Substitute Goods, Auctions, and Equilibrium *

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May 28, 2008

*We are grateful to anonymous referees, John Quah, Kevin Roberts, as well as participants of the Nuffield College Theory Workshop for useful comments. All remaining errors are our own. Financial support for Milgrom's research was provided by the National Science Foundation under grant ITR-0427770.

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Abstract

This paper identifies *two* notions of substitutes for auction and equilibrium analysis. Weak substitutes, the usual price-theory notion, guarantees monotonicity of tâtonnement processes and convergence of clock auctions to a *pseudo-equilibrium*, but only strong substitutes, which treats each unit traded as a distinct good with its own price, guarantees that every pseudo-equilibrium is a Walrasian equilibrium, that the Vickrey outcome is in the core, and that the "law of aggregate demand" is satisfied. When goods are divisible, weak substitutes along with concavity guarantees all of the above properties, except for the law of aggregate demand.

JEL Classification: D44, C62.

Keywords: Gross substitutes, tâtonnement, Pseudo-Equilibrium, Law of Aggregate Demand.

1 Introduction

Recent years have seen the growing use of simultaneous clock auctions for substitute goods like electrical power, natural gas, and certain agricultural contracts. The first step in creating such an auction is to define categories of goods that are to be regarded as homogeneous, even if they are not perfectly uniform. These may include goods available at different but nearby times and places or with different but similar specifications. Once categories of goods are defined, the standard definition of substitutes can be applied: roughly, two categories of goods are demand-side substitutes if increasing the price of one does not reduce the demand for the other.

It can turn out, however, that changes in the inclusiveness of a product definition affects whether the goods are substitutes according to the standard definition. It is convenient to say that goods are *weak substitutes* when the substitutes condition is satisfied for a given classification of the goods and *strong substitutes* when each item is regarded as distinct, potentially having its own price. Even in ordinary-looking problems, these two notions of substitutes have very different implications for the analysis of auctions and competitive equilibrium. It can even happen that identical goods fail to be substitutes for one another according to the standard definition.

We illustrate the possibilities with some simple examples. Consider a firm producing an output whose price is normalized to one. Suppose that the firm's output $f(x, y)$ is a function of two types of discrete inputs $x \in \{0, 1\}$ and $y \in \{0, 1, 2\}$ according to

$$f(x, y) = \min(10x + 3y, 4 + y),$$

which we tabulate below:

f	$y = 0$	$y = 1$	$y = 2$
$x = 0$	0	3	6
$x = 1$	4	5	6

The firm chooses x and y to maximize $f(x, y) - rx - wy$. In this example, because f is submodular, the two inputs are substitutes, that is, when comparing any two price vectors p and p' for which the firm's optimum is unique, if $p \geq p'$ and $p_i = p'_i$, then the demand for good i is weakly higher at prices p.

Next, consider an alternative formulation in which the two units of input y are treated as distinct. Let $y = y_1 + y_2$ and suppose $y_1, y_2 \in \{0, 1\}$. In this formulation, the prices are also potentially distinct, so the firm maximizes $f(x, y_1 + y_2) - rx - w_1y_1 - w_2y_2$. It is as if we had distinguished blue and red versions of the input, where the color is devoid of any consequences for production. It is easy to check that if the input prices are $(r, w_1, w_2) = (1.5, 1.5, 1.5)$, then the firm's unique profit-maximizing input vector is $(0, 1, 1)$, but if $(r, w_1, w_2) = (1.5, 1.5, 2.5)$, then the profit-maximizing choice is $(1, 0, 0)$. This demonstrates that an increase in the price of input y_2 reduces the demand for input y_1: different units of the same type of good may fail to be substitutes.

Examples of this sort are hardly rare. For instance, an airline that is acquiring landing slots at a hub airport may wish to have some number N of slots, for illustration $N = 2$, within a particular period, say from 2:00pm to 2:15pm or from 3:00pm to 3:15pm, so that passengers traveling on the same continuing flight can be scheduled to arrive at that flight at about the same time. Landing slots in the two time periods are weak substitutes if when slots at 2-2:15 are expensive, the airline will substitute slots at 3-3:15. Slots within a given time period, however, need not be substitutes. As in our example, the airline may demand both or neither. As in our example, this can happen even with diminishing returns to additional slots in the same time period. Because clock auctions have been proposed for just this sort of application, it is important to investigate how these auctions perform in settings where slots are weak substitutes but not strong substitutes.

Despite the practical significance of the weak substitutes condition for applications like landing slots, previous studies of ascending clock auctions, which have focused exclusively on the strong substitutes condition. Ausubel [1, p. 617] mentions that his clock auction design, which applies when goods are distinct and substitutes, can also be applied when there are multiple units of each good, without inquiring about whether the meaning of the substitutes condition may change.[1] Gul and Stacchetti [5] restrict their auction design to nonidentical goods, in effect assuming strong substitutes.

One important difference between the weak and strong substitutes arises when studying the existence of market-clearing prices. Using models in which goods are priced individually, Gul and Stacchetti [5] and Milgrom [12] display monotonic auction processes that converge to exact or approximate market-clearing prices.[2] In those formulations, substitutes means strong substitutes: the equilibrium existence results do not extend to the case of weak substitutes.

