn from a uniform distribution on $(0,\beta^{-1}(b))$. Since his expected payment when he wins must be the same as the corresponding second-price auction, $\beta_2(t)=t/2$.
  - A bidder in the first round of type *t* who learns that his type is highest expects to pay *t*/3, so $\beta_1(t)=t/3$.

The RCA Transponder Sale

- Sotheby's (1981)

Order	Winning Bidder	Price Obtained
1	TLC	14,400,000
2	Billy H. Batts	14,100,000
3	Warner Amex	13,700,000
4	RCTV	13,500,000
5	HBO	12,500,000
6	Inner City	10,700,000
7	UTV	11,200,000
Total		90,100,000

Iterated Expectations

- Suppose that $H(x,y,z)$ is a function of three groups of variables, distributed with a positive, continuous joint density *f*.
- Definitions:

$$f(y,z)=\int f(s,y,z)ds;\ f(z)=\int f(r,z)dr$$

$$E[H\,|\,y,z]=\left(\int H(s,y,z)f(s,y,z)ds\right)/f(y,z)$$

$$E[H\,|\,z]=\left(\int\int H(s_1,s_2,z)f(s_1,s_2,z)ds_2ds_1\right)/f(z)$$

- Claim ("iterated expectations"):

$$E[H\,|\,z]=E[E[H\,|\,y,z]\,|\,z]$$

Proof

- Just calculate:

$$
\begin{aligned}
E[H \mid z] &= \left(\int\int H(s_1,s_2,z)f(s_1,s_2,z)ds_1ds_2\right)/f(z) \\
&= \int\left(\int H(s_1,s_2,z)f(s_1,s_2,z)/f(s_2,z)ds_1\right)f(s_2,z)ds_2/f(z) \\
&= \int E[H \mid y,z]f(y,z)dy/f(z) \\
&= \int\int E[H \mid y,z]f(s_1,y,z)ds_1dy/f(z) \\
&= E[E[H \mid y,z] \mid z]
\end{aligned}
$$

Weber's Martingale Theorem

- Background assumptions and notation
  - Standard symmetric, independent private values model.
  - k items sold sequentially, with each bidder eligible to win just one.
  - I_n denotes the information available to each bidder after the sale of item n.
- <u>Theorem</u>. If each bidder's bid at any round is an increasing function of his type, then

$$E[p_n \mid I_{n-1}] = E[v(t^{(k+1)}) \mid I_{n-1}].$$

If the auction is a first or second-price auctions with prices publicly announced, then the sequence of prices forms a "martingale": $E[p_n|p_1,\ldots,p_{n-1}]=p_{n-1}$.

Proof

- Let I_n be the information available when item n+1 is to be sold. By the revenue equivalence theorem, the expected average payments by winners of the last k-n items, is $E[v(t^{(k+1)}) \mid I_n]$.
- The expected average payment at round n for items sold in rounds starting at n+1 is

$$E\left[E[v(t^{(k+1)}) \mid I_{n+1}] \middle| I_n\right] = E[v(t^{(k+1)}) | I_n]$$

so that price must also be expected for round n.
  - Notice that the price formula describes a *martingale* if p_n is "adapted to" I_n.

Topic #6: Single Crossing

Issues

- For several auctions, like the first-price and all-pay, we have so far derived candidate equilibrium strategies by assuming that the equilibrium is increasing and applying necessary conditions (first-order conditions or the envelope theorem).
- Questions to be resolved:
  1. Are there other equilibria in which bids are not increasing functions of values?
  2. What about sufficient conditions? Are the candidate strategies actually best replies, and hence equilibrium strategies?
- We approach these questions by…
  - developing "comparative statics" conditions that imply bid is increasing in type, regardless of how others may bid.
  - Then, surprise! The same conditions are part of a set of sufficient conditions for use in the equilibrium theory.

What is Most Important

- Proofs are included for completeness, but what you need to know, and what may be covered on the exam, is…
  - Main definitions
    - "single crossing differences"
    - "increasing differences"
  - Main theorems
    - Monotonic selection theorem
    - Finite sufficiency theorem
    - Constraint simplification theorem
  - Applications of the theorems to auction theory

"Law of Demand"

- The ideas developed here, though abstract, can be understood as extensions of the law of demand.
- Think of a profit-maximizing firm and let

$$x \in \arg\max_{y \in X} p \cdot y$$

$$x' \in \arg\max_{y \in X} p' \cdot y$$

- Then, $p \cdot x \geq p \cdot x'$ and $p' \cdot x' \geq p' \cdot x$.
  So, $p \cdot (x - x') \geq 0$ and $p' \cdot (x - x') \leq 0$
  Subtracting the second inequality from the first leads to
- The "Law of Demand": $(p - p') \cdot (x - x') \geq 0$

Building Intuition

- Varying a single price
  - Raising the price of an output leads to more production of that output.
  - Raising the price of an input leads to less use of that input.
  - Raising several prices leads to a change in the same general "direction" as the price change.
- Other parameters and problems
  - How do these ideas generalize to non-linear objective?
  - One important case is when a parameter change acts "like a price increase" by increasing the marginal return to some choice variable, independently of other choices.

Single Crossing Properties

1. $f:\Re\to\Re$ has the single crossing property if for all $x>y$,
   1. $f(y)>0 \Rightarrow f(x)>0$ and
   2. $f(y)\geq 0 \Rightarrow f(x)\geq 0$. ("strict" single crossing adds that $f(x)>0$)
2. $g:\Re^2\to\Re$ has the (strict) single crossing differences property if for all $x>y$, $f(t)\equiv g(x,t)-g(y,t)$ has the (strict) single crossing property.
3. $g:\Re^2\to\Re$ has the **smooth** single crossing differences property if, in addition to single crossing differences, it satisfies

$$g_1(x,t)=0 \Rightarrow (\forall\delta>0)\, g_1(x,t+\delta)\geq 0\geq g_1(x,t-\delta)$$

Monotonic Selections

- For $X\subset\Re$, define $X^*(t,X)=\arg\max_{x\in X} f(x,t)$
- Monotonic Selection Theorem. The following two are equivalent:
  - for all finite X, *every selection* $x(t)$ from $X^*(t;X)$ is non-decreasing
  - f satisfies the strict single crossing differences property.

Proof

- Let $x(t)$ be a selection from $X^*(t,X)$ that is not nondecreasing: for some $t_0<t_1$, $x(t_0)=x_0>x_1=x(t_1)$. Then
  1. $f(x_0,t_0)-f(x_1,t_0)\geq 0$
  2. $f(x_0,t_1)-f(x_1,t_1)\leq 0$

  which contradicts strict single crossing differences.
- Conversely, suppose single crossing differences fails. Then for some $t_0<t_1$ and $x_0>x_1$, inequalities 1 and 2 hold. Then taking $X=\{x_0,x_1\}$, $x(t_0)=x_0$ and $x(t_1)=x_1$,we have a selection from $X^*(t,X)$ that is decreasing. **QED**

Raising "Marginal Returns"

- Intuitively, increasing a price increases the optimal choice by raising a marginal return.
  - The law of demand conclusion is robust: it does not depend on other prices or choices.
- Next questions:
  - What conditions in the new theory correspond to this "robustness" in traditional demand theory?
    - We consider one variation: robustness relative to added terms in the objective.
  - How can we strengthen the single crossing differences condition to make it "robust"?
    - By formalizing "raising a marginal return"

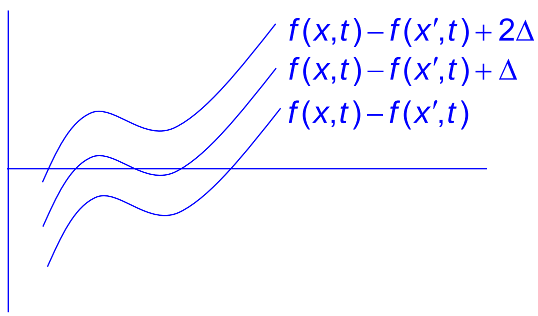
Increasing Differences

- A simple case of single crossing occurs when the differences are strictly monotonic.
- Definition. The function $f(x,t)$ has *increasing differences* if
  $x > x' \Rightarrow f(x,t) - f(x',t)$ is increasing in t.
- Theorem. The objective function $f(x,t)+g(x)$ has the strict single crossing differences property for all $g:\Re\to\Re$ if and only if f has increasing differences.

Proof

- The relevant difference function is:
  $$h_g(t) = f(x,t) - f(x',t) + [g(x) - g(x')]$$
- Since g is arbitrary, h_g satisfies strict single crossing for all functions g if and only if
  $$f(x,t) - f(x',t) + \Delta$$
  satisfies the property for all real numbers Δ, which holds if and only if $f(x,t)-f(x',t)$ is increasing in t. **QED**

Picture Proof



A Surprise

- The conditions studied in this set of slides imply monotonicity of the optimal choice.
- …but they also, surprisingly, contribute to identifying necessary and sufficient conditions for optimality.

Finite Sufficiency Theorem

- Theorem. Suppose that $f(x,t)$ has single crossing differences. Suppose $x:[0,1]\to X$
  1. maps [0,1] onto a finite set X,
  2. is nondecreasing, and
  3. satisfies the envelope formula:

  $$f(x(t),t)-f(x(0),0)=\int_0^t f_2(x(s),s)ds$$

  Then $x(t)$ is a selection from $X^*(t,X)$.

Proof Sketch

- Let $X=\{x_1,\ldots,x_N\}$ and assume (1)-(3).
- $x(\cdot)$ nondecreasing & onto $\Rightarrow$ there exist $0=t_0,\ldots,t_N=1$ with $x(t)=x_n$ for $t\in(t_{n-1},t_n)$.
- Integral formula $\Rightarrow f(x(t),t)$ continuous $\Rightarrow$ for $k=1,\ldots,n$,

  $$f(x_k,t_k)=\lim_{t\uparrow t_k} f(x(t),t)=\lim_{t\downarrow t_k} f(x(t),t)=f(x_{k+1},t_k)$$

- $f(x_{k+1},t_k)=f(x_k,t_k)$ and single crossing of differences

  $\Rightarrow$ for $t>t_k$, $f(x_{k+1},t)\geq f(x_k,t)$ and for $t<t_k$, $f(x_{k+1},t)\leq f(x_k,t)$

  $\Rightarrow$ for $t\in(t_{n-1},t_n)$, $f(x_n,t)\geq f(x_{n-1},t)\geq f(x_{n-2},t)$...
  and $f(x_n,t)\geq f(x_{n+1},t)\geq f(x_{n+2},t)$...

  QED

General Sufficiency Theorem

- Theorem. Suppose that $g(x,t)$ has smooth single crossing differences. Suppose $x(\cdot)$
  1. maps [0,1] onto X,
  2. is nondecreasing,
  3. is the sum of a jump function and an absolutely continuous function, and
  4. satisfies the envelope formula:

  $$g(x(t),t)-g(x(0),0)=\int_0^t g_2(x(s),s)ds$$

  Then $x(\cdot)$ is a selection from $X^*(t,X)$.

"Constraint Simplification"

- Theorem. Let $f(x,t)$ be continuously differentiable with strict single crossing differences and satisfy the conditions of the envelope theorem. Let X be the range of $x(\cdot)$ and suppose that it is finite. Then, $x(\cdot)$ is a selection from $X^*(t,X)$ if and only if:

  (1) $x(\cdot)$ is nondecreasing and

  (2) $f(x(t),t)-f(x(0),0)=\int_0^t f_2(x(s),s)ds$

- Remarks.
  - The text includes a version of this theorem in which the restriction that the range of x is finite is relaxed.
  - In our applications, we will ignore that restriction.

Proof Sketch

- That (1) and (2) are necessary follows from the envelope theorem and the monotonic selection theorem.
- The converse follows from the general sufficiency theorem.

QED

FOC⇒Envelope Formula

- Theorem. Suppose that $f(x,t)$ and $x(t)$ are both continuously differentiable and that for all $t\in[0,1]$, $f_1(x(t),t)=0$. Then, the envelope integral formula holds:

$$V(t)-V(0)\equiv f(x(t),t)-f(x(0),0)=\int_0^t f_2(x(s),s)ds$$

- Proof. This follows from the Fundamental Theorem of Calculus and the observation that the total derivative of f(x(t),t) is:

$$\tfrac{d}{dt}f(x(t),t)=f_1(x(t),t)x'(t)+f_2(x(t),t)=f_2(x(t),t)$$

QED

Extra Result:
Mirrlees-Spence Condition

Mirrlees-Spence Condition

- Applies to $U(x,y,t):\Re^3\to\Re$.
- Mirrlees-Spence condition:
  1. U_1 exists.
  2. U_2 exists and is everywhere positive or everywhere negative.
  3. For all x and y, the following ratio is non-decreasing in t:

$$\frac{U_1(x,y,t)}{|U_2(x,y,t)|}$$

M-S Single Crossing

- Theorem. Suppose that $h(x,y,t):\Re^3 \to \Re$ is twice continuously differentiable with $h_2 \neq 0$ and $|h_1|$ bounded and for every $(x,x',y,t) \in \Re^4$ there exists $y' \in \Re$ such that $h(x,y,t)=h(x',y',t)$. Then, the following are equivalent:
  - h satisfies the Mirrlees-Spence condition
  - For every continuously differentiable function f, $g^f(x,t)=h(x,f(x),t)$ satisfies the *smooth single crossing differences* condition.

Guessing and Verifying Equilibrium

A Standard Symmetric Model

Symmetric "Independent Private Values Model"

- N bidders
- There may be one or more items for sale, but each buyer wants just one.
- Bidder types are uniformly distributed on [0,1].
- Bidder values are $v(t)$, where v is increasing and differentiable.
- A strategy is a mapping β:[0,1]→{strategies in the mechanism}.
- Losers' payoffs are equal to minus any amounts they pay. The winner's payoff is $v(t)$ minus any amounts he pays. (Risk neutrality)

Vickrey's Payoff Equivalence

- *Theorem. In the symmetric independent private values model, the expected price and the expected bidder payoffs are the same for the simplified English, Dutch, first-price and second price auctions.*
  - Vickrey showed similar results for various auction rules with N bidders and M objects (limited to 1 per bidder). This "revenue equivalence" result remained a puzzle from 1962 until 1979.

Increasing Differences?

- Suppose an "auction" is a mechanism in which actions are bids and, for any given bids by others, bidder *i*'s probability of winning is an increasing function of its own bid *b.*
  - Denote that probability by $\pi(b)$.
  - Denote the bidder's expected payment by $P(b)$.
  - Assume that $v(\cdot)$ is increasing
- Bidder *i*'s best bid when its type is *t* solves:

$$\beta(t) \in \arg\max_b v(t)\pi(b) - P(b)$$

Increasing Differences Verified

- Comparing two bids $b > b'$, the difference in expected payoff is the following increasing function of *t*:

$$v(t)\left(\pi(b) - \pi(b')\right) - \left(P(b) - P(b')\right)$$

- <u>Theorem</u>. For any "auction" as defined on the preceding slide, each bidder's objective function, as a function of its bid and type (*b*,*t*), has increasing differences.

Conclusions…

- In every auction in the class studied, regardless of the strategies adopted by others, bidder *i*'s best response is a strategy β that is a non-decreasing function of *t*.
  - So, limiting attention to increasing bidding functions was "reasonable."
  - …but why would we expect β to be <u>strictly</u> increasing?
- …but can we conclude that such functions are actually equilibrium strategies?