To illustrate the problem using our example, suppose that good y is treated as a single class and that the available supply for the two classes of goods is given by the vector $(1, 2)$. Suppose that firm 1 has production function f as before, and that there is a second firm with production function $g(x, y) = 1_{x=1} + 4 \times 1_{y \geq 1}$ (thus, firm 2 has zero marginal value from getting a second unit of good y). At the unique efficient allocation, firm 2 uses one unit of y and firm 1 uses one unit of each good. To induce firm 2 to make its efficient choice, the price of input x must be $p_x \geq 1$, but in that event, there is no price p_y at which firm 1 demands $(1, 1)$: market clearing prices do not exist. As the example

[1] Bikhchandani and Mamer [3, p. 391] make a similar point: "In case there are multiple units of some objects, one can expand the commodity space by treating each unit of an object as a different commodity. It may be verified that market clearing prices exist in the original economy with multiple units per object if and only if market clearing prices exist in the new economy with one unit per object. Moreover, the sets of market allocations supported by equilibrium prices in the two economies (which may be empty sets) are identical, except for relabelling."

[2] These analyses are descendants of the Kelso and Crawford [7] analysis of strict core allocations in a labor market.

illustrates, such failure is related to a "hole" in firm 1's demand. Despite concavity of firm 1's underlying valuation, its demand for good y lies in $\{0, 2\}$ for $p \geq (1, 0)$, even at those prices where firm 1's demand is multi-valued. When such holes are absent, we say that firm 1's valuation satisfies the *consecutive integer* property. We will show that this property plus concavity is one of several alternatives that precisely differentiate weak and strong substitutes and imply the existence of a market clearing price.

In our example, if the supply vector is anything else besides $(1, 2)$, then not only does a market clearing price vector exist, but more is true. First, the set of market clearing price vectors is a sublattice. Second, a continuous *tâtonnement* or *clock auction* process beginning with low prices converges monotonically upward to the minimum market clearing price vector. A similar process beginning with high prices converges monotonically downward to the maximum market clearing price vector. Similar conclusions have been derived in the past using strong substitutes, but not for the weak substitutes of this example.

How does the clock auction perform when there are no market clearing prices? Suppose that firms 1 and 2 have the same valuations as above, and supply is still $(1, 2)$ so that no clearing price exists. We initially set the input price vector at some low positive prices $(\varepsilon, \varepsilon)$. At those prices there is strict excess demand for good y until p_y increases above 1. There is strict excess demand for good x until p_x reaches 1. At $p_x = 1$, firm 2 is indifferent between 0 and 1 unit of x so x is not necessarily in excess supply any more. For $p_x = 1$, firm 1 becomes indifferent between one unit of x and two units of y when p_y reaches 1.5. For $(p_x, p_y) = (1, 1.5)$, aggregate demand consists of the four bundles $(1, 1)$, $(2, 1)$, $(0, 3)$ and $(1, 3)$, which contain supply $(1, 2)$ in their convex hull. We call the corresponding price pair a *pseudo-equilibrium* price vector and argue below that such prices are significant for both equilibrium theory and auction design.[3]

Examples like the preceding one are also potentially significant for the design of *activity rules* in auctions. These are rules that typically prevent a bidder from increasing its total demand for all products) as prices increase.[4] In our example, at prices $(p_x, p_y) = (0.5, 1.5)$, firm 1 demands $(1, 0)$ while at prices $(p_x, p_y,) = (1.5, 1.5)$, firm 1 demands $(0, 2)$. Suppose these two price vectors represent successive prices in an ascending auction; the firm's total demand rises from 1 unit to 2 units. Hatfield and Milgrom [6] had shown that the strong substitutes property implies that, for a profit-maximizing firm, *the sum of the quantities of goods demanded does not increase* as prices rise, a result they called the *law of aggregate demand*. Only When that property is satisfied is it true that standard activity never interfere straightforward reporting of demands, so it is significant that the property is implied by strong substitutes but not by weak substitutes.

These examples herald general results, which are the subject of this paper. Section 2 defines weak-substitute valuations and strong-substitute valuations. Section 3 shows the

[3] See Definition 11. We will also find that the set of pseudo-equilibria coincides with the equilibrium set when the latter is non-empty. In practice, pseudo-equilibria represent the closest one can come to full market clearing. As in the example, small supply adjustments can sometimes be sufficient to convert a pseudo-equilibrium to a real equilibrium, and real sellers are sometimes willing and able to adjust supply to accomplish just that.

[4] An exception is the revealed-preference activity rule of Ausubel and Milgrom [2].

the substitutes characterizes these properties of valuations in terms of the firm's dual profit function, which provides a fruitful way to is a fruitful perspective because equilibrium, when it exists, coincides with the . Section 4 further studies the relationship between the two concepts of substitutes and other properties of demand.