Characterizing Equilibrium

- Assume (for now) symmetric, increasing equilibrium strategies
  - second price auction: β_S(t)=v(t)
  - first-price auction: β_F(t)=?
  - all-pay own-bid auction ("lobbying"): β_{AO}(t)=?
  - all-pay second-bid auction ("mating fight"): β_{AS}(t)=?
  - Cook County tax sale, jump bid strategy: β_{Cj}(t)=?
- Method:
  - infer the strategy from the payoff equivalence result
  - verify that single crossing conditions apply
  - conclude that the strategy profile is, in fact, an equilibrium

First Price Auction

- Assume the symmetric independent private values model with an increasing equilibrium bidding strategy.
- Since the highest bidder wins, by Myerson's lemma, each type's expected payment must be the same as in the second-price auction.

$$t^{N-1}\beta(t) = \int_0^t v(s)\,ds^{N-1}$$

$$\beta(t) = t^{1-N}(N-1)\int_0^t v(s)s^{N-2}ds$$

$$= E\left[v(\max(t^2,\ldots,t^n)) \mid \max(t^2,\ldots,t^n) \le t\right]$$

- That is necessary. But is it sufficient? Is $\beta(\cdot)$ as determined above actually an equilibrium strategy?

Verifying Equilibrium

- By inspection and construction,
  - single crossing differences (because increasing differences)
  - $\beta(\cdot)$ is strictly increasing
  - $\beta(\cdot)$ verifies the envelope formula
- A separate argument shows that bids outside the range of $\beta(\cdot)$ cannot lead to higher expected payoffs:
  - Bidding more than the upper bound of the range of $\beta(\cdot)$ is strictly worse than bidding the upper bound.
  - Bidding less than the lower bound of the range of $\beta(\cdot)$ is no better than bidding the lower bound
- Hence, $\beta(\cdot)$ is indeed a symmetric equilibrium strategy of the first price auction model.

Is the "Extra" Argument Needed?

- The theorem assumes that β is "onto" and is silent about comparisons with bids outside the range of β.
- Example 1:
  - The strategy according to which everyone bids $10 satisfies monotonicity and the envelope formula.
    - » It is the equilibrium of the game when the only feasible action is to bid $10.
  - It is not an equilibrium of the game in question.
- Example 2:
  - The equilibrium strategy of the first-price auction game in which bids must be whole numbers satisfies monotonicity and the envelope formula.
  - It is not an equilibrium of the game in question.

Two All-Pay Auctions

- If there is a symmetric increasing equilibrium in the all-pay-own-bid auction, then it must satisfy:

$$\beta(t) = \int_0^t v(s)\,ds^{N-1}$$

- In the both-pay second bid ("war of attrition") game, it must satisfy:

$$\int_0^t v(s)ds = \beta(t)(1-t) + \int_0^t \beta(s)ds$$

$$\therefore \beta(t) = \int_0^t \frac{v(s)}{1-s}ds$$

Verification, Step 1

- The following computation verifies that the proposed strategy for the war-of-attrition matches the expected payments of the 2nd price auction:

$$\beta(t)(1-t)+\int_0^t \beta(s)ds=(1-t)\int_0^t \frac{v(s)}{1-s}ds+\int_0^t\int_0^s \frac{v(r)}{1-r}drds$$
$$=(1-t)\int_0^t \frac{v(s)}{1-s}ds+\int_0^t\int_r^t ds\frac{v(r)}{1-r}dr$$
$$=(1-\cancel{t})\int_0^t \frac{v(s)}{1-s}ds+\int_0^t(\cancel{t}-r)\frac{v(r)}{1-r}dr$$
$$=\int_0^t \frac{v(s)}{1-s}ds-\int_0^t s\frac{v(s)}{1-s}ds$$
$$=\int_0^t v(s)ds$$

"Both Pay" Auction Verifications

- Conditions to check for both auctions:
  - $\beta(\cdot)$ is increasing
  - By construction, $\beta(\cdot)$ verifies the envelope or expected payment restriction.
  - There is no better "bid" outside the range of β.
- Therefore, proposed strategies are equilibria, respectively, of…
  - the war of attrition
  - the both-pay own bid auction

Cook County Tax Sale

- Bidding begins at a very high price and proceeds by oral outcry. The low bid wins; the minimum bid is 0.
- Cost of supplying service can be positive (up to c_{max}) or negative!
- Strategies:
  - Bidder with costs c>0 bids down to c. Write this as $\beta(c)=c$.
  - Bidder with costs c<0 bids down to some amount B>0 and then jumps to 0. Write this as $\beta(c)=-B$.
- Assume there exists a symmetric equilibrium with $\beta(\cdot)$ increasing.
- Problem: *Derive the equilibrium bid function $\beta(\cdot)$.*
  - Comment: Similar modeling challenges arise in modeling the use of "buy prices" at eBay.

Cook County, equilibrium

- Apply the revenue equivalence theorem.
  - For positive cost types, bid as in a second-price auction.
  - For negative cost types, revenue must be the same as in a second-price auction, so the equilibrium bid solves:

$$\int_c^{c_{max}} sf(s)ds=\int_c^{-\beta(c)} sf(s)ds \Leftrightarrow \int_c^{-\beta(c)} sf(s)ds=0$$

- There is any solution for the lowest cost type if

$$\int_{c_{min}}^{c_{max}} sf(s)ds \ge 0$$

and in that case there is a unique solution for every cost type.

Cook County, verification

- This is an equilibrium because…
  - Increasing differences is verified
  - The bidding strategy is increasing
  - Revenue equivalence is verified (so the envelope identity holds)
  - There is no better bid outside the range of the equilibrium strategy. (Verify this as an exercise!)

Discarding Risk Neutrality

- Suppose a bidder with value v chooses a bid b to maximize $U(v;b)F(b)$. If we don't assume risk neutrality, can we still be sure that $\beta^*(v)$ must be non-decreasing, regardless of F?
- A monotonic transformation preserves the optimizer:
  $$\log\left[U(v;b)F(b)\right] = \log\left[U(v;b)\right] + \log(F(b))$$
- The function $\beta^*(v;F)$ is nondecreasing for all F *if and only if* $\ln(U(\cdot;\cdot))$ has increasing differences.
- In a symmetric model where $\log(U)$ has increasing differences, an increasing bid function that satisfies envelope and boundary conditions is necessarily an equilibrium strategy.

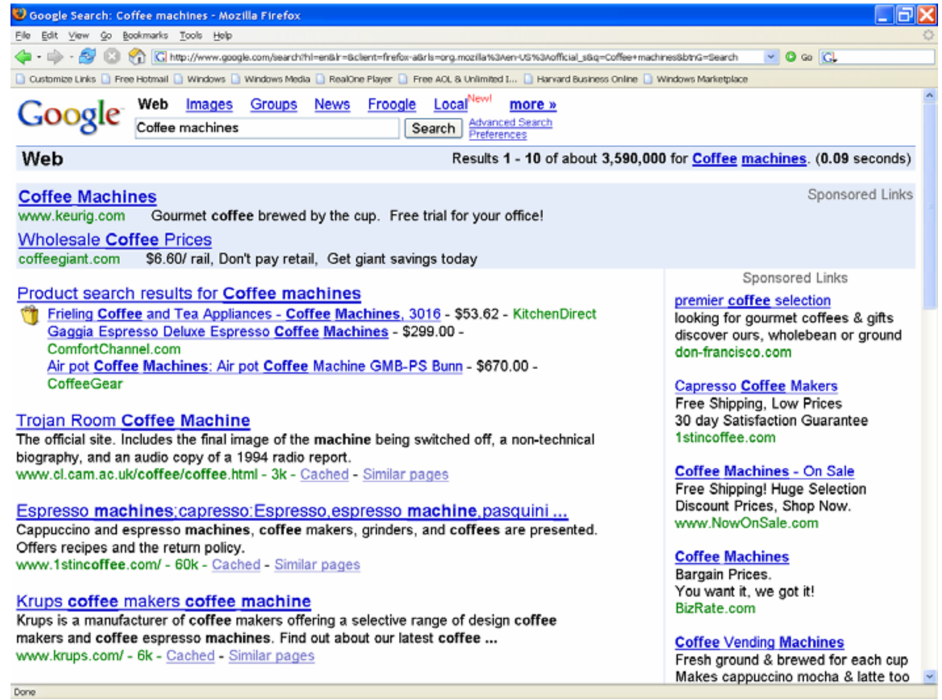
Review

- Time allotted to review and prepare for mid-term exam.
- Bring your questions!

Topic #7: Google's AdWords Auction

Today's Project

- Reverse engineer Google's AdWords auction.
- Make recommendations for a better AdWords auction to Google or one of its competitors.



Google Links for Browsing

- https://adwords.google.com/select
- Where will my ads appear?
  - Adword advantages
  - Program comparison
- Getting started
  - Editorial guidelines
  - Step-by-step
  - Optimization tips
  - Keyword tools
- Vulnerabilities
  - http://www.theregister.co.uk/2005/02/03/google_adwords_attack/
  - …

Botnets strangle Google Adwords campaigns

By John Leyden (john.leyden at theregister.co.uk)

Published Thursday 3rd February 2005 17:14 GMT

Security researchers have discovered a way to shut down or seriously impair a Google Adwords advertising campaign by artificially inflating the number of times an ad is displayed. By running searches against particular keywords from compromised hosts, attackers can cause click-through percentage rates to fall through the floor.

This, in turn, causes Google Adwords to automatically disable the affected campaign keywords and prevent ads from being displayed. By disabling campaign keywords using the technique, cybercrimals could give their preferred parties higher ad positions at reduced costs, according to click fraud prevention specialists Clickrisk.

"By disabling targeted keywords across many advertisers' campaigns simultaneously by artificially inflating the number of times an ad is displayed an attacker can secure a higher ad position," explains Clickrisk.com chief exec Adam Sculthorpe. The attack - dubbed keyword hijacking - is difficult to prevent because it takes advantage of a design feature of Google Adwords rather than a flaw, he added. Clickrisk came across the attack in investigating why the click through rates of one of its clients - which had been running at a steady rate - dropped to zero for no apparent reason. Subsequent monitoring and forensic testing revealed that a botnet made up of open proxies in China was responsible for the attack.

High—cost-per-click (CPC) advertisers in niche markets are particular vulnerable to the keyword hijacking attack. "Once keywords are disabled they can't be re-enabled and attacks can go undetected for some time," Sculthorpe told *El Reg*. When keywords are disabled an advertiser must erase all campaigns featuring the affected keywords and create a new campaign as a workaround.

Although the true scope of the problem remains unclear, Clickrisk security analysts believe the keyword hijacking attack may be widely exploited. Clickrisk advises users to monitor click-through rates and traffic levels, log into Google Adwords campaign frequently and check that keywords are not disabled.

The incidence of click fraud risk exposure in general is on the rise. According to Clickrisk's chief risk officer, Jack Bensimon, "our clients have experienced substantial losses ranging from 20 – 65 per cent of their total click costs." Bensimon believes that "managing business risk is a critical component of online advertising" and further recommends that "online marketers should be vigilant and regularly monitor keywords". ®

Google's Rules

- Each bidder can specify a rich rule for determining how to bid as a function of the search terms and the site from which the search originates.
  - Google sets a reserve or minimum price for each search term.
- Google estimates the "click-through rate" that each bidder would have if it were listed in the first spot.
- Google ranks the bids according to the product of the click-through-rate and the bid; it assigns ad spots in that order.
- Google is paid only if a bidder's link is clicked. In that case, it receives the smallest price the bidder could have bid to get its ranking.

Example

- I bid $1 and have an estimated click-through rate of .50.
- You have bid $2 and have an estimated click-through rate of .2.
- The reserve price is 0.1.
- My score is .5; yours is .4, so my ad ranks first.
  - I could have won with a bid as low as .81, so that is what I pay if my link is clicked.
  - You could have had your spot for as low as .1, so that is what you pay if your link is clicked.

Questions

- What are the key properties of this mechanism?
- Why is Google paid only for clicks?
  - By comparison, television ads are priced according to "impressions" (how many times they are seen).
  - By comparison, consignment stores are paid according to sales.
- Incentives
  - Under what assumptions does Google's pricing scheme lead to a dominant strategy of bidding "honestly"?
  - Under what assumptions is the auction outcome efficient?
  - Under what assumptions is the auction revenue-maximizing?
  - Are these assumptions realistic?

Toward a Better Design

- If the assumptions are unrealistic,
  - how might a bidder exploit the differences?
  - how might a competitor create a better design?
- What constraints, if any, would you expect to be imposed on your design…
  - by competition from other search engines?
  - by legal considerations?
- How might Google accommodate market power of advertisers?
  - Quantity discounts?

Sample Assumptions

- Bidders value clicks
  - Without regard to the source of the click
  - Without regard to the position of the ad
- The auction is honest and trusted
  - Click through rates are genuine
  - No shill bidders, false clicks, manipulated prices
- Private values (no adverse selection)
- Searcher behavior
  - Searcher clicks only on the first listing
  - Searcher clicks on the first relevant listing
  - Searcher inspects the top two listings equally

Topic #8: Revenue Comparisons

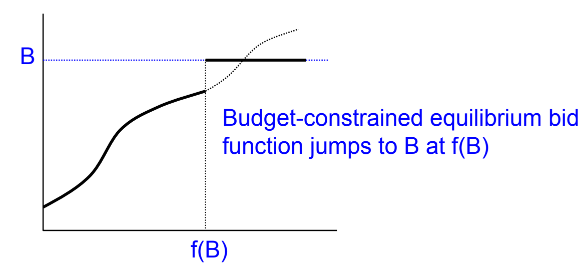
Budget Constraints, 1

- Benchmark model with budget constraint B
  - Symmetric, independent private values
- In a second-price auction, each player has a dominant strategy: $\beta(t)=\min(v(t),B)$.
- With no budget constraint, the first-price auction equilibrium strategy is:

$$\beta(s) \equiv E\left[\max(v(t^2),\ldots v(t^N)) \mid \max(t^2,\ldots t^N) < s\right]$$

Budget Constraints, 2

- Seeking equilibrium of the first price auction with budget $B<\beta(1)$.
  - Strategy must be monotonic. Let us guess that…
    - » a bidder with a "high" type ($t > f(B)$) bids the budget B.
    - » a bidder with a "low" type ($t < f(B)$) bids the same as if the budget were absent



Budget Constraints, 3

- The type $f(B)$ is the unique one that is just indifferent between two bids. It is the t that solves:

$$(v(t)-\beta(t))t^{N-1}$$
$$=(v(t)-B)\sum_{n=0}^{N-1}\frac{1}{n+1}\binom{N-1}{n}(1-t)^n t^{N-n-1}$$

Budget Constraints, 4

- Theorem. (Che & Gale) A symmetric equilibrium in the budget constrained first-price auction with budget B is:

$$\hat{\beta}(s)=\begin{cases}\beta(s) \text{ for } s\le f(B)\\ B \text{ for } s>f(B)\end{cases}$$

  The equilibrium strategy has a jump discontinuity. Equilibrium revenues are the same as for a second price auction with the larger budget $v(f(B))>B$.
- Proof. By construction, the expected payments are the same as in a second price auction with budget $v(f(B))$. Hence, the envelope condition is satisfied. The strategy is non-decreasing. There is no more profitable bid outside the range of the strategy. **QED**

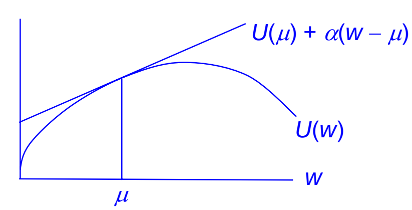
Jensen's Inequality

- For any concave function U and any random variable X with finite expectation $E[X]=\mu$:

$$E\left[U(X)\right]\le U\left(\mu\right)=U\left(E[X]\right)$$

Proof of Jensen's Inequality

- Proof: Since U is concave, for any μ there exists another α such that for all w, $U(w)\le U(\mu)+\alpha(w-\mu)$.
  - If U is differentiable at μ, then one can take $\alpha=U'(\mu)$.