Section 5 treats the implications of weak and strong substitutes for aggregate demand. We show that the strong substitutes condition is sufficient and necessary (in a quantified sense) for the robust existence of market-clearing prices and for Vickrey payoffs to be in the core. The weak substitutes condition implies that the set of pseudo-equilibrium price vectors is a non-empty sublattice and that this set coincides exactly with the set of equilibrium prices whenever any equilibrium exists.

Section 6 presents our analysis of clock auctions when bidders have weak-substitute valuations. We first introduce a continuous-time model and show that weak substitutes is necessary and sufficient for the monotonicity of a certain continuous tâtonnement-like clock auction and implies that a continuous descending or ascending clock auctions terminates at a pseudo-equilibrium. We then show how the analysis can be applied to the case in which prices follow small bid increments and bidders only need to announce one optimal bundle, rather than their entire indifference set of optimal bundles.

Section 7 analyzes the case of divisible goods and compares it the discrete case. We find that the law of aggregate demand and its variants, which were necessary in the discrete case for such important conclusions as that competitive equilibrium exists and that the Vickrey outcome is in the core, are entirely dispensable in the continuous case. Section 8 concludes.

2 Definitions

Consider an economy with K goods, in which good k is available in N_k units. Let $\mathcal{X} = \Pi_{k\in\mathcal{K}}\{0,1,\ldots,N_k\}$ and $\tilde{\mathcal{X}} = \Pi_{k\in\mathcal{K}}\{0,1\}^{N_k}$ represent the space of possible bundles of the exchange economy in its *multi-unit* and *binary* formulations. The multi-unit formulation only considers the aggregate quantity of each good, while the binary formulation treats each unit of each good as a distinct good available in binary (0 or 1) amount. To any bundle of the binary formulation, we can associate a bundle in the multi-unit formulation. This correspondence φ is defined formally as $x_k = \varphi_k(\tilde{x}) = \sum_{j=1}^{N_k} \tilde{x}_{kj}$, i.e. by counting all items in the binary vector $\tilde{x}$ corresponding to good k.

Definition 1 (Multi-Unit Valuation) *A multi-unit valuation v is a mapping from $\mathcal{X}$ into $\mathbb{R}$.*

Definition 2 (Binary Valuation) *A binary valuation $\tilde{v}$ is a mapping from $\tilde{\mathcal{X}}$ into $\mathbb{R}$.*

The binary valuation $\tilde{v}$ *corresponds to* the multi-unit valuation v, if for every $\tilde{x}$, $\tilde{v}(\tilde{x}) = v(\varphi(\tilde{x}))$. We denote by $\mathcal{V}$ the space of multi-unit valuations and $\tilde{\mathcal{V}}$ the space of *corresponding* binary valuations. Corresponding binary valuation treat all units of any given good symmetrically, and hence $\tilde{\mathcal{V}}$ is strictly included in the set of all binary valuations. Similarly, let $\mathcal{P} = \mathbb{R}_+^K$ and $\tilde{\mathcal{P}} = \Pi_{k \in \mathcal{K}} \mathbb{R}_+^{N_k}$ denote the respective price spaces of the multi-unit and binary economies. The first price space permits only linear prices for each category of goods, while the second allows nonlinear prices for each class of goods. With nonlinear prices, an agent who buys x units of some good will buy the x cheapest ones. Consequently, the marginal price faced by such agent is weakly increasing. Throughout the paper, we assume that agents have quasi-linear utilities.

ASSUMPTION 1 (QUASI-LINEARITY) The utility of an agent with multi-unit valuation v acquiring a bundle x at price p is $u(x,p) = v(x) - px$. Similarly, the utility of an agent with binary valuation $\tilde{v}$ acquiring a bundle $\tilde{x}$ at price $\tilde{p}$ is $\tilde{u}(\tilde{x},\tilde{p}) = \tilde{v}(\tilde{x}) - \tilde{p}\tilde{x}$.

Given a binary valuation $\tilde{v}$ and a price vector $\tilde{p} \in \tilde{\mathcal{P}}$, define the demand of the agent at price $\tilde{p}$ by $\tilde{D}(\tilde{p}) = \arg\max_{\tilde{x} \in \tilde{\mathcal{X}}} \{\tilde{v}(\tilde{x}) - \tilde{p}\tilde{x}\}$.

Similarly, we define the multi-unit demand D of an agent with valuation v as $D(p) = \arg\max_{x \in \mathcal{X}} \{v(x) - px\}$.

With quasi-linear preferences, there is no distinction to be made between gross and net substitutes, so we drop the modifier and make the following definitions.

DEFINITION 3 (STRONG-SUBSTITUTE VALUATION) *A multi-unit valuation v is a* strong-substitute valuation *if its binary form $\tilde{v}$ satisfies the* binary substitutes property*: for any prices $\tilde{p}$ and $\tilde{q}$ in $\tilde{\mathcal{P}}$ such that $\tilde{p} \leq \tilde{q}$, and $x \in \tilde{D}(\tilde{p})$, there exists a bundle $\tilde{x}' \in \tilde{D}(\tilde{q})$ such that $\tilde{x}'_{kj} \geq \tilde{x}_{kj}$ for all (k,j) such that $\tilde{p}_{kj} = \tilde{q}_{kj}$.*

DEFINITION 4 (WEAK-SUBSTITUTE VALUATION) *A multi-unit valuation v is a* weak-substitute valuation *if it satisfies the* multi-unit substitutes property*: for all prices p and q in $\mathcal{P}$ such that $p \leq q$ and $x \in D(p)$, there exists a bundle $x' \in D(q)$ such that $x'_k \geq x_k$ for all k in $\mathcal{K} = \{\kappa \in \mathcal{K} : p_\kappa = q_\kappa\}$.*

The strong substitutes condition is at least weakly more restrictive than the weak substitutes condition, because the latter applies only for linear prices while the former applies also for nonlinear prices. Moreover, the weak substitutes condition only compares units of distinct goods, while the strong substitutes condition requires that units of the *same* good be substitutes. Section 1 illustrates that the two conditions are not equivalent. In particular, weak-substitute valuations can violate the law of aggregate demand, but strong substitute valuations cannot. An even simpler illustration of the difference is the case of multiple units of a single good. In that case, the weak substitutes condition is vacuous, while strong substitutes imposes concavity, as shown by Theorem 9.

3 Duality Results

To any multi-unit valuation v we associate the dual profit function $\pi : \mathcal{P} \to \mathbb{R}$ such that $\pi(p) = \max_{x \in \mathcal{X}} \{u(x,p) = v(x) - px\}$. Similarly, to any binary valuation $\tilde{v}$ we associate the dual profit function $\tilde{\pi}(\tilde{p}) = \max_{\tilde{x}\in\tilde{\mathcal{X}}}\{\tilde{u}(\tilde{x},\tilde{p}) = \tilde{v}(\tilde{x}) - \tilde{p}\tilde{x}\}$.

Definition 5 (Multi-Unit Concavity) *A multi-unit valuation is concave if it can be extended to a concave function on $\mathbb{R}^K$.*

Theorem 1 *Let v be a multi-unit valuation and π be its dual profit function. Then, for all $x \in \mathcal{X}$, $v(x) \leq \min_{p\in\mathcal{P}}\{\pi(p) + px\}$. Moreover, v is concave if and only if*

$$v(x) = \min_{p\in\mathcal{P}}\{\pi(p) + px\} \text{ for all } x \in \mathcal{X}. \quad (1)$$

Proof. The first claim follows from the definition of π. The second claim is proved by applying the separating-hyperplane theorem. ■

Ausubel's and Milgrom's dual characterization of strong substitute valuations extends straightforwardly to the cases treated here.

Theorem 2 *v is a weak-substitute valuation if and only if π is submodular, and this holds if and only if the dual profit function $\tilde{\pi}$ of the corresponding binary form $\tilde{v} = \phi(v)$ is submodular on the restricted domain where goods of the same type have equal prices. In addition, v is a strong-substitute valuation if and only if the dual profit function $\tilde{\pi}$ of its binary form $\tilde{v} = \phi(v)$ is submodular.*

Proof. The proofs of the two statements follow the proof of Theorem 10 in Ausubel and Milgrom [2]. ■

The preceding theorem relies on the idea that one can characterize weak substitutes by focusing on the subset $\mathcal{P}_L$ of the price space in the binary formulation $\tilde{\mathcal{P}}$ in which goods of the same type have the same price. This subset is isomorphic to the set $\mathcal{P}$ of linear prices used in the multi-unit economy. The weak-substitute property then corresponds to the requirement that the dual profit function is submodular on $\mathcal{P}_L$, while the strong-substitute property requires submodularity on the whole price space. An immediate consequence of this alternative formulation is the following:

Theorem 3 *Any strong-substitute valuation is also a weak-substitute valuation.*

The converse is not true. For example, suppose there is only one type of good, so that every valuation v is a weak-substitute valuation. Let $v(0) = 0$, $v(1) = 1$ and $v(2) = 3$ and suppose prices are $(p_1, p_2) = (1.4, 1.4)$, at which both units are demanded. Increasing p_1 to 1.7 would reduce demand to 0, thus violating the strong-substitute property. The same example establishes that a multi-unit valuation can be submodular even when the related binary valuation is not.

We have seen than weak-substitute valuations need not be submodular. The following result shows that adding the requirement that v is concave does yield submodularity.

Theorem 4 *Any concave weak-substitute valuation is submodular.*

Proof. From Theorem 1 $v(x) = \min_{p \in \mathcal{P}}\{\pi(p) + px\} = \max_p\{-\pi(p) - px\}$. From Theorem 2, π is submodular. Therefore, v is the maximum over p of a function that is supermodular in p and $-x$, which implies that v is supermodular in $-x$ or, equivalently, submodular in x. ■

THEOREM 5 *Let $\tilde{v}$ be a strong-substitute valuation. Then* $\tilde{v}(\tilde{x}) = \min_{\tilde{p} \in \tilde{\mathcal{P}}}\{\tilde{\pi}(\tilde{p}) + \tilde{p}\tilde{x}\}$.