- Let $\mu=E[X]$. Then, $E\left[U(X)\right]\le E[U(\mu)+\alpha(X-\mu)]$ $=U(\mu)+\alpha\left(E[x]-\mu\right)=U(\mu)$.

Iterated Expectations

- Law of Iterated Expectations: for any random variables X and Y and any function H such that $E[H(X,Y)]$ is finite, $E\left[E[H(X,Y)\,|\,Y]\right]=E[H(X,Y)]$.
- Proof:

$$
\begin{aligned}
E\left[E[H(X,Y)\,|\,Y]\right] &= \int\left(\int H(x,y)f_x(x\,|\,y)dx\right)f_y(y)dy \\
&= \int\left(\int H(x,y)\frac{f(x,y)}{f_y(y)}dx\right)f_y(y)dy \\
&= \int\int H(x,y)f(x,y)dxdy \\
&= E[H(X,Y)]
\end{aligned}
$$

Risk Averse Sellers, 1

- Theorem. In the benchmark model, if the bidders are risk neutral but the seller is risk averse, then the seller's expected utility of income is higher in the first price auction than in the second price auction.

Risk Averse Sellers, 2

- Proof. Let U denote the seller's concave utility function. Let $t^{(n)}$ denote the n^{th} order statistic among the types. When $t^{(1)}=s$, the winning bid is: $\beta(s)=E\left[v(t^{(2)})\,|\,t^{(1)}=s\right]$.
- So, the expected utility of revenue is:

$$
\begin{aligned}
E\left[U(\beta(t^{(1)}))\right] &= E\left[U(E\left[v(t^{(2)})\,|\,t^{(1)}\right])\right] \\
&\geq E\left[E\left[U(v(t^{(2)}))\,|\,t^{(1)}\right]\right] \\
&= E\left[U(v(t^{(2)}))\right].
\end{aligned}
$$

QED

Bidders Not Risk Neutral, 1

- Suppose the bidders are not risk neutral: set U(0)=0.
  - A bidder's maximization problem:

$$\max_b U(v(t)-b)(\beta^{-1}(b))^{N-1}$$

$$\max_b \ln U(v(t)-b)+(N-1)\ln(\beta^{-1}(b))$$

  - Note: $b^*(t)$ is nondecreasing if $\ln(U(v(t)-b))$ has increasing differences, that is, if $\ln(U(\cdot))$ is concave.
- First-order & equilibrium conditions

$$\frac{-U'(v(t)-b)}{U(v(t)-b)}+\frac{N-1}{\beta^{-1}(b)}\frac{d}{db}\beta^{-1}(b)=0;\quad \frac{-U'(v(t)-\beta(t))}{U(v(t)-\beta(t))}+\frac{N-1}{t\beta'(t)}=0$$

Bidders Not Risk Neutral, 2

- Theorem. Suppose that $\ln(U(\cdot))$ is concave and differentiable, where U is the bidder's utility function. Then, the unique symmetric equilibrium strategy of the first-price auction is the solution to the following differential equation and boundary condition:

$$\frac{N-1}{t\beta'(t)} = \frac{U'\left(v(t)-\beta(t)\right)}{U\left(v(t)-\beta(t)\right)}, \qquad (*)$$
$$\beta(0) = v(0).$$

Bidders Not Risk Neutral, 3

- The Constraint Simplification Theorem applies:
  - As noted, if $\ln(U(\cdot))$ is concave, then $\beta'>0$, so the proposed strategy is increasing.
  - If $\ln(U(\cdot))$ is concave, then the bidder's problem satisfies increasing differences.
  - The bidder's first-order condition implies that the envelope formula is satisfied.
- By inspection, no bid outside the range of β can pay more for any type than some bid in the range of β.
- Uniqueness follows because
  - no ties are possible at equilibrium
  - no other boundary condition is possible.

QED

Risk ***Averse*** Bidders

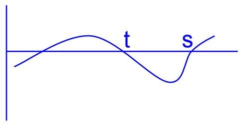
- Theorem. Suppose that the bidder utility function U is differentiable and ***strictly concave***. Then for every type $t>0$, the equilibrium bid $\beta(t)$ in the first-price auction is greater than for the case of risk-neutral bidders. In particular, expected seller revenues are greater for the first-price auction than for the second-price auction.

Math: Single Crossing Lemma

- Lemma. Suppose that $f:\Re\to\Re$ is a differentiable function with the property that for all t, either $f(t) > 0$ or $f'(t) > 0$. Then f has the strict single crossing property.
- Remarks: The lemma comes in several versions.
  - "Weak" version of the lemma: if for all t, either $f(t) > 0$ or $f'(t) \geq 0$, then f has weak single crossing property.
  - This lemma is used repeatedly in the text to make various revenue comparisons.

Proof of lemma

- Suppose to the contrary for some s with $f(s) \le 0$ there exists some $t<s$ such that $f(t) \ge 0$, and let t be the greatest such number.
  - Such a t exists, because $f'(s) > 0$.

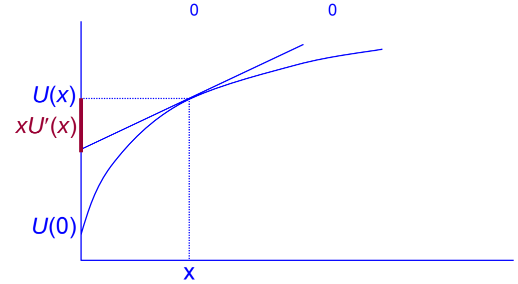


- Then by the Mean Value Theorem, there exists $r \in (t,s)$ such that $f'(r) = [f(s)-f(t)]/(s-t) \le 0$.
- By construction, however, $f(r) \le 0$, and these two combine to contradict the hypothesis.

Math Review

- If U is a concave function and $x>0$, then

$$U(x) - U(0) = \int_0^x U'(s)ds \ge \int_0^x U'(x)ds = xU'(x)$$



Equilibrium Formula Repeated

- Using the first-order conditions, we derived the equilibrium differential equation:

$$\frac{N-1}{t\beta'(t)} = \frac{U'\left(v(t)-\beta(t)\right)}{U\left(v(t)-\beta(t)\right)},$$

$$\beta(0) = v(0).$$

- This applies for risk-averse and risk-neutral bidders, and even for risk-loving bidders if $\ln(U(\cdot))$ is concave.
- Let β_{RN} and β denote the equilibrium bid functions for the risk neutral and risk-averse cases.

Proof of Revenue Comparison

- By formula (*), $\beta(0) = \beta_{RN}(0) = v(0)$
- U satisfies $U(0)=0$, so $U'(x)/U(x) < 1/x$ for $x>0$.

$$\frac{N-1}{t\beta'(t)} = \frac{U'\left(v(t)-\beta(t)\right)}{U\left(v(t)-\beta(t)\right)}, \quad \frac{N-1}{t\beta'_{RN}(t)} = \frac{1}{v(t)-\beta_{RN}(t)}$$

- Comparing derivatives, for all t, either $\beta(t) > \beta_{RN}(t)$ or $\beta'(t) > \beta'_{RN}(t)$.
- So, by the single crossing lemma, for all $t > 0$, $\beta(t) - \beta_{RN}(t) > 0$. **QED**

Extra Material

Not covered in 2005

Correlated Types, 1

- Suppose the types are correlated:
  - A bidder's maximization problem:

$$\max_b (v(t)-b)F\left(\beta^{-1}(b)\,|\,t\right)$$

$$\max_b \ln(v(t)-b)+\ln F\left(\beta^{-1}(b)\,|\,t\right)$$

  - Note: $b^*(t)$ is nondecreasing if $\ln(F(x|t))$ has increasing differences.
- First-order & equilibrium conditions

$$\frac{-1}{v(t)-b}+\frac{f\left(\beta^{-1}(b)\,|\,t\right)}{F\left(\beta^{-1}(b)\,|\,t\right)}\frac{d}{db}\beta^{-1}(b)=0;\quad \frac{-1}{v(t)-\beta(t)}+\frac{f\left(t\,|\,t\right)}{F\left(t\,|\,t\right)\beta'(t)}=0$$

Correlated Types, 2

- Theorem. Suppose that $\ln(F(x|t))$ is differentiable with increasing differences, where F is the conditional distribution of the highest type among other bidders. Then, the unique symmetric equilibrium strategy of the first-price auction is the solution to the following differential equation:

$$\frac{\beta'(t)}{v(t)-\beta(t)}=\frac{f\left(t\,|\,t\right)}{F\left(t\,|\,t\right)}, \qquad (**)$$

$$\beta(0)=v(0).$$

Correlated Types, 3

- The Constraint Simplification Theorem applies:
  - By inspection, $\beta'>0$, so the proposed strategy is increasing.
  - By construction, it solves the bidder's first-order condition, so it satisfies the envelope formula.
  - Log objective, below, satisfies increasing differences:

$$\log\left(v(t)-b\right)+\log F(\beta^{-1}(b)\,|\,t)$$

- By inspection, no bid outside the range of β can pay more profitable than a bid in the range of β.
- Uniqueness follows from necessary conditions:
  - no ties are possible at equilibrium: hence β is increasing
  - no other boundary condition is possible.

QED

"Affiliation" Aside

- Notation: (A particular conditional distribution)

$$G_s(r \mid t) = \begin{cases} F(r \mid t)/F(s \mid t) \text{ if } r < s \\ 1 \text{ otherwise} \end{cases}$$

- Theorem. If $\ln(f(x|t))$ has increasing differences, then for all r and s, $G_s(r|t)$ is non-increasing in t.
- Remarks.
  - The theorem concerns "first-order stochastic dominance."
  - The condition on $\ln(f)$ is also called "affiliation" of the density function. It means that the ratio $f(s|t)/f(r|t)$ is increasing in t for all $s>r$.

Affiliation Proof

- For $t > t'$, using the assumed increasing differences of $\ln(f(u|t))$,

$$\frac{G_s(r \mid t)}{1-G_s(r \mid t)} = \frac{\int_0^r \frac{f(u \mid t)}{F(s \mid t)}du}{\int_r^s \frac{f(u \mid t)}{F(s \mid t)}du} = \frac{\int_0^r \frac{f(u \mid t)}{f(r \mid t)}du}{\int_r^s \frac{f(u \mid t)}{f(r \mid t)}du}$$

$$\le \frac{\int_0^r \frac{f(u \mid t')}{f(r \mid t')}du}{\int_r^s \frac{f(u \mid t')}{f(r \mid t')}du} = \frac{G_s(r \mid t')}{1-G_s(r \mid t')}$$

QED

Correlated Types, 4

- Theorem. Suppose that $\ln(F(x|t))$ is differentiable with increasing differences. Then for every type of bidder, the equilibrium payoff is weakly lower in the second-price auction than in the first-price auction.
- Proof. The proof involves comparing the expected payoffs to bidders in the first and second price auctions, using the envelope theorem and the single crossing lemma.

Proof, Details

- Let the expected payment made in the Vickrey auction by type t bidding $v(s)$ be $\hat{b}(s \mid t)$.
  - Notice that $\frac{\partial}{\partial t}\hat{b}(s \mid t) \ge 0$. (by previous slide.)
- Then, expected payoffs in the first- and second-price auctions are:

$$V_{SP}(t) = \max_s \left(v(t) - \hat{b}(s \mid t)\right) F(s \mid t) \text{ and}$$

$$V_{FP}(t) = \max_s \left(v(t) - b(s)\right) F(s \mid t)$$

- By the envelope theorem,

$$V_{SP}(t) \le V_{FP}(t) \text{ or } V'_{SP}(t) \le V'_{FP}(t)$$

- Apply the single crossing lemma. **QED**

Revenue Ranking with Correlated Types

Correlation: First-Order Condition

- Suppose the (two bidders') types are correlated:
  - A bidder's maximization problem:

$$\max_b (v(t)-b)F\left(\beta^{-1}(b)\,|\,t\right)$$

$$\max_b \ln(v(t)-b)+\ln F\left(\beta^{-1}(b)\,|\,t\right)$$

  - If β is increasing and $\ln(F(x|t))$ has increasing differences, then the log-objective has increasing differences.
- Verifying increasing differences

$$\frac{\partial^2}{\partial b\partial t}\left[\ln(v(t)-b)+\ln F\left(\beta^{-1}(b)\,|\,t\right)\right]$$

$$=\frac{\partial}{\partial b}\left[\frac{v'(t)}{v(t)-b}\right]+\left.\frac{\partial^2}{\partial s\partial t}\right|_{s=\beta^{-1}(b)}\ln F(s\,|\,t)\frac{\partial\beta^{-1}}{\partial b}\geq 0$$

Correlation: First-Order Condition

- Suppose the types are correlated:
  - A bidder's maximization problem:

$$\max_b (v(t)-b)F\left(\beta^{-1}(b)\,|\,t\right)$$

$$\max_b \ln(v(t)-b)+\ln F\left(\beta^{-1}(b)\,|\,t\right)$$

  - If β is increasing and $\ln(F(x|t))$ has increasing differences, then the log-objective has increasing differences.
- First-order & equilibrium conditions

$$\frac{-1}{v(t)-b}+\frac{f\left(\beta^{-1}(b)\,|\,t\right)}{F\left(\beta^{-1}(b)\,|\,t\right)}\frac{d}{db}\beta^{-1}(b)=0;\quad \frac{-1}{v(t)-\beta(t)}+\frac{f(t\,|\,t)}{F(t\,|\,t)\beta'(t)}=0$$

Equilibrium Characterization

- Theorem. Suppose that $\ln(F(x|t))$ is differentiable with increasing differences. Then, the following is a symmetric equilibrium strategy in the model with correlated types:

$$\beta(t)=v(t)-\int_0^t L_t(s)dv(s)$$

$$\text{where } L_t(s)=\exp\left(-\int_s^t\frac{f(r\,|\,r)}{F(r\,|\,r)}dr\right)\text{ for } t>s>0$$

- Proof Sketch:
  - Verify single crossing differences.
  - Verify monotonicity.
  - Verify first-order condition (β solves the diff eq).
  - Apply sufficiency theorem.

A Revenue Comparison

- Theorem. Suppose that ln($F(x|t)$) is differentiable with increasing differences. Then for every type of bidder, the equilibrium payoff is weakly lower in the second-price auction than in the first-price auction.

Proof Sketch

- Let the expected payment made in the Vickrey auction by type t bidding $v(s)$ and winning be $\hat{\beta}(s\,|\,t)$.
  - By stochastic dominance, $\frac{\partial}{\partial t}\hat{\beta}(s\,|\,t) \geq 0$.
- Then, expected payoffs in the first- and second-price auctions are:

$$V_{SP}(t) = \max_s \left(v(t) - \hat{\beta}(s\,|\,t)\right) F(s\,|\,t) \text{ and}$$

$$V_{FP}(t) = \max_s \left(v(t) - \beta(s)\right) F(s\,|\,t)$$

- By maximization and the envelope theorem,

$$V_{SP}(t) \geq V_{FP}(t) \Rightarrow \beta(t) \geq \hat{\beta}(t\,|\,t) \Rightarrow V'_{SP}(t) \leq V'_{FP}(t)$$

- Apply the single crossing lemma. **QED**

Topic #9: Modeling Costly Entry

Motivation

- The single most important determinant of success in many auctions is participation.
  - Do enough bidders participate?
  - Do the right bidders participate?
- Auction rules that make bidder payoffs low can discourage participation, so there is potentially a trade-off.
  - How should an auctioneer balance the need to attract participants against other objectives?
  - Particularly, how does entry affect the optimal reserve price?
- Are *auctions*, in which all bidders are gathered together, more effective than negotiations when participation is costly?

Costly Sequential Entry, 1

- The model (McAfee-McMillan) is as follows:
  - valuations are drawn iid from a distribution *F*, but are not freely known to bidders.
  - the seller has a zero cost of supply.
  - the seller commits to auction rules and a reserve price r.
  - bidders make entry decisions in sequence, each knowing the rules and the past history of entry decisions.
  - a bidder who enters incurs entry cost *c* to learn its own valuation.
  - consider a “sequential entry equilibrium” in which the first bidders enter and learn their values as long as expected net profits are non-negative, while other potential entrants stay out.

Costly Sequential Entry, 2

Theorem. If the reserve price is zero and a second-price auction is used, the number of entrants *N* at the “sequential entry equilibrium” is the number that maximizes expected total surplus net of entry costs.

Costly Sequential Entry, 3

Proof. There are several steps to the proof, as follows.

- ***Lemma**. The incremental expected contribution to expected surplus is declining in the number of bidders*.
  - The k^{th} entrant’s contribution to total surplus when its value is *v* and the highest opposing value is y is given by $(v-y)1_{\{x>y\}}$, which is a nonincreasing function of y.
  - The maximum order statistic from a sample of size k is everywhere weakly larger than the maximum order statistic from the subsample.
  - Hence, $E\left[\left(V-V_{k-1}^{(1)}\right)1_{\{V>V_{k-1}^{(1)}\}}\right]$ is decreasing in k.

Costly Sequential Entry, 4

Continuation of Proof...

- *When the reserve is zero, a bidder who expects to be the last entrant has expected profit from the (2^{nd}-price) auction equal to his expected contribution to expected social surplus.*
  - By inspection of the surplus formula.
- *When the reserve is zero, a bidder enters if and only if its expected profit from the auction exceeds the entry cost.*

■

Costly Sequential Entry, 5

Theorem. Let N be the optimal number of entrants. Then, the second price auction with zero reserve and entry fee

$$e = \tfrac{1}{N} E\left[X_N^{(1)} - X_N^{(2)} \right] - c$$

is an auction that maximizes the seller's expected total revenue at a sequential entry equilibrium.

Proof. With that fee, the N^{th} entrant has expected net profits of zero. Then, by definition, there are N entrants at the sequential entry equilibrium, so total surplus is maximized. By symmetry, all bidders have expected net profits of zero. Hence the seller's expected revenue equals the maximum expected total surplus. ■

Zero Reserve?

- Whoa!
  - In optimal auction theory, the seller sets the reserve to exclude bidders for whom the "marginal revenue" is negative.
  - According to the preceding result, the seller optimally sets the reserve to zero.
- How can we reconcile these two very different conclusions?
  - Does it depend on the sequential nature of the entry?

Costly Simultaneous Entry, 1

- The model (Levin-Smith) is as follows:
  - valuations are drawn iid from a distribution F, but are not freely known to bidders.
  - the seller has a zero cost of supply.
  - the seller commits to auction rules and a reserve price r.
  - the K potential bidders make their entry decisions simultaneously, knowing the only rules of the auction and the reserve.
  - each entrant incurs entry cost c to learn its own valuation.
  - after entry, bidders learn (alternately, do not learn) the number of entrants
  - consider a "symmetric simultaneous entry equilibrium" in which each bidder enters with probability p

Costly Simultaneous Entry, 2

- **Theorem**. In the symmetric simultaneous entry equilibrium, the expected-revenue-maximizing reserve price is zero. At this price, the seller captures the entire social surplus.
- **Proof**. At a mixed strategy equilibrium, the bidders' expected net profits must be zero, so the seller captures the total surplus.

  At a reserve price of zero, a bidder enters if and only if its expected profit, which is the same as its expected contribution to total surplus, exceeds its entry cost. So, the probability of entry p that maximizes expected total surplus is an equilibrium probability of entry. Since this zero-profit equilibrium is unique, the equilibrium p maximizes expected total surplus and hence seller revenue. ■

Costly Simultaneous Entry, 3

- **Theorem**. The seller's expected revenue in an auction with a random number of bidders with mean N is less than in an auction with N bidders exactly.
- **Proof**. Apply Jensen's inequality. ■
  - **Thus, random participation is bad for sellers.**
  - **Do we see mechanisms to reduce it?**

A Search Theory Result

- **Sequential search model**.
  - infinitely many items
  - each item searched costs *c*
  - value of each item searched is distributed as *F*
  - only one item may be taken
- **Theorem**. Let V* be the optimal value of this search problem. Then, the optimal policy is to search sequentially until an item of value at least V* is found and then to take that item. Also, V* is the unique solution to:

$$V^* = -c + V^* F(V^*) + \int_{V^*}^{\infty} v dF(v).$$

- **Proof**. Exercise.

Auction Entry as Search

- **Sequential auction entry model**.
  - auctioneer controls entry process
  - entrant incurs cost *c* to learn its value
  - bidder's value is distributed as *F*
  - only one item may be sold
- **Theorem** (Riley & Zeckhauser, 1983). Let *V** be the optimal value of the *search* problem. Then, the maximum expected revenue in the auction problem is also *V**. The optimal policy is to set a reserve equal to *V** and to sell at that price to the first entrant willing to pay it.
- **Proof**. Exercise. ■

Two-Stage Procedures

- In auctions for business assets, bidders incur costs
  - "due diligence," investigating the condition of the assets.
  - analysis, evaluating business plans that use the assets
- The sale is often conducted in two stages:
  - in stage #1, potential bidders are identified and make preliminary bids to "indicate interest."
    - these "indicative" bids are used to "screen" bidders
  - in stage #2, bids represent binding offers.
    - buyers who are invited to stage #2 are offered extensive access to voluminous, confidential business data

Modeling "Indicative Bidding"

- Exploratory Model
  - each bidder $j=1,\ldots,N$ learns its value v_j
  - bidders make indicative bids $\beta(v_j)$
  - seller selects the top $n\geq 2$ bidders to proceed to stage 2
  - those n bidders each incur a due diligence cost c
  - those n bidders participate in a second price auction
- Two Common Questions:
  - Why does the seller want to limit the number of bidders at stage 2?
  - Why does a bidder not bid an infinite amount at the indicative stage?

Can Indicative Bidding Work?

- **Theorem** (Lixin Ye, 2000). In the exploratory model, there exists no pure symmetric equilibrium bidding strategy.
- **Proof**. We show here only that there exists no *increasing* equilibrium bid function b. Suppose otherwise. Then, a bidder of type v does better to bid $\beta(v-c)$ at the indicative stage, because such a bid loses only when there is some other bidder with value $v' \in (v-c,v)$, and in that case a successful bid would incur a loss in the continuation game. ■

Discussion

- Ye further shows that even if there is information learned before the second stage, equilibrium generically fails to exist.
- Reason: pure increasing equilibrium strategies exist only if for all v, a marginal entrant of type v is indifferent about entering. While this condition may hold for certain specialized models, it fails generically.

Topic #10: Drainage Tract Model

Motivation

- In auctions for mineral rights, timber rights, etc, bidders often have similar values for what is being sold but have different estimates of it.
  - How much oil does the structure hold?
  - How much timber of what types is on that land?
- This can lead to the "winner's curse," which is the tendency of a bidder to win only when its estimate is highest.
- Should the seller worry about this?
  - Should the seller try to mitigate the curse?
  - What policies might the seller use to do that?
  - Are there strategies a bidder can use to exploit the curse to increase its profits?

Math Used in this Section

- Envelope formula
- Jensen's inequality
- Law of Iterated Expectations
  - Variant: $E\left[\Pr\{A \mid X\}\right] = \Pr\{A\}$
  - Proof:

$$\begin{aligned}\Pr\{A\} &= 1\cdot\Pr\{A\} + 0\cdot\Pr\{A^C\} \\ &= E[1_A] = E\left[E[1_A \mid X]\right] \\ &= E\left[1\cdot\Pr\{A \mid X\} + 0\cdot\Pr\{A^C \mid X\}\right] \\ &= E\left[\Pr\{A \mid X\}\right]\end{aligned}$$

Winner's Curse: A Definition

- Model and Idea:
  - The value V of a certain item, say the right to extract oil from some tract, is the same for all bidders
  - Each bidder j makes an unbiased estimate X_j of V.
  - Each bidder bids the same increasing function of its estimate.
  - Then, the winner's estimate is the highest of the X_j.
  - A selection bias results for the winner:

$$E[\max(X^1,\ldots,X^N) \mid V] > \max\left(E[X^1 \mid V],\ldots,E[X^N \mid V]\right) = V$$

- The winner's curse is this selection bias.

Types of Tracts

- Wildcat drilling
  - Drilling in a previously unexplored area.
- Drainage tracts
  - Wells near previously drilled wells.
  - Drilling experience provides superior information about geologic structure, likelihood of finding hydrocarbons.
  - A bidder who has drilled a nearby tract will be called a "neighbor" while others are "non-neighbors"
- We study a drainage tract model introduced by Wilson.
  - Equilibrium characterization by Engelbrecht-Wiggans, Milgrom and Weber.
  - Random reserve theorem by Hendricks & Porter.
  - Other theorems by Milgrom and Weber.

Formulation & Equilibrium

- Elements:
  - A first-price, "common value" model with two bidders in which the actual value of winning is the same for both.
  - The "neighbor" observes a signal t^1 about V and the "non-neighbor" observes an uninformative signal t^2.
  - Without loss of generality, we let t^1 and t^2 be uniformly distributed on (0,1), take $v(s)=E[V|t^1=s]$, and assume that $v(s)$ is nondecreasing.
- **Theorem**: *This auction game has an essentially unique Nash equilibrium. Both players use the same strategy:*

$$\beta(s) = \frac{1}{s}\int_0^s v(r)dr = E[v(t^1) \mid t^1 < s]$$

  - "Essentially unique" means that the distribution of bids and the payoffs for each type are the same at all equilibria.
  - *Surprise!! Bid distributions are exactly the same!!*

Intuition

- The non-neighbor (uninformed) bidder
  - cannot make less than zero,
  - must be indifferent among its bids,
  - cannot make more than zero from the lowest bid in the "support" of its bid distribution, an
  - must have the same support for its bid distribution as does the neighbor.
- So, when the non-neighbor's bid of $b = \beta(s)$ wins, the conditional expected value of V must be b.
  - $\beta(s) = E[v(t^1) \mid t^1 < s] = \frac{1}{s}\int_0^s v(r)dr$

Proof: Equilibrium

- Verifying that 2 is playing a best reply.
  - By construction of 1's strategy, bidder 2's maximum expected profit is 0.
  - Bidder 2 earns zero from the prescribed strategy.
- Verifying that 1 is playing a best reply.
  - We apply the sufficiency theorem.
    - » Increasing differences is verified for the usual reason.
    - » The bid function is nondecreasing.
    - » Since the bidder maximizes $(v(t)-b)\beta^{-1}(b)$ by choosing $b=\beta(t)$, expected profits satisfy the envelope formula:

$$\pi(s) = s\left(v(s) - \tfrac{1}{s}\int_0^s v(r)dr\right) = sv(s) - \int_0^s v(r)dr = \int_0^s rv'(r)dr$$

    - » No bid outside the range of β is better. **QED**

Proof Sketch: Uniqueness

- Neighbor's strategy is uniquely determined by non-neighbor's zero expected profit condition.
  - Therefore, the unique equilibrium strategy for player 1 is $\beta^1(s)=E[v(t^1)|t^1<s]$.
- Also, since β^1 must be optimal for the neighbor, its expected profits must satisfy the envelope formula:

$$\int_0^t G(\beta(s))v'(s)ds = (v(t) - \beta(t))G(\beta(t))$$

$$= \left(v(t) - \frac{1}{t}\int_0^t v(s)ds\right)G(\beta(t))$$

- With the boundary condition $G(\beta(1))=1$, this must have a unique solution, and one solution is: $G(\beta(t))=t$.

Verifying “One Solution”

- With $G(\beta(t)) \equiv t$, we may substitute and use integration by parts to get:

$$
\begin{aligned}
(v(t)-\beta(t))G(\beta(t)) &= \left(v(t)-\frac{1}{t}\int_0^t v(s)ds\right)t \\
&= tv(t)-\int_0^t v(s)ds \\
&= \int_0^t sv'(s)ds \\
&= \int_0^t G(\beta(s))v'(s)ds.
\end{aligned}
$$

N Non-neighbors: Intuition

- The non-neighbors must still expect to earn zero when they win, so <u>the neighbor’s strategy should be independent of the number *N*</u>.
- The distribution of the <u>maximum</u> non-neighbor bid must be unchanged, for otherwise the neighbor would want to bid differently.
  - If there is a random reserve, then it is the distribution of the maximum of the non-neighbor bid or the reserve that must be the same.

N Uninformed bidders

- **<u>Theorem</u>**. *Suppose the seller sets a reserve r according to distribution G satisfying $G(r)>Pr\{\beta(t^1)<r\}$. Then the profile $(\beta,F_1,\ldots,F_N)$ is a Nash equilibrium of the model with N uninformed bidders if and only if β is the pure strategy given by*

$$\beta(s)=E[v(t^1)|t^1<s]$$

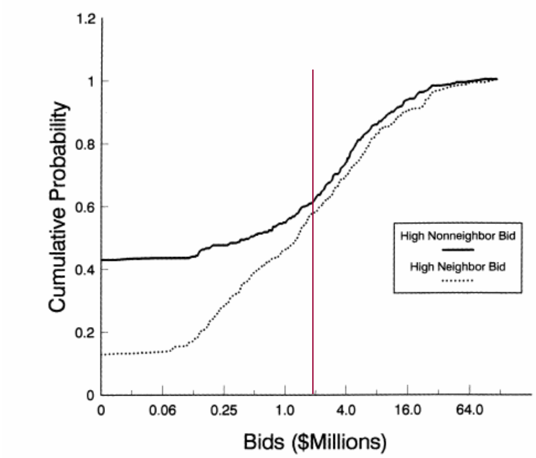
and the “distribution matching” condition holds:

$$F_1(b)\cdots F_N(b)G(b)=Prob\{\beta(t^1)<b\}$$

Empirical Success?

- Surprising predictions
  1. Equal bid distribution for informed and highest uninformed bidder
  2. Informed bid independent of the number of uninformed bidders
- Findings
  1. Fits the data well for higher bids above the range of reserve prices (~$2 million), but not for lower bids.
  2. In a regression test to predict winning bids, coefficient of number of non-neighbors is close to zero.