Proof. Given $\tilde{x}$, define $\tilde{p}$ as $\tilde{p}_a = 0$ if $\tilde{x}_a = 1$ and $\tilde{p}_a = \infty$ if $\tilde{x}_a = 0$. Clearly, $\tilde{x} \in \tilde{D}(\tilde{p})$. The rest of the proof is identical to the proof of Theorem 1. ■

Underlying Theorem 4 is the fact that concavity allows v to be expressed by formula (1). As Theorem 5 shows, concavity is not required in the binary form to obtain that equation. Combining this observation with the same argument as in the proof of Theorem 4, provides a direct explanation of why strong substitutes valuations are submodular.

4 Relations between Concepts of Substitutes

Gul and Stacchetti [4] introduced the *single-improvement property* for binary valuations, which requires that if some vector x is not demanded at price vector p, then there is a vector y that is strictly preferred to x and entails increasing the demand for at most one good and decreasing the demand for at most one other good, as follows.

DEFINITION 6 (BINARY SINGLE-IMPROVEMENT PROPERTY) *A binary valuation $\tilde{v}$ satisfies the single-improvement property if for any price vector $\tilde{p}$ and $\tilde{x} \notin \tilde{D}(\tilde{p})$, there exists $\tilde{y}$ such that $u(\tilde{y}, \tilde{p}) > u(\tilde{x}, \tilde{p})$, $\|(\tilde{y} - \tilde{x})_+\|_1 \leq 1$, and $\|(\tilde{x} - \tilde{y})_+\|_1 \leq 1$.*

Gul and Stacchetti also showed that this single-improvement property is equivalent to the strong substitutes property:

THEOREM 6 (GUL AND STACCHETTI) *A monotonic valuation is a strong-substitute valuation if and only if it satisfies the binary single-improvement property.*

We now extend these results to multi-unit economies.

DEFINITION 7 (MULTI-UNIT SINGLE-IMPROVEMENT PROPERTY) *A valuation v satisfies the multi-unit single-improvement property if for any p and $x \notin D(p)$, there exists x' such that $u(x', p) > u(x, p)$, $\|(x' - x)_+\|_1 \leq 1$, and*[5] $\|(x - x')_+\|_1 \leq 1$.

The only difference in the definitions of binary and multi-unit single-improvement properties resides in the price domain where the property has to hold.

Throughout the paper, we will denote by e_k the vector of $\mathbb{R}^K$ whose k^{th} component equals one and whose other components equal zero.

[5] Here the norm is defined on $\mathbb{R}^K$, whereas it was defined on $\mathbb{R}^{\sum_k N_k}$ in the binary setting.

THEOREM 7 *If v satisfies the multi-unit single-improvement property then it is a weak-substitute valuation.*

Proof. Suppose by contradiction that the weak-substitute property is violated: there exist p, k, a small positive constant ε, and a bundle x such that $x \in D(p)$ and for all $y \in D(p + \varepsilon e_k)$, there exists $j \neq k$ such that $y_j < x_j$. Set $\hat{p} = p + \varepsilon e_k$. We have $x \notin D(\hat{p})$ and $y_k < x_k$ for all $y \in D(\hat{p})$ (since D(p) clearly contains bundles with strictly less than x_k units of good k). Therefore x is only dominated by bundles y that have strictly less units of at least two goods, implying that $\|(x - y)_+\|_1 \geq 2$, which violates the single-improvement property. ■

The converse in not true. In the first example of Section 1, the valuation is submodular in a two-good economy, thus satisfies the weak substitutes property. However, for $r = 0.2$ and $w = 0.3$, the bundle $(1, 0)$ is only dominated by the bundle $(0, 2)$, which violates the single-improvement property. Let $\wedge$ and $\vee$ respectively denote the "meet" and "join" operators for the lattice structure induced by the usual order on $\mathcal{X}$.

DEFINITION 8 (MULTI-UNIT SUBMODULARITY) *A multi-unit valuation v is submodular if for any vectors x and x' of $\mathcal{X}$, $v(x) + v(x') \geq v(x \wedge x') + v(x \vee x')$.*

The next theorem contains a key result for the existence of Walrasian equilibria in multi-unit economies. The proof uses Gul and Stacchetti's characterization theorem (Theorem 6) and thus requires monotonicity of v. Throughout the rest of the paper, we assume that v is nondecreasing.

ASSUMPTION 2 *Agent valuations are nondecreasing.*

THEOREM 8 *If v is a strong-substitute valuation, then any bundle x is optimal at some linear price.*

Proof.

Let x be any bundle, and $\tilde{x}$ be a binary representation of this bundle. From Theorem 5, we have

$$v(x) = \tilde{v}(\tilde{x}) = \min_{\tilde{p}} \{\tilde{\pi}(\tilde{p}) + \tilde{p}\tilde{x}\}. \tag{2}$$

Since v is a strong substitutes valuation, $\tilde{\pi}$ is submodular, so the objective in (2) is submodular. By a theorem of Topkis [18], the set M of minimizers of a submodular function is a sublattice and, since the objective is continuous, the sublattice is closed. Therefore, it has a largest element $\tilde{p}$. We will first show that for any good k consumed in positive amount (i.e. $x_k \geq 1$), $\tilde{p}_{ki} = \tilde{p}_{kj}$ whenever $\tilde{x}_{ki} = \tilde{x}_{kj} = 1$. That is, the restriction to good k of $\tilde{p}$ must be linear for the x_k cheapest units of that good. We will then show that this price linearity can be extended to the entire supply of good k without affecting optimality of the bundle.