Hendricks-Porter-Wilson



All bids are represented in 1972 dollars.

FIGURE 1.—Distribution of bids.

Seller's Expected Receipts

- **Theorem**. For the uninformed bidder, equilibrium expected profits are zero. Let $F = v^{-1}$ be the distribution of neighbor's estimate $v(t^1)$. Then the neighbor's expected profits are:

$$\pi = \int_0^\infty F(z)(1-F(z))dz$$

and the seller's expected receipts are $E[v(t^1)]-\pi$.

Proof

- With value v and hence type $F(v)$, the neighbor maximizes $\beta^{-1}(b)(v-b)$ by choosing $b=\beta(F(v))$.
- By the envelope theorem, the neighbor's equilibrium expected profits with value estimate v are therefore:

$$\int_0^v F(z)dz$$

- The bidder's expected profits are

$$\int_0^\infty \left(\int_0^v F(z)dz\right) f(v)dv = \int_0^\infty \left(\int_z^\infty f(v)dv\right) F(z)dz$$

$$= \int_0^\infty F(z)(1-F(z))dz$$

- The last assertion of the theorem is immediate. ■

Revealing Royalty Reports

- Suppose the neighbor observes X and Y and the reports X to the seller. (For example, X may be a report of oil extracted from a nearby property as required to determined the seller's royalty payment.)
- **Theorem**. The policy of revealing the report X reduces the neighbor's expected profit (and raises the seller's expected receipts rise by an equal amount).

Proof

- The neighbor's expected profit is:

$$E\left[\int_0^\infty F(z\mid X)(1-F(z\mid X))dz\right]=\int_0^\infty E[\underbrace{F(z\mid X)}_{\uparrow \text{ in } F(z|X)}\underbrace{(1-F(z\mid X))}_{\downarrow \text{ in } F(z|X)}]dz$$

$$\le \int_0^\infty E[F(z\mid X)](1-E[F(z\mid X)])dz$$

$$=\int_0^\infty F(z)(1-F(z))dz$$

  - The inequality can step follows from majorization (alternatively, Jensen's inequality applied to the concave function $\varphi(y)=y(1-y)$ evaluated at the random variable $F(z|X)$).
  - The math fact that $F(z)=E[F(z|X)]$ is the statement that $E[\Pr\{V\le z|X\}] = \Pr\{V\le z\}$, which we have seen is a variant of the law of iterated expectations.

The Value of Publicity

- …to the **informed** bidder—the neighbor.
- Let A be the neighbor and B the non-neighbor. We endogenize A's information choice.
- **Theorem**. Consider A's profit when A:
  (i) Observes X only and has B know that
  (ii) Observes X and Y and has B act as if only X was observed.
  (iii) Observes X and Y and has B know that A observes X and Y

  A's expected profits are higher for option (ii) than for option (i). Option (iii) has a higher conditional expected payoff than (ii) for every realization of $E[V|X,Y]$.
- Suggested Intuition: A more severe winner's curse causes the non-neighbor to bid "less aggressively" (in a special, but relevant, sense).

Aside: Math Review

- Theorem: For every convex function f,

$$E\left[f(E[V\mid X,Y])\right]\ge E\left[f(E[V\mid X])\right].$$

- Proof. Let f be convex. Apply the law of iterated expectations, Jensen's inequality, and iterated expectations a second time to get:

$$E\left[f(E[V\mid X,Y])\right]=E\left[E\left[f(E[V\mid X,Y])\right]\mid X\right]$$
$$\ge E\left[f(E[E[V\mid X,Y]\mid X])\right]$$
$$=E\left[f(E[V\mid x])\right]$$

Math Review Application

- Recall that
  - The random variables $E[V|X]$ and $E[V|X,Y]$ both have mean $E[V]$.
  - The mean of a positive random variable with distribution F is $\int_0^\infty (1-F(s))ds$.
- Let F and G be the cdf's of $E[V|X]$ and $E[V|X,Y]$. Consider the convex function $f(v)=\max(0,v-x)$. Then, the math review theorem implies that for all x,

$$\int_x^\infty (s-x)dG(s)\ge \int_x^\infty (s-x)dF(s)\Rightarrow \int_x^\infty (s-x)d(G-F)(s)\ge 0$$

$$\Rightarrow \int_x^\infty (G(s)-F(s))ds\le 0\Rightarrow \int_0^x (G(s)-F(s))ds\ge 0$$

Proof (Value of Publicity)

- Let G denote the distribution of E[V|X,Y] and F the distribution of E[V|X]. By the envelope theorem, when the estimate is v, profits in the two cases are:

$$\int_0^v G(z)dz \text{ and } \int_0^v F(z)dz$$

- By the math review application, the first of these is larger.

The Value of Secrecy

- … to the **uninformed** bidder—the non-neighbor.
- **Theorem**. Let A observe X and Y and consider B's profit when…
  (i) B observes nothing and A knows that.
  (ii) B observes X and A knows that.
  (iii) B observes X+ε, where ε is independent noise, and A knows that.
  (iv) B observes X but A bids as if B observed nothing.

  B's expected profits are zero under options (i) and (ii) and positive under options (iii) and (iv).
- Interpretation: The non-neighbor prefers to hide what it knows.

More Profit Expressions

- Define functions, make assumptions:
  - h(x)=E[V|X=x]. Assume h'(x)>0.
  - k(x,y)=E[V|X=x,Y=y]. Assume k_x>0.
- A neighbor of type x who bids as if it were of type z earns (h(x)-b(z))F_x(z). At equilibrium, z*=x. Hence, by the envelope theorem, the expected profits of type x are:

$$\int_0^x F_X(s)h'(s)ds$$

- *Ex ante* expected profits are therefore:

$$\int_0^\infty \left(\int_0^x F_X(s)h'(s)ds \right) f_X(x)dx = \int_0^\infty \left(\int_s^\infty f_X(x)dx \right) h'(s)F_X(s)ds$$
$$= \int_0^\infty (1-F_X(s))F_X(s)h'(s)ds$$

Information Revelation

- Analogously, if the seller observes and announces that Y=y, expected profits are then

$$\int_0^x F_X(s\mid y)k_x(s,y)ds$$

- When the seller's policy is to announce Y, *ex ante* expected profits are

$$E_Y\left[\int_0^{\overline{V}} (1-F_X(s\mid Y))F_X(s\mid Y)k_x(s,Y)ds\right]$$

- Define Δ to be that expected profit minus the expected profit when the seller reports no information:

$$\Delta = E_Y\left[\int_0^{\overline{V}} (1-F_X(s\mid Y))F_X(s\mid Y)k_x(s,Y)ds\right]$$
$$-\int_0^{\overline{V}} (1-F_X(s))F_X(s)h'(s)ds$$

Decomposing Effects

- Suppose the neighbor observes X, the seller observes Y, and the seller's policy is to report information, the neighbor's expected profits change by an amount Δ=W+P that reflects two different effects:
  - W: the *weighting effect* -- Y reduces (or, if negative, increases) the weight of the private information X in the "multiple regression" estimate of V. Denote this effect on profits by ***W***.
  - P: the *publicity effect* -- Y conveys information about X, making X less private and reducing A's information rents. Denote this effect on profits by ***P***.

Two Effects

- Define the weighting effect W and the publicity effect P by:

$$\Delta = W + P$$

$$P = \int_0^\infty \underbrace{\left[F_X(x)(1-F_X(x)) - E[F_X(x \mid Y)(1-F_X(x \mid Y))\right]}_{\text{Integrand is positive!}} h'(x)\,dx \geq 0$$

$$W = \int_0^\infty E\left[(h'(x)-k_x(x,Y))F_X(x \mid Y)(1-F_X(x \mid Y))\right]dx$$
$$= E\left[(h(X)-k(X,Y))(2F_X(X \mid Y)-1)\right]$$

  - The second expression for W comes from integrating by parts.
- Discuss economic ideas captured by the decomposition.

Example: "Neutral Information"

- Suppose $V=X+Y$ where X and Y are independent.
- Then,
  - W=0: revealing Y does not effect the weight accorded to X in estimating V, that is, $h'(x) \equiv k_x(x,y) \equiv 1$.
  - P=0: revealing Y conveys no information about X.
- Therefore, revealing Y does not affect expected profits or expected revenues.

Informational Substitutes

- Suppose that X and Y are distributed according to a joint density f with log(f) strictly supermodular:

$$\frac{\partial^2 \log(f(x,y))}{\partial x \partial y} > 0 \text{ everywhere}$$

- **Theorem**. *Assume that $k_x>0$, $k_y>0$ and* log(f) *is strictly supermodular. Then, P>0 and W>0.*
  - *Revealing information then reduces the informed bidder's expected profits and increases the seller's expected receipts by P+W.*

Proof that W ≥ 0

- We already know that P≥0. Calculating,

$$W = E\left[(h(X)-k(X,Y))(2F_X(X\mid Y)-1)\right]$$
$$= E\left[E\left[\underbrace{(h(X)-k(X,Y))}_{\text{Decreasing in } Y}\underbrace{(2F_X(X\mid Y)-1)}_{\text{Decreasing in } Y}\mid X\right]\right]$$
$$> E\left[E[h(X)-k(X,Y)\mid X]E[(2F_X(X\mid Y)-1)\mid X]\right]$$
$$= E\left[0\cdot E[(2F_X(X\mid Y)-1)\mid X]\right]$$
$$= 0$$

- The inequality is by majorization. ■

Informational Complements

- Suppose X=V+Y, where V and Y are independent.
- Then,
  - It is obvious that the reported information is useless to the uninformed bidder. Hence, the situation can formally be mapped into that covered by our results. Hence, the informed bidder's profits rise from the revelation of Y.
  - Evidently, P<0, so W>0.

Topic #11: Auctions with Weak and Strong Bidders

Variant of Maskin-Riley Model

- Bidder j with value function v_j:[0,1]→$\Re$, j=1,2.
- The value functions are increasing & differentiable, and the reserve r is in the range of both functions.
- Consider increasing strategies β_j satisfying β_1(r) = β_2(r) = r.
- Defining the "matching function"

$$m(t) = \beta_2^{-1}(\beta_1(t))$$

First-order conditions

- Bidder 1 of type t can be thought of choosing its probability s of winning by solving

$$\max_s s\left(v_1(t)-\beta_2(s)\right)$$

and bidder 2 solves a similar problem. At equilibrium, we must have s=m(t), so the first-order conditions for bidders 1 and 2 are:

$$0=v_1(m^{-1}(s))-\beta_2(s)-s\beta_2'(s)$$
$$0=v_2(m(t))-\beta_1(t)-t\beta_1'(t)$$

Unique Equilibrium

- Theorem (Maskin-Riley). The system of equations below has a unique solution (β_1,β_2), and it describes the unique equilibrium of the auction game:

$$m(t)=\beta_2^{-1}(\beta_1(t))$$
$$0=v_1(m^{-1}(s))-\beta_2(s)-s\beta_2'(s)$$
$$0=v_2(m(t))-\beta_1(t)-t\beta_1'(t)$$
$$r=\beta_1(v_1^{-1}(r))=\beta_2(v_2^{-1}(r))$$
$$\beta_1(1)=\beta_2(1)$$

Ranking Bid Distributions

- Theorem. Suppose that for all $t\in(0,1)$, $v_1(t)>v_2(t)$. Then, for all $t\in(0,1)$, $\beta_1(t)>\beta_2(t)$.
- Proof. Let $h(t)=\beta_1(1-t)-\beta_2(1-t)$.
For any t such that $h(t)$=0, $\beta_1(1-t)-\beta_2(1-t)$ & $m(t)=t$.
Hence,

$$h'(t)=\beta_2'(1-t)-\beta_1'(1-t)$$
$$=\frac{1}{1-t}(v_1(1-t)-v_2(1-t))>0$$

- Since $h(0)$=0, $h(t)$>0 for all t>0 (a single crossing lemma).

QED

Ranking value→bid functions

- Theorem. Suppose that values are drawn from the distributions F(•|0) for the "weak" bidder and F(•|1) for the "strong" bidder, where log(F(v|s)) is supermodular. Then for each possible value, the strong bidder bids less than the weak bidder.
- Proof. Exercise: Use the first order conditions to apply the single crossing lemma to the following function:

$$h(\bar{v}-t)=\beta_0(t)-\beta_1(t)$$

Ranking Profits

- Theorem. Under the hypotheses of the previous theorem, the equilibrium expected profit of a "strong" bidder with any value v is higher in the second-price auction than in the first price auction. The reverse inequality holds for the weak bidder.
- Proof. The strong bidder's probability of winning is lower in the first-price auction than in the second-price auction. Apply Myerson's lemma.
  - A symmetric argument applies for the weak bidder.

Topic #12: Multi-Item Auctions

Selling Related Items

- Often, auctions sell multiple related items
- Sometimes, all of the items are substitutes
  - At Stanford University, empty lots on the "hill site" were sold for development by individual faculty
  - The US Treasury (and many other treasuries) sell debt instruments at auction
- Sometimes, some items may be complements
  - Spectrum licenses in adjacent geographic areas
  - A pair of matched art objects
- A sequence of unrelated auctions may perform poorly.

Three Cases

1. Homogeneous items, like electricity or pollution abatement.
   - All units are identical and we may aim to trade each one at the same price.
2. Heterogeneous substitutes, like Stanford's hill-site lots for faculty homes.
3. Heterogeneous complements, where items may be sold in sets and individual item prices may not be workable.

Identical Items and Diminishing Returns

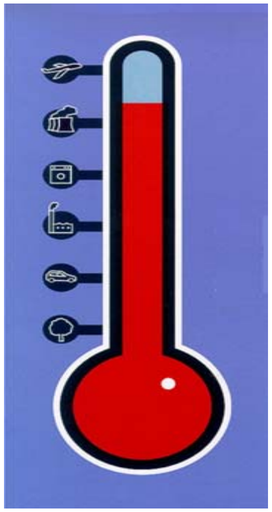
"Homogeneous" Items

- Consider a sale of K identical items.
- If each bidder has a diminishing marginal value for items, then
  - the items are (perfect) substitutes
  - the market clears at any price that is
    - not greater than the marginal value of the K^{th} item
    - not less than the marginal value of the $K+1^{th}$ item
- For this problem, we can conceive of the auction as finding the market clearing price.

Two Uniform Price Auctions

- A sealed-bid auction, in which participants bid only once and their bids are used to
  - Determine the uniform price
  - Determine the bidders' quantities
- A "clock" auction in which
  - the auctioneer announces a sequence of prices
  - bidders name quantities
  - auction ends when a market-clearing price has been found

British CO_2 Auctions



- **Greenhouse Gas Emissions Trading Scheme Auction United Kingdom March 11-12, 2002**
  - **38 bidders**
  - **34 winners**
  - **4 million metric tons of CO_2 emission reductions**

Greenhouse Auction Rules

- Auctioneer calls prices
  - Starts high
  - Prices can only decrease
- Bidders announce tons of CO_2 they will abate at that price.
  - Tons abated can only decline as prices decrease.
- Auctioneer
  - multiplies tons of abatement times price
  - if total cost exceeds budget, lowers the price
  - when total cost first falls short of budget, auction ends and that allocation is implemented

Electricity Auctions

- Suppose a state wishes to contract for electricity for distribution to residents.
- It may run a clock auction, similar to greenhouse gas auction.
  - The main difference is that the quantity demanded, rather than the budget, is fixed.