To show the first claim, suppose by contradiction that $\tilde{p}_{ki} \neq \tilde{p}_{kj}$ for some units i, j of some good k such that $\tilde{x}_{ki} = \tilde{x}_{kj} = 1$. Then the price vector $\tilde{p}'$ equal to $\tilde{p}$ except for units

i and j of good k, where $\tilde{p}_{ki}$ and $\tilde{p}_{kj}$ are swapped, is also a minimizer of (2). Therefore $\tilde{p} \vee \tilde{p}' > \tilde{p}$ is also in M, which contradicts maximality of $\tilde{p}$. We have thus shown that $\tilde{p}$ is linear on the "support" of $\tilde{x}$: for each good k there exists a price p_k such that $\tilde{p}_{ki} = p_k$ for all i such that $\tilde{x}_{ki} = 1$. Obviously, $\tilde{p}_{kl} = +\infty$ whenever $\tilde{x}_{kl} = 0$, so the $\tilde{p}$ is not linear on the whole supply. To prove the theorem, therefore, we need to show that price linearity can be extended to those units of good k that are not consumed.

We now prove that x is optimal for the linear price vector $p = (p_k)_{k\in\mathcal{K}}$, where $p_k = +\infty$ when $x_k = 0$, $p_k = 0$ when $x_k = N_k$, and p_k is defined as above when $1 \leq x_k \leq N_k - 1$. That is, we can impose $\tilde{p}_{kl} = p_k$ for all units, including those for which $\tilde{x}_{kl} = 0$, and preserve optimality of x.

To show this, we first observe that for any good k such that $x_k \in \{1, N_k - 1\}$, the firm must be indifferent, at $\tilde{p}$, between x and some bundle y^k such that $y^k_k < x_k$, otherwise it would be possible to increase p_k, which would contradict maximality of $\tilde{p}$. We can choose y^k so that it is optimal if we slightly increase the price of some particular unit of good k. Since $\tilde{v}$ is a strong substitute valuation, we can choose y such that $y^k_k = x_k - 1$, and $y^k_j \geq x_j$ for all j. Since $\tilde{p}_{kl} = +\infty$ outside of the support of $\tilde{x}$, we necessarily have $y^k_j = x_j$ for $j \neq k$. This shows that $y^k = x - e_k$. Such indifference bundles exist for all goods k such that $1 \leq x_k \leq N_k - 1$.

Now, for all goods such that $x_k \in [1, N_k - 1]$, reset all unit prices outside the support of $\tilde{x}$ from $+\infty$ to p_k. This change does not affect optimality of x among bundles z such that $z \leq x$, and it does not affect indifference between x and the bundles y^k. For any good k, consider the bundle $z^k = x + e_k$. Since $\tilde{v}$ is submodular, Theorem 11 implies that v is component-wise concave (see p. 13). Therefore, $v(z^k) - v(x) \leq v(x) - v(y^k) = p_k$, which implies that z^k is weakly dominated by x. Now for two goods $k \neq j$ such that $x_k \geq 1$ and $x_j < N_j$, consider the bundle $z^{kj} = x - e_k + e_j$. We claim that z is also weakly dominated by x. To see this, we use the following Lemma, whose proof is in the Appendix.[6]

LEMMA 1 *If v is a strong-substitute valuation, k and j are two goods and x is a bundle such that $x_k \leq N_k - 1$ and $x_j \leq N_j - 2$, then $v(x+e_k+e_j) - v(x+e_k) \geq v(x+2e_j) - v(x+e_j)$.*

Applying Lemma 1 to the bundle $x - e^j - e_k$ yields $v(x) - v(y^j) \geq v(z^{kj}) - v(y^k)$, which implies, along with $v(x) = v(y^j) + p_j = v(y^k) + p_k$, that $v(x) - p_k \geq v(z^{kj}) - p_j$. Thus, x weakly dominates z. This shows that $\tilde{x}$ has no single improvement. From Theorem 6, $\tilde{v}$ satisfies the single-improvement property. Therefore, $\tilde{x}$ must be optimal at the linear price $\tilde{p}$ such that $\tilde{p}_{kl} = p_k$ for all $l \in \{1, \ldots, N_k\}$. Equivalently, the bundle x is optimal at price $p = (p_k)$, which concludes the proof. ■

We can now state the properties of strong-substitute valuations in linear-pricing economies.

THEOREM 9 *Suppose that v is a strong-substitute valuation. Then it satisfies the following properties:*

[Concavity] *v is concave.*

[6] As can be easily checked, the proof of Lemma 1 is independent of the proof of the present theorem.

[Weak-Substitute Property.] *For any* $p \in \mathcal{P}$, $k \in \mathcal{K}$, $\varepsilon > 0$, *and* $x \in D(p)$, *there exists* $x' \in D(p + \varepsilon e_k)$ *such that* $x'_j \geq x_j$ *for all* $j \neq k$.