Strategic Equivalence

- Suppose bidders in the auction observe only the prices and that prices decline in a fixed sequence.
- Then, a pure strategy is a function mapping the current price and the bidder's own past quantities into a current quantity.
  - A reduced strategy is a map from the current price into the current quantity.
- The clock auction is strategically equivalent to a sealed bid auction in which
  - a bid is a "supply curve"
  - the auctioneer posts its demand curve
  - the price is determined so that supply = demand.

Rules

- Auctioneer sets supply Q(p)
  - Initially, assume an inelastic supply Q.
- Each bidder j
  - Has a value function $V_j(q)$ for goods acquired
  - Bids a schedule of prices and quantities (p_{jk},q_{jk}), $k=1,2,\ldots K_j$.
- Pseudo-Vickrey rules: The auctioneer
  - Allocates goods to the Q highest bids
  - Sets the price equal to the highest rejected bid.
  - (A similar analysis applies if price = lowest accepted bid)

Example & lessons

- Rules: 10 items for sale; seller takes 10 highest bids, reserve price = 1.
- Bidders: 10 bidders, each with a value of 100+ε for each item for as many items as it can get.
- A "collusive-seeming" Nash equilibrium
  - Each bidder bids 100 for a single item and 1 for each additional item.
  - Equilibrium price = highest rejected bid = 1
  - There are many other equilibria, and all are robust to model variations.
  - There is "demand reduction" to exert market power.

Increasing Elasticity

- Suppose the seller increases supply, but makes it elastic. Seller promises to
  - Sell 10 if the price is at least 1.
  - Sell 11 if the price is at least 40.
  - Sell 12 if the price is at least 70.
  - Sell 13 if the price is at least 85.
- Every equilibrium has Q = 13 and price ≥ 85.
- Elastic supply eliminates bad equilibria.
- Effect can be counter-intuitive:
  - In the example, <u>increasing</u> supply <u>increases</u> the price.
  - In general, it is increasing elasticity of supply that increases the price, because it makes "implicit collusion" harder to sustain.

General Lessons

- Every uniform price auction encourages some form of demand reduction.
  - Idea: similar to traditional monopoly theory.
- The "collusive seeming" equilibria depend on an finding a point at which prices are low and yet very sensitive to demand variations.
  - Making supply elastic eliminates such points and drastically reduces the ability of "moderately-sized bidders" to sustain low prices in uniform price auctions.

Heterogeneous Items: Substitutes

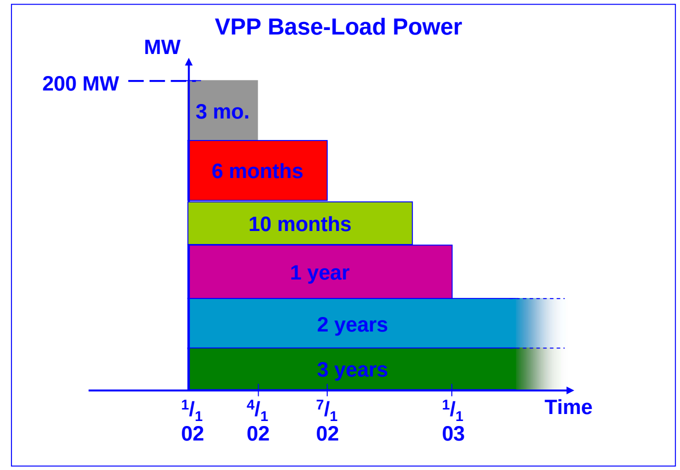
EDF Generation Capacity Auction

EDF Electricité de France

PRICEWATERHOUSECOOPERS

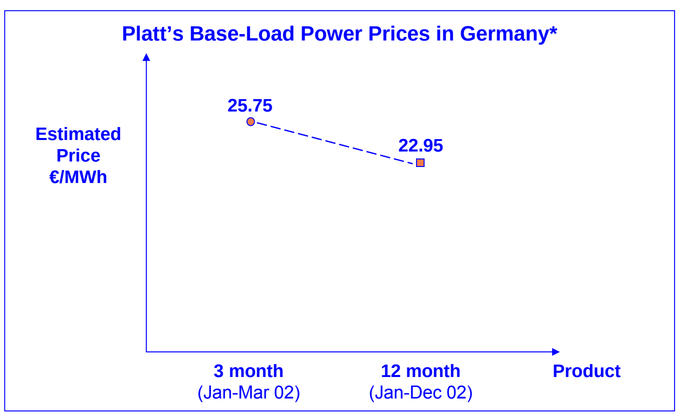
αγορα
market design inc.

Product Group A



Relative Values

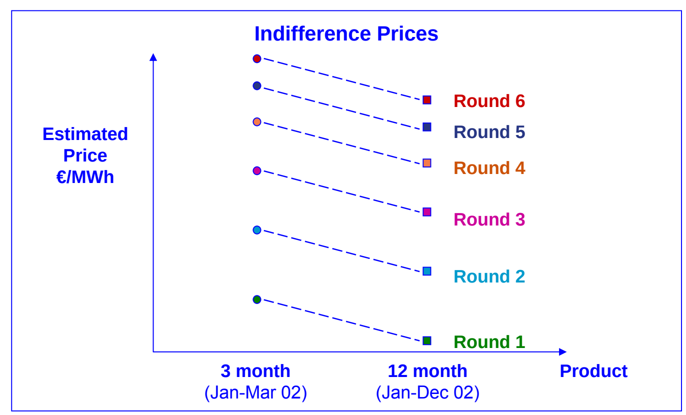
- Products have different expected values:



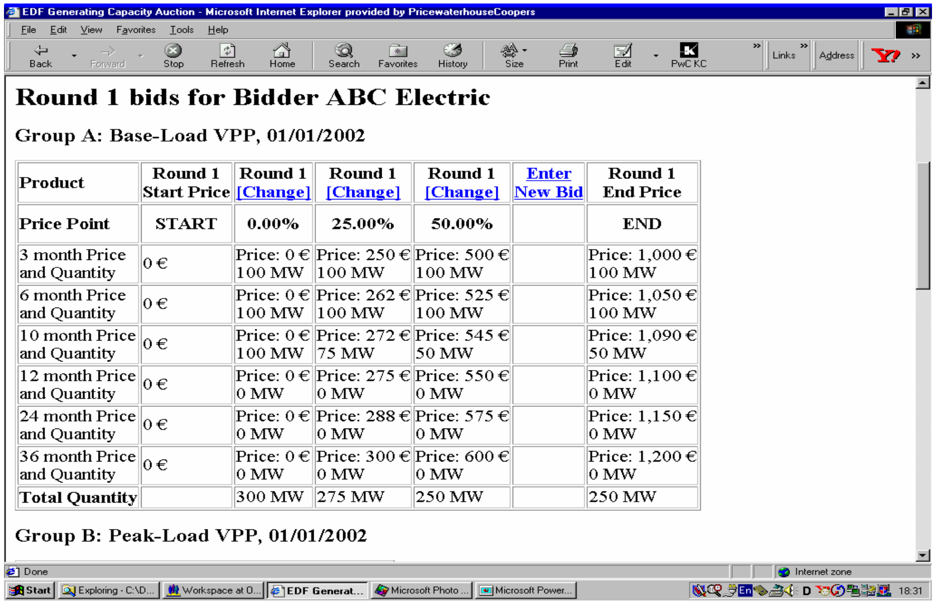
**Source: Platt's European Power Daily, 17 July 2001*

Indifference Prices for EDF Auction

- Applied within the product group:



Round 1: Bid Example



Round 1 bids for Bidder ABC Electric

Group A: Base-Load VPP, 01/01/2002

Product	Round 1 Start Price	Round 1 [Change]	Round 1 [Change]	Round 1 [Change]	Enter New Bid	Round 1 End Price
Price Point	START	0.00%	25.00%	50.00%		END
3 month Price and Quantity	0 €	Price: 0 € 100 MW	Price: 250 € 100 MW	Price: 500 € 100 MW		Price: 1,000 € 100 MW
6 month Price and Quantity	0 €	Price: 0 € 100 MW	Price: 262 € 100 MW	Price: 525 € 100 MW		Price: 1,050 € 100 MW
10 month Price and Quantity	0 €	Price: 0 € 100 MW	Price: 272 € 75 MW	Price: 545 € 50 MW		Price: 1,090 € 50 MW
12 month Price and Quantity	0 €	Price: 0 € 0 MW	Price: 275 € 0 MW	Price: 550 € 0 MW		Price: 1,100 € 0 MW
24 month Price and Quantity	0 €	Price: 0 € 0 MW	Price: 288 € 0 MW	Price: 575 € 0 MW		Price: 1,150 € 0 MW
36 month Price and Quantity	0 €	Price: 0 € 0 MW	Price: 300 € 0 MW	Price: 600 € 0 MW		Price: 1,200 € 0 MW
Total Quantity		300 MW	275 MW	250 MW		250 MW

Group B: Peak-Load VPP, 01/01/2002

EDF Auction

- The market determines a single price
  - All other prices are fixed relative to the single price by the auction design.
- Bidders can substitute among different types of contracts.
- Rules prescribe that a bidder's total demand cannot increase as prices rise.

Simultaneous Ascending Auctions and Market Clearing

SAA: Basic Rules

- Bidding on all licenses occurs simultaneously, in rounds.
  - The auction is typically run electronically to permit tracking of multiple licenses.
- All bids become public information at the end of the round.
- The "*standing high bid*" on each license plus a percentage becomes the minimum bid for the next round.
- The "*standing high bidder*" is initially the FCC. Higher bids make new standing high bidders
  - Time stamps or (pseudo-)randomization to break ties.
- Auction ends when there is no new bid for any license.
- Large penalties for non-payment

The original FCC report can be found at
www.milgrom.net/fcc auction 1994 r&o.pdf

Activity Rules

- To make the simultaneous ascending auction practical, the FCC adopted the Milgrom-Wilson activity rule.
  - All subsequent similar auctions have employed some activity rule.
- Milgrom-Wilson activity rule
  - A bidder *j* begins the auction with some eligibility $e_j(1)$.
  - "Activity" at a round consists of new bids and standing high bids from the prior round
    - Activity measured in licenses, or POPs, or "points"
    - A bidder's activity in round n may not exceed its eligibility at that round $A_j(n) \le e_j(n)$
  - A bidder's "eligibility" evolves as $e_j(n+1)=\min(e_j(n), \alpha A_j(n))$, where α is close to but possibly larger than 1.
- This rule requires bidders who hope to acquire licenses to be active early in the auction to speed the process to completion.

Advantages Claimed

- Reveals information "soon enough to be useful to bidders to implement back-up strategies."
- Information revelation mitigates inefficiency due to the winner's curse
- Activity rule
  - Mitigates worst-case, slow bidding, scenario
  - Improves information flow: bidders track the "eligibility ratio" (ratio of total eligibility points to total points offered)

Formulation

- $\{1,\ldots,L\}$ is a set of *indivisible* licenses with typical subset *S*.
- Bidders' payoffs are the value of licenses acquired minus the amount paid $v_j(S)-m_j$.
  - Assume free disposal
- Demand "correspondence" is

$$D_j(p) = \arg\max_S v_j(S) - p(S)$$

  - Limit attention to prices with unique demands and treat D_j as a demand function.
- "Personalized price" p^{jn}_k for bidder *j* on item k at round n is the lowest price at which *j* might conceivably acquire k
  - the high bid if *j* is the *standing high bidder* on k
  - the high bid plus one increment otherwise

Definitions

- Say that bidder *j demands* set *S* at price vector *p*, if $S \subseteq D_j(p)$.
- Licenses are *substitutes* (standard definition) if:
  - $\left(k \in D_j(p), p' \ge p, p'_k = p_k\right) \Rightarrow k \in D_j(p')$
- Examples
  - a bidder who wants just one license.
  - a bidder who doesn't care about which license and has declining marginal values for licenses.
- Say that bidder *j bids straightforwardly* if,
  - whenever *j* is standing high bidder after round *n* on $S_j \subseteq D_j(p^{jn})$,
  - she makes the minimum bid at round *n*+1 to be active on $D_j(p^{jn})$.

Substitutes: Straightforward Bidding

- Theorem: *Assume that*
  - *all the licenses are substitutes for bidder j and*
  - $S_j^n \subseteq D_j(p^{jn})$.
- *If, at round n+1, bidder* j *bids straightforwardly, then, regardless of the bids made by other bidders,* $S_j^{n+1} \subseteq D_j(p^{j,n+1})$.
- Corollary. *If bidder j bids straightforwardly at every round during the auction, then for all n,* $S_j^n \subseteq D_j(p^{jn})$. *(At every round it demands its licenses at its end-of-round personalized prices.)*

Proof of Theorem Sketched

- Since $S_j^n \subseteq D_j(p^{jn})$ and *j* bids straightforwardly, *j* is active and makes minimum bids for $D_j(p^{jn})$.
- By the rules of the auction, a bidder can become high bidder only on what she bids for, so $S_j^{n+1} \subseteq D_j(p^{jn})$.
- By construction of personalized prices,
  - prices can only rise: $p^{j,n+1} \geq p^{j,n+1}$.
  - but *j*'s personalized prices does not rise for $k \in S_j^{n+1}$, so $p^{j,n+1}(k) = p^{j,n+1}(k)$.
- Hence, by substitutes, $S_j^{n+1} \subseteq D_j(p^{j,n+1})$

QED

Discussion

- When the theorem applies, a bidder who bids just for what she wants never gets stuck with an unwanted package.
  - But this depends very much on the assumption that goods are substitutes.
- Example:
  - Bidder 1's package values are:
    - » 5 for either license alone
    - » 20 for licenses A and B (the licenses are complements!)
  - Bidder 1 is the standing high bidder on both at prices of 8 and 8.
  - Bidding on license A ceases, but the price of license B is bid gradually up to 15.
- In the substitutes case, does the auction uncover market clearing prices?

Example…Auction Ending

- Values:
  - 1 would pay 17 for A or 22 for B or 34.5 for both.
  - 2 would pay 20 for A or 20 for B or 37.5 for both.

Round	A's price: p_A	B's price: p_B	A's High Bidder	B's High Bidder
25	11	16	1	1
26	12	17	2	2
27	13	17	1	2
28	14	17	2	2
29	14	18	2	1

Describing Outcomes

- We describe the auction outcome as an exact competitive equilibrium for a nearby set of values.
- The nearby values are constructed as follows:
  - Identify the goods that bidder *j* wins at the auction.
  - Define *j*'s modified values for any set of goods *T* to be the original value minus one bid increment for each good in *T* that *j* does not win.
- Notice: *j*'s net values (value-minus cost) in the *nearby* situation using the *final auction prices* is the same as her net value in the old economy using personalized prices.

Substitutes: Competitive Equilibrium

- *Theorem:* Suppose the licenses are substitutes and that all bidders bid straightforwardly. Let (p^*,S^*) be the final standing high bids and license assignment and suppose the minimum bid increment vector is q. Then (p^*,S^*) is a competitive equilibrium for a nearby economy with individual valuations defined by:

  $$\hat{v}_j(T) = v_j(T) - q \cdot 1_{T \setminus S_j^*}$$

  The final assignment "nearly" maximizes total value:

  $$\max_S \sum_j v_j(S_j) \le \sum_j v_j(S_j^*) + \varepsilon q \cdot 1_L$$

  Corollary. If the minimum bid increment vector is εq and ε is sufficiently small, then the final license assignment $S^*(\varepsilon)$ is a total-value-maximizing assignment for the original valuations.

Proof Sketch

- By the previous theorem, at termination of the auction, every bidder demands the licenses it is assigned. By straightforward bidding, no bidder demands even more licenses at its personalized prices. Hence,

  $$S_j^* = D_j(p^{*j})$$

- If p^{*j} denotes the final "personalized prices," then by construction:

  $$D_j(p^{*j}) = \hat{D}_j(p^*)$$

- The Corollary follows by observing that the choice set is finite, so any ε-optimal allocation must be exactly optimal when ε is sufficiently small.

Example, continued…

Round	A's price: p_A	B's price: p_B	A's High Bidder	B's High Bidder
29	14	18	2	1

- Values:
  - 1 would pay 17 for A or 22 for B or 34.5 for both.
  - 2 would pay 20 for A or 20 for B or 37.5 for both.
- Nearby "pseudo-values"
  - 1 would pay 16 for A or 22 for B or 33.5 for both.
  - 2 would pay 20 for A or 19 for B or 36.5 for both.
- The final prices (14,18) and allocation clear the market using the nearby values.
  - 1 would earn 2 from A, ***4 from B***, or 1.5 from AB.
  - 2 would earn ***6 from A***, 1 from B, or 4.5 from AB.

Summary of non-strategic theory

- **Theorem:** *Suppose the licenses are substitutes for bidders and that all bid "straightforwardly." Then*
  - *(Arbitrage/Uniform Prices) The final prices for identical items will differ by at most one bid increment.*
  - *(Efficiency) If the bid increments are sufficiently small, the final license allocation will be efficient.*
  - *(Competitive Equilibrium) The final prices will be market clearing prices for bidder values "close to" the actual values (in which the values of items not acquired are reduced by one bid increment).*
- …should we care about the non-strategic theory?

Exposure Problem in the Netherlands

- A simultaneous ascending auction completed February 18, 1998 after 137 rounds.
- Raised NLG 1.84 billion.
- Prices per band in millions of NLG
  - Lot A: 8.0
  - Lot B: 7.3
  - Lots 1-16: 2.9-3.6
- Bad outcomes?
  - No arbitrage: why not?
  - Stranded bands:
    - » Orange/Veba was last to drop out on the large licenses, obtained only 2 small bands
    - » Only one new competitor-obtained sufficient small licenses (TeleDanmark acquired 5) for viable business

Equilibrium Example

- Suppose the following
  - There are ten identical licenses for sale
  - There are ten identical bidders with these values
    - » 10 for one license.
    - » Generally, (10-½(n-1))n for n licenses.
- Subgame perfect equilibrium strategy
  - Bid for the one cheapest license so long as the price is less than 10.
- Generally, uniform price auctions admit "collusive seeming equilibrium"
- In this example, other equilibria are hard to rationalize if bidders have perfect information.

Topic #13:
Package Auctions

An Example

- There are two items for sale, A and B, and two bidders with values as follows.

	A	B	AB
1	0	0	12
2	10	10	10

First Issue: Market-Clearing Prices

- There are two items for sale, A and B, and two bidders with values as follows.

	A	B	AB
1	0	0	12
2	10	10	10

- The efficient (value-maximizing) outcome assigns both items to 1.
- Any market clearing price vector must support the efficient allocation and so must satisfy $p_A \geq 10$, $p_B \geq 10$, $p_A + p_B \leq 12$.
- Therefore, no market clearing “item” prices exist.

Introducing the “Core”

- The “core”—to be defined mathematically later, is the relevant generalization of a competitive outcome.
- In words, an allocation is in the core if there is no set or “coalition” of players that could make a deal on their own from which all of them would benefit.
  - The core is also sometimes defined to be the set of imputed payoff profiles, or “imputations,” which correspond to core allocations.
  - Example:
    - If bidder 2 buys A for a price of 6, the imputed payoff profile (listing the seller first) is (6,0,4).
    - This is not a core imputation, because the coalition {S,1} could do better by S selling AB to 1 for a price of 8.

The Empty Core Problem

- Same example: There are two items for sale, A and B, and two bidders with values as follows.

	A	B	AB
1	0	0	12
2	10	10	10

- With just one seller, the imputation (10,2,0) is in the core.
- But if goods A and B belong to different sellers, then the core is empty, because…
  - Bidder 2 must get 0
  - Coalition of either seller and bidder 2 must get 10
  - So, each seller must get 10, but only 12 is available.

Thinking About Empty Cores

- In relation to the Coase Theorem
  - Logically impossible for every coalition to bargain to its optimum, regardless of "transaction costs"
  - Borrow from political science the idea of favoring small coalitions ⇒ expect inefficiency even without physical "externalities"
- In relation to costless renegotiation
  - Why not expect small coalitions, once formed, to renegotiate to form larger coalitions?
  - Problem: Still have an empty core with respect to initial negotiation.

What is Done in Practice?

- Single sellers often use *pay-as-bid package auctions,* in which bidders bid for packages.
  - London bus routes (Cantillon & Pesendorfer)
  - Sears truck routes (Ledyard et al)
  - Chilean school lunches (Epstein et al)
  - IBM procurements (Hohner et al)
  - Portland General Electric generating assets (Milgrom)
  - 150 package procurements run by *CombineNet* (Sandholm)
- Multi-seller exchange problems exhibit symptoms of failure
  - Compare cell phones in the US and Europe
  - Real estate development patterns in US
  - Designing for the US spectrum-exchange problem

"Pay-as-Bid" Package Auctions

- Theory: Canonical Rules
  - Each bidder bids a separate price for each package it may want to buy.
  - Seller may accept at most one bid per bidder
  - Seller may impose constraints, such as
    - » Minimum quantity sold to minority-owned bidders
    - » Maximum concentration ratio of sales
    - » Procurement sales: geographic diversification of supply
  - Auctioneer selects the feasible combination of offers that is optimal according to some objective.
  - Each winning bidder pays the price it bid.

The Core, in Mathematics

- Notation:
  - X denotes the set of feasible allocations, $x \in X$
  - N denotes the set of bidders and seller, $j \in N$
  - $u_j(x_j)$ the valuation function for bidder
  - $S \subset N$ a typical coalition.
- Coalitional value function:

$$w(S) \equiv \begin{cases} 0 & \text{if } seller \notin S \\ \max_{x \in X} \sum_{j \in S} u_j(x_j) \text{ subject to } x_{-S} = 0 & \text{if } seller \in S \end{cases}$$

- (N,w) is the coalitional game derived from trade between the seller and bidders.
- "Core" imputations are defined as follows:

$$Core(N,w) = \left\{ \pi \geq 0 \middle| \sum_{j \in N} \pi_j = w(N), (\forall S) \sum_{j \in S} \pi_j \geq w(S) \right\}$$

Pay-as-Bid Theory

- Definition. A core allocation is "*bidder optimal*" if there is no other core allocation that is strictly preferred by every bidder.
- Theorem (Bernheim-Whinston). The full-information, coalition-proof equilibrium outcomes of the pay-as-bid package auction are exactly the bidder optimal core allocations.
  - Special case: one good, Bertrand equilibrium
  - At equilibrium, for every package, each bidder bids its value minus its core payoff ("profit target strategy").

Pay-as-Bid Auction Example

- Values

	A	B	AB
1	0	0	12
2	10	10*	10
3	10*	8	11

- Equilibrium bids: "constant profit targets" but no bid is less than zero. Particular equilibrium with π=(12,0,3,5).

	A	B	AB
1	0	0	12
2	7	7*	7
3	5*	3	6

Interpretation?

- Theorem (Bernheim-Whinston). The full-information, coalition-proof equilibrium outcomes of the pay-as-bid package auction are exactly the bidder optimal core allocations.
- How should we interpret this?
  - Competitive payoffs (Bertrand analogy)?
  - Limited participation problems?
  - Why are these equilibria interesting?
    - » Infeasibility of the full-information strategy
    - » Equilibrium selection criterion
    - » Still multiple equilibria!
  - Why is the design popular? Theorem is a poor answer.

Vickrey Auction

	A	B	AB
1	0	0	12
2	10	10	10

- The Vickrey auction (Vickrey-Clarke-Groves pivot mechanism):
  - Assign goods efficiently, so bidder 1 is the sole winner.
  - Set each winning bidder's price equal to the opportunity value of the goods acquired. In this case, the opportunity value is 10.
  - Losers pay zero.
- Known properties
  - Truthful reporting is a dominant strategy.
  - Outcome is efficient.
  - No other mechanism has these three properties for all valuations (Green-Laffont/Holmstrom theorems).

Why Not Vickrey Auctions?

	A	B	AB
1	0	0	12
2	10	10	10

- Vickrey payoffs (10,2,0) are in the core in this example.
  - Outcome is efficient.
  - No coalition can block.
- So, why don't we see more Vickrey Auctions?
  - Concerns about complexity?
    - …but compare pay-as-bid package auctions
  - Concerns about privacy? (Rothkopf, Teisberg, Kahn)
    - …but eBay, Amazon are Vickrey-like auctions
    - …and especially Google's ad placement auctions

Decisive Fault

- The *decisive fault* is that Vickrey outcomes may lie far outside the core due to too-low seller revenues.
- A 3-bidder example.

	A	B	AB
1	0	0	10
2	10	10	10
3	10	10	10

  - Bidders 2 and 3 win items at Vickrey price 2
  - Seller payoff is just 4
- The core: coalition of seller, bidder 1 can get 12
  - Vickrey outcome is not in the core
  - When 12 is replaced by 10… seller revenue falls to zero.

Vickrey Payoffs: Theory

- Bidder j's Vickrey payoff is $v_j = w(N)-w(N-j)$, his marginal contribution to the coalition of the whole.
- Theorem (Bikchandani-Ostroy, Ausubel-Milgrom). A bidder's Vickrey payoff is $v_j = \max\{r_j | r \in Core(N,w)\}$.
- Corollary 1. If the Vickrey payoff vector is in the core, then it is the unique bidder optimal core allocation.
- Corollary 2. If the Vickrey payoff vector is not in the core, then for every $r \in Core(N,w)$, $v_0 < r_0$.
  - Interpretation: When the Vickrey payoff is not in the core, the seller's Vickrey revenue is uncompetitively low.

Related Vickrey Problems

	A	B	AB
1	0	0	12
2	10	10	10
3	10	10	10

- "Monotonicity problem": Adding bidder 3 reduces revenues from 10 to 4, and that is problematic in practice because…
  - Seller might seek to exclude bidder 3, or to disqualify the bid after it is made.
  - Bidder 2 could profitably sponsor a fake bidder 3.
  - Lowered revenues expose the auction to ridicule.

Scope of the Problems

- How widespread are these problems?
  - When do core outcomes exist? (With one seller: Always!)
  - When are Vickrey outcomes in the core?
  - When do competitive equilibria exist?
- Two "positive" results:
  - Theorem (Ausubel-Milgrom): If goods are substitutes for all bidders, then Vickrey outcomes are core outcomes.
  - Theorem (Milgrom, Gul-Stacchetti): If goods are substitutes for all bidders, then competitive equilibria exist.
- So, the these two problems vanish when goods are substitutes.

Converse Theorems

- Theorem (Ausubel-Milgrom). If $V \not\subset V_{sub}$ and $V_{add} \subset V$, then there exists a profile of valuations from V such that the Vickrey outcome is not a core outcome.
- Theorem (Milgrom, see also Gul-Stacchetti). If there are at least three bidders, $V \not\subset V_{sub}$ and $V_{add} \subset V$, then there exists a profile of valuations from V such that no competitive equilibrium exists.
- Definitions
  - V is the set from which individual bidders' valuations of goods are drawn
  - V_{add} is the set of "additive" valuations
    - » the value of a package is the sum of the item values
  - V_{sub} is the set of substitutes valuations
    - » the implied demand function satisfies the substitutes condition

A Third Package Auction

- Package extensions of the English auction include Parkes iBundle 3 and Ausubel-Milgrom ascending proxy auction
- Ausubel-Milgrom rules
  - Bidders report maximum bids to a proxy bidder.
  - Auction initiates with bids of zero by all bidders for all packages
  - Auctioneer "holds" its most preferred feasible collection of bids.
    - » Typically, total sales revenue determines the preference
    - » Tie-breaking rule makes auctioneer preferences strict
  - At each round,
    - » Bidders with bids being held do nothing
    - » For others, proxy bidders makes the most "profitable" new bid, or no bid if none is profitable.
  - Bids accumulate: the auctioneer may choose from all previously submitted bids.
  - Auction ends when there are no new bids.

Proxy Auction Example

- Values

	A	B	AB
1	0	0	12
2	10	10	10

- Time path of bids

	Bidder 1	Bidder 2		
Round	AB	A	B	AB
1	1*	1	1	1
2	1	2*	2	2
3	2*	2	2	2
4	2	3*	3	3
…	…	…	…	…
19	10*	10	10	10

Algorithm Property

- Theorem (Ausubel-Milgrom). The ascending proxy auction terminates at an *efficient outcome* (cf Parkes) and, what is more, at a *core allocation,* both with respect to the reported preferences.
- Proof Idea. At termination of the algorithm
  - Allocation is feasible
  - Allocation is unblocked
    - Each bidder has made every offer that he prefers.
    - No feasible combination of those offers is also preferred by the seller.

Ex Post Equilibrium

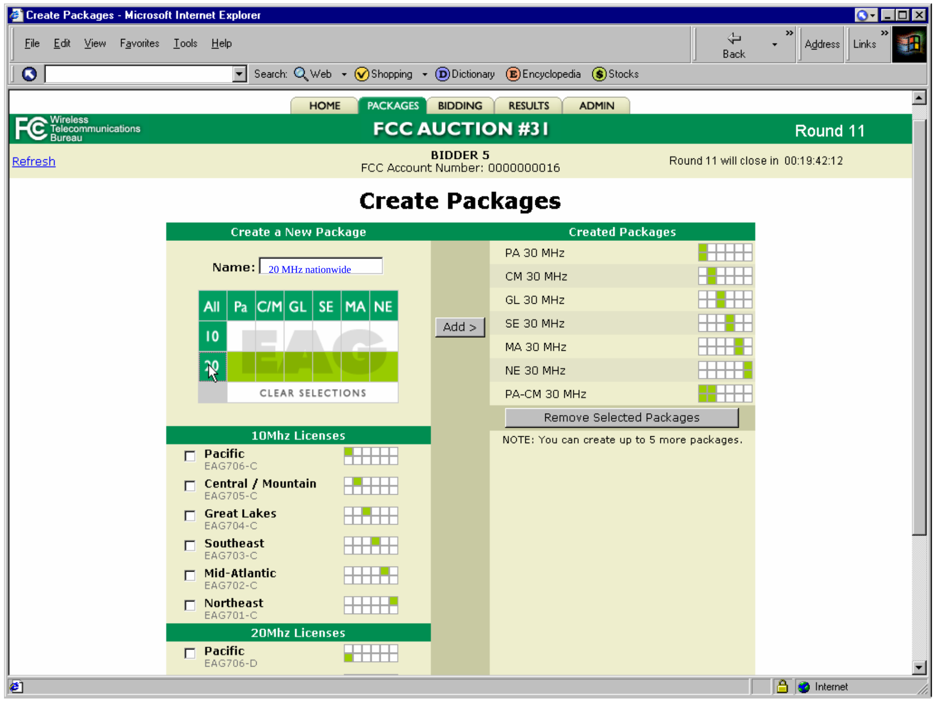
- Theorem (Ausubel-Milgrom). If goods are substitutes for all bidders or if there are just two bidders, then truthful reporting of values is an *ex post* equilibrium.
  - This means that, after learning the other bids, no bidder could ever profit by changing her own bids.
- This theorem extends the familiar connection between ascending auctions for one item and the dominant strategy Vickrey auction to a multi-item setting, provided the goods are substitutes.