[Law of Aggregate Demand.] *For any* $p \in \mathcal{P}$, $k \in \mathcal{K}$, $\varepsilon > 0$, *and* $x \in D(p)$, *there exists* $x' \in D(p + \varepsilon e_k)$ *such that* $\|x'\|_1 \leq \|x\|_1$.

[Consecutive-Integer Property.] *For any* $p \in \mathcal{P}$ *and* $k \in \mathcal{K}$, *the set* $D_k(p) = \{z_k : z \in D(p)\}$ *consists of consecutive integers.*

Proof. Theorem 3 implies that v satisfies the weak-substitute property, and Hatfield and Milgrom [6] show that v must satisfy the law of aggregate demand. Therefore, it remains to show that v is concave and satisfies the consecutive-integer property.

We first show that v is concave. Theorem 8 implies that for any x there exists p such that $\pi(p) = v(x) - px$, where π is the dual profit function defined in Section 3. From the first part of Theorem 1, $v(x) \leq \min_p \pi(p) + px$. Combining the two equations above yields $v(x) = \min_p \pi(p) + px$ for all x. Applying the second part of Theorem 1 then proves that v is concave.[7]

Last, we show the consecutive-integer property. Suppose by contradiction that there exist p, k, and two bundles x and y in $D(p)$ such that $x_k \geq y_k + 2$ and $z \in D(p) \Rightarrow z_k \notin (y_k, x_k)$. Consider the binary price vector $\tilde{p}$ that is linear and equal to p_j for all good $j \neq k$, and that equals p_k for the first x_k units of good k and $+\infty$ for the remaining units of good k. Clearly, there exist binary forms $\tilde{x}$ and $\tilde{y}$ of x and y that belong to $\tilde{D}(\tilde{p})$, and there is no bundle $\tilde{z}$ in $\tilde{D}(\tilde{p})$ such that $z_k \in (y_k, x_k)$. If the price of one unit of good k is slightly increased, the demand for good k thus falls directly below z_k, implying that the demand of another unit of good k, whose price had not increased, has strictly decreased, which violates the strong-substitute property for $\tilde{v}$. ■

The consecutive-integer property is not implied by concavity of v. For example, in a (multi-unit) two-good economy, concavity is compatible with the demand set $D(p) = \{(1,0),(0,2)\}$. However, this demand set violates the consecutive-integer property: the set $D_2(p) = \{0,2\}$ does not consist of consecutive integers. The consecutive-integer property rules out valuations causing a sudden decrease in the consumption of a good (independently of the consumption of other goods). For example, there are no prices at which the firm is indifferent between bundles containing, say, 5 and 10 units of a good, but strictly prefers these bundles to any bundle containing between 6 and 9 units of that good. In that sense, there are no "holes" in the demand set with respect to any good. In terms of demand, the property implies a progressive reaction to price movements: as the price of a good increases, the optimal demand of that good decreases unit by unit. By contrast, concavity is not required for the law of aggregate demand.

THEOREM 10 *If* v *is a weak-substitute valuation that satisfies the consecutive-integer property, then it satisfies the law of aggregate demand.*

Proof. See the Appendix.

[7]As can be easily verified, the proof of Theorem 1 is independent of the present proof.

The weak-substitute property and the law of aggregate demand do not imply the consecutive-integer property. For example, in an economy with one good available in two units, consider the non-concave valuation $v(0) = 0$, $v(1) = 1$, and $v(2) = 4$. v is trivially a substitutes valuation, and satisfies the law of aggregate demand. However, at price $p = 2$, the demand set is $\{0, 2\}$, which violates the consecutive-integer property. This is also an example of a weak-substitute valuation that is not concave.

To obtain sharp results, we consider the concept of component-wise concavity, which is weaker than concavity and entails diminishing marginal returns in each component separately.

Definition 9 (Component-wise Concavity) *A multi-unit valuation v is component-wise concave if for all x and k, $v(x_k + 1, x_{-k}) - v(x) \geq v(x_k + 2, x_{-k}) - v(x_k + 1, x_{-k})$.*

Theorem 11 *A multi-unit valuation v is submodular and component-wise concave if and only if its binary form $\tilde{v} = \phi(v)$ is submodular.*

Proof. By a theorem of Topkis [18], it is sufficient to consider binary bundles x and y that differ in just two components. If the two components represent the same good, then submodularity of the binary form is the same as component-wise concavity. If the two components represent different goods, then submodularity of the binary form is implied by submodularity of the multi-unit form (and conversely). ■

The last three properties listed in Theorem 9 describe the demands corresponding to a strong-substitute valuation in linear-pricing economies. Even though strong-substitute valuations are defined by their demands in response to nonlinear prices, the identified properties turn out to be sufficient to characterize strong substitutes. That is the essential content of Theorem 12 below.

Before proving this theorem, we state a new "minimax" result, in which one of the choice set is a lattice and the other choice set consists of nonlinear prices. The proof of this result is in the Appendix.