Selected Nash equilibrium

- Theorem (Ausubel-Milgrom). For every bidder optimal core payoff vector π, there is a full information Nash equilibrium with payoffs π at which the maximum bids reported to the proxy are identical to the coalition-proof equilibrium bids in the pay-as-bid package auction.
  - If bidders play this way, their final bids wind up being the same as in the pay-as-bid auction, so the outcomes and payoffs are also the same.

A Practical Design?

- The ascending proxy auction…
  - matches the excellent performance of the Vickrey design on environments where goods are substitutes.
  - avoids the worst low revenue outcomes and monotonicity problems of the Vickrey auction when goods are not substitutes.
  - matches the pay-as-bid package auction in terms of full information equilibrium (but it is not clear why that should be important).
- Are these the right criteria for evaluating designs?


Create Packages - Microsoft Internet Explorer
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Wireless Telecommunications Bureau
FCC AUCTION #31
Round 11
Refresh
BIDDER 5
FCC Account Number: 0000000016
Round 11 will close in 00:19:42:12
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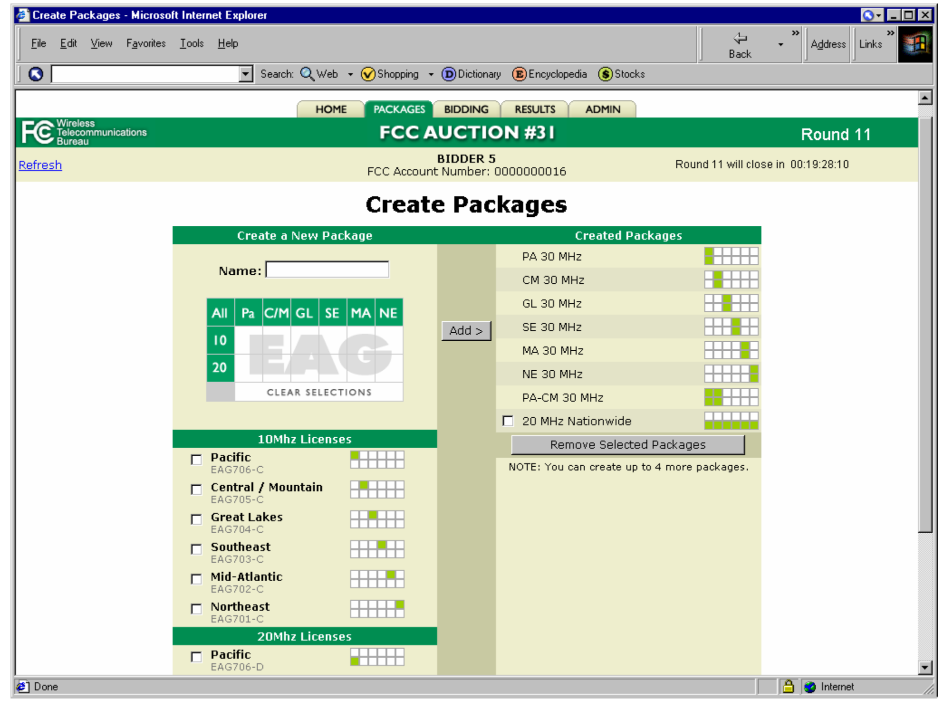
Name: 20 MHz nationwide
All Pa C/M GL SE MA NE
10
20
CLEAR SELECTIONS
Add >
Created Packages
PA 30 MHz
CM 30 MHz
GL 30 MHz
SE 30 MHz
MA 30 MHz
NE 30 MHz
PA-CM 30 MHz
Remove Selected Packages
NOTE: You can create up to 5 more packages.
10Mhz Licenses
Pacific EAG706-C
Central / Mountain EAG705-C
Great Lakes EAG704-C
Southeast EAG703-C
Mid-Atlantic EAG702-C
Northeast EAG701-C
20Mhz Licenses
Pacific EAG706-D
Internet

Create Packages - Microsoft Internet Explorer
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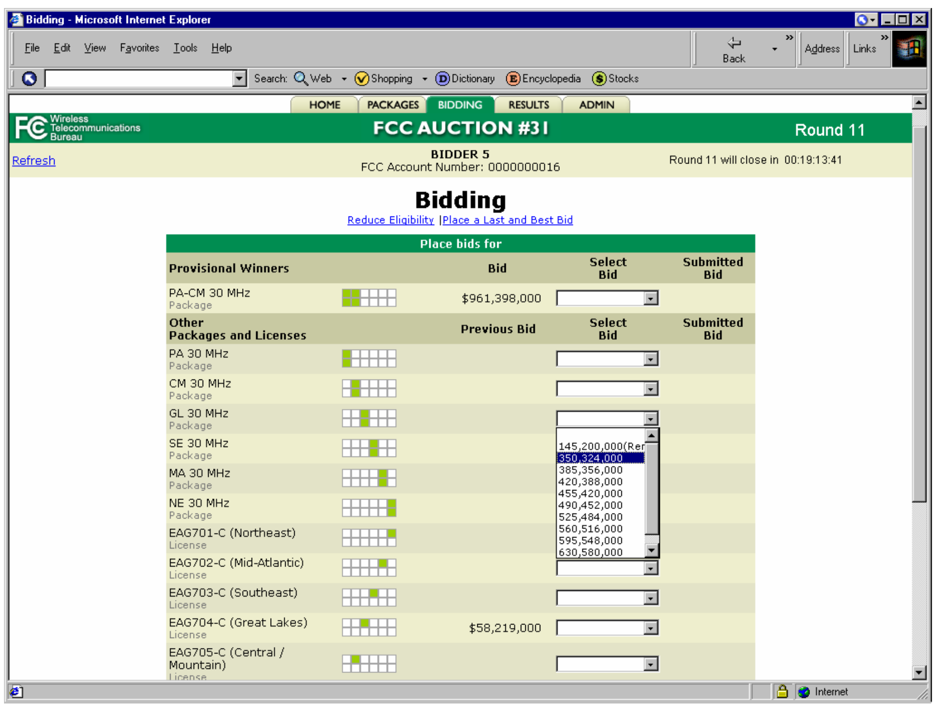
BIDDING
RESULTS
ADMIN
Wireless Telecommunications Bureau
FCC AUCTION #31
Round 11
Refresh
BIDDER 5
FCC Account Number: 0000000016
Round 11 will close in 00:19:28:10
Create Packages
Create a New Package
Name:
All Pa C/M GL SE MA NE
10
20
CLEAR SELECTIONS
Add >
Created Packages
PA 30 MHz
CM 30 MHz
GL 30 MHz
SE 30 MHz
MA 30 MHz
NE 30 MHz
PA-CM 30 MHz
20 MHz Nationwide
Remove Selected Packages
NOTE: You can create up to 4 more packages.
10Mhz Licenses
Pacific EAG706-C
Central / Mountain EAG705-C
Great Lakes EAG704-C
Southeast EAG703-C
Mid-Atlantic EAG702-C
Northeast EAG701-C
20Mhz Licenses
Pacific EAG706-D
Done
Internet

Bidding - Microsoft Internet Explorer
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RESULTS
ADMIN
Wireless Telecommunications Bureau
FCC AUCTION #31
Round 11
Refresh
BIDDER 5
FCC Account Number: 0000000016
Round 11 will close in 00:19:13:41
Bidding
Reduce Eligibility | Place a Last and Best Bid
Place bids for
Provisional Winners
Bid
Select Bid
Submitted Bid
PA-CM 30 MHz Package
$961,398,000
Other Packages and Licenses
Previous Bid
Select Bid
Submitted Bid
PA 30 MHz Package
CM 30 MHz Package
GL 30 MHz Package
SE 30 MHz Package
MA 30 MHz Package
NE 30 MHz Package
EAG701-C (Northeast) License
EAG702-C (Mid-Atlantic) License
EAG703-C (Southeast) License
EAG704-C (Great Lakes) License
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EAG705-C (Central / Mountain) License
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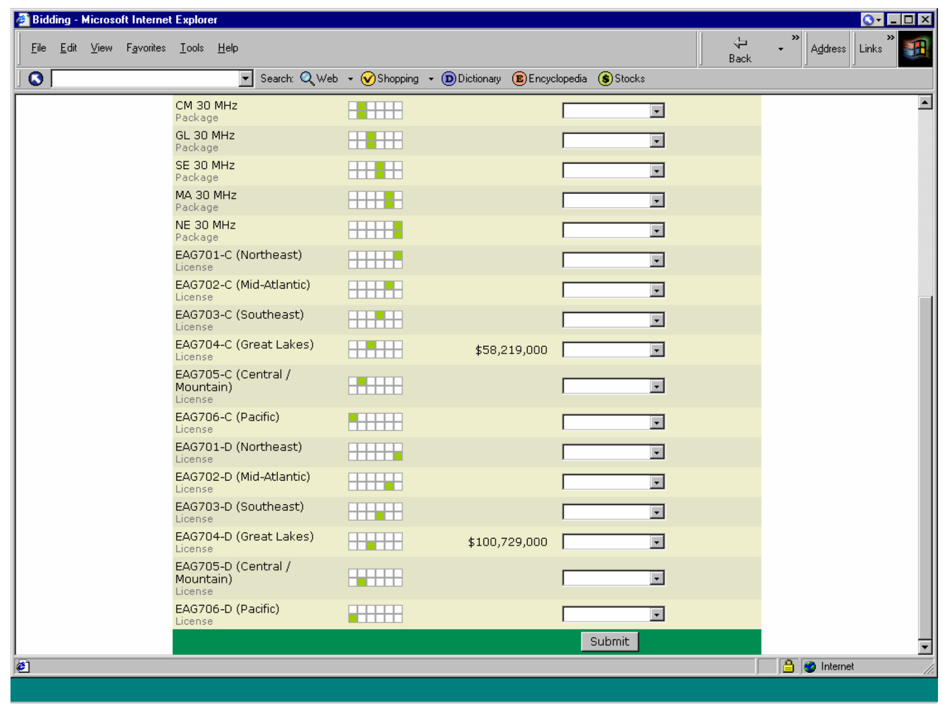
350,324,000
385,356,000
420,388,000
455,420,000
490,452,000
525,484,000
560,516,000
595,548,000
630,580,000
Internet

Bidding - Microsoft Internet Explorer
File Edit View Favorites Tools Help
Back
Address
Links
Search: Web Shopping Dictionary Encyclopedia Stocks
CM 30 MHz Package
GL 30 MHz Package
SE 30 MHz Package
MA 30 MHz Package
NE 30 MHz Package
EAG701-C (Northeast) License
EAG702-C (Mid-Atlantic) License
EAG703-C (Southeast) License
EAG704-C (Great Lakes) License
$58,219,000
EAG705-C (Central / Mountain) License
EAG706-C (Pacific) License
EAG701-D (Northeast) License
EAG702-D (Mid-Atlantic) License
EAG703-D (Southeast) License
EAG704-D (Great Lakes) License
$100,729,000
EAG705-D (Central / Mountain) License
EAG706-D (Pacific) License
Submit
Internet

Package Auction Experiments

- Variety
  - Environments
  - Rules
- Hypothesis: bidders use proxy-like strategies
  - describes much of bidder behavior in some experiments (Plott and Salmon).
  - applies with mixed success to bidders in spectrum auctions (Plott and Salmon).

FCC-Cybernomics Experiment

Complementarity Condition	None	Low	Medium	High
Efficiency				
SAA (No packages)	97%	90%	82%	79%
SAAPB ("OR" bids)	99%	96%	98%	96%
Rounds				
SAA (No packages)	8.3	10.0	10.5	9.5
SAAPB ("OR" bids)	25.9	28.0	32.5	31.8

Faulty Experiments?

- Inadequacies of package auction experiments
  - No detailed data saved about values and bids.
  - Poor measures of efficiency
  - No measures of revenue adequacy
  - No measures of problem complexity
  - No measures to characterize strategy
- Theory and evidence
  - Are outcomes not merely efficient, but also in the core?
  - How is bidder behavior be characterized?
    - How variable are the mark-ups on different packages?
    - How low are losers' lowest mark-ups?

Package Exchanges

- Definition: An exchange with multiple buyers, multiple sellers, and package bids (and possibly with some players who buy and sell different items).
- Applications:
  - Securities trading, with packages consisting of orders to buy and sell related securities.
  - Spectrum trading, in an attempt to shift the broadcast bands to higher value uses.

Exchange Design Principles?

- Assume a direct revelation form.
- Principles/constraints of the *threshold* design
  - Budget must be balanced.
  - Outcome must be *ex post* individually rational, using the reported values.
  - Outcome must maximize the *ex post* value of trade, using the reported values.
- Objective of the *threshold* design
  - Payments should minimize the maximum *ex post* gain from deviations from truth-telling, given these constraints and others' reported values.

Threshold Exchange

- Theorem. There is a unique direct mechanism determined by the threshold constraints and objective. Each bidder reports its type and the goods are allocated to maximize total value. A bidder *j* receives a "payoff" equal to max(0,Pivot Mechanism Payoff-C), where C is an amount determined to make the budget balance.
  - Parkes, Kalagnanam and Eso (2002)
  - In any mechanism satisfying the constraints, a deviator can earn at least its pivot payoff.
  - So, in this mechanism, the maximum gain from a deviation is C.

Examples

- Non-combinatorial
  - Two person exchange: buyer and seller
    - If the buyer's reported value exceeds the seller's, then trade takes place at a price equal to the mean value.
  - Dividing a partnership
    - Low value partner sells to high value partner at a price equal to the mean of the two reported values.
  - N buyers and 1 seller
    - If the seller's value is not the highest one, then the price is the mean of the two highest reported values.
- Combinatorial exchanges
  - finding a profitable deviation can be an NP-complete problem, so if C is not too large… incentives are pretty good!

Conclusion

- Package auctions are finding increasing use for hard resource allocation problems.
- Vickrey package auctions are impractical because they too often lead to low revenue, non-core outcomes.
- New designs attractively compromise incentive and distributional properties.
- Package exchanges are fundamentally hard due to empty cores, but some interesting new ideas are being studied.

The End

Notes for Homework Sessions

Order Statistic Probabilities

- If N bidders' types are independently distributed uniformly on (0,1), with order statistics $t^{(1)}$, $t^{(2)}$…, then for $y<x$,

$$\Pr\left\{t^{(1)} \le x\right\} = x^N$$

$$\Pr\left\{t^{(2)} \le s \mid t^{(1)} = x\right\} = \min\left(1,(s/x)^{N-1}\right)$$

$$\Pr\left\{t^{(1)} \le x, t^{(2)} \le y\right\} = \int_0^x \min\left(1,(y/s)^{N-1}\right) ds^N$$

$$= \left[\int_0^y ds^N + \int_y^x (y/s)^{N-1} ds^N\right] \text{ if } y < x$$

$$= y^N + Ny^{N-1}(x-y)$$