If x is a multi-unit bundle and $\tilde{p}$ is a nonlinear price vector, let $(\tilde{p}, x)$ denote the cost of acquiring bundle x under $\tilde{p}$. That is,

$$(\tilde{p}, x) = \sum_{k \in \mathcal{K}} \sum_{i=1}^{x_k} \tilde{p}_{k(i)},$$

where $\tilde{p}_{k(i)}$ is the price of the i^{th} cheapest unit of good k.

Proposition 1 (Minimax) *Suppose that v is a concave weak-substitute valuation satisfying the consecutive-integer property, and let $\tilde{p}$ be a nonlinear price vector. Then,*

$$\max_x \min_p \{\pi(p) + px - (\tilde{p}, x)\} = \min_p \max_x \{\pi(p) + px - (\tilde{p}, x)\}$$

Theorem 12 *Let v be a multi-unit valuation. The following properties are equivalent.*

(i) v is a strong-substitute valuation.

(ii) v is a concave weak-substitute valuation, and satisfies the consecutive-integer property.

Proof. We know from Theorem 9 that (i) implies (ii). We now show that (ii) implies (i). From Theorem 2, it is enough to show that $\tilde{\pi}$ is submodular. Consider any nonlinear price vector $\tilde{p}$. We have

$$\tilde{\pi}(\tilde{p}) = \max_{\tilde{x}}\{\tilde{v}(\tilde{x}) - \tilde{p}\tilde{x}\} = \max_{x}\{v(x) - (\tilde{p}, x)\}.$$

Since v is concave, Theorem 1 implies that

$$\tilde{\pi}(\tilde{p}) = \max_{x}\{\min_{p}\{\pi(p) + px\} - (\tilde{p}, x)\} = \max_{x}\{\min_{p}\{\pi(p) + px - (\tilde{p}, x)\}\}.$$

From Proposition 1, the max and min operators can be swapped:

$$\tilde{\pi}(\tilde{p}) = \min_{p}\{\max_{x}\{\pi(p) + px - (\tilde{p}, x)\}\} = \min_{p}\{\pi(p) + \max_{x}\{px - (\tilde{p}, x)\}\}.$$

As can be easily verified, the inner maximum equals

$$\sum_{k\in\mathcal{K}}\sum_{i=1}^{N_k}(p_k - \tilde{p}_{ki})_+.$$

Therefore,

$$\tilde{\pi}(\tilde{p}) = \min_{p}\left\{\pi(p) + \sum_{k\in\mathcal{K}}\sum_{i=1}^{N_k}(p_k - \tilde{p}_{ki})_+\right\}.$$

Since v is a weak-substitute valuation, π is submodular by Theorem 2. Moreover, the function $(x, y) \to (x - y)_+$ is submodular as a convex function of the difference $x - y$. Therefore, $\tilde{\pi}(\tilde{p})$ is the minimum over p of an objective function that is submodular in p and $\tilde{p}$, which shows that it is submodular in $\tilde{p}$.[8] ■

It turns out that, *given* concavity and the weak-substitute property, the law of aggregate demand is equivalent to the consecutive integer property. Some of the main results above are combined and extended in the following theorem.

Theorem 13 (Equivalence of Substitute Concepts) *Let v be a multi-unit valuation. The following statements are equivalent.*

(i) v satisfies the binary single-improvement property.

(ii) v is a strong-substitute valuation.

(iii) v is a concave weak-substitute valuation and satisfies the consecutive-integer property.

[8] See Topkis [17].

(iv) v is a concave weak-substitute valuation and satisfies the law of aggregate demand.

(v) v is concave and satisfies the multi-unit single-improvement property.

Proof. $(i) \Leftrightarrow (ii)$ is Gul and Stacchetti's theorem (see Theorem 6). $(ii) \Leftrightarrow (iii)$ is a restatement of Theorem 12. Theorem 10 shows that (iii) implies (iv). For the converse, the weak-substitute property implies[9] for all p that any edge E of $D(p)$ has direction e_i or $e_i - \alpha e_j$ for some goods i, j. In the first case, concavity implies that all integral bundles on the edge belong to the demand. In the second case, $\alpha = 1$. Otherwise, slightly modifying the price would reduce demand to that edge, and increasing p_i if $\alpha > 1$ or p_j if $\alpha < 1$ would violate the law of aggregate demand. This, along with concavity, implies that the consecutive-integer property holds along all edges, and thus for $D(p)$. $(i)-(iv)$ implies (v): (i) clearly implies the multi-unit single-improvement property, and (iii) implies concavity. We conclude by showing that (v) implies (iii). We already know from Theorem 7 that if v satisfies (v), then it is a weak-substitute valuation. Therefore, there only remains to show that v satisfies the consecutive-integer property. Suppose it doesn't. There exists a price vector p, a good k, and a unit number d such that $D_k = \{z_k : z \in D(p)\}$ is split by d: the sets $D_k^- = D_k \cap [0, d-1]$ and $D_k^+ = D_k \cap [d+1, N_k]$ are disjoint and cover D_k. Now slightly increase p_k. The new demand set D' is such that $D'_k \subset D_k^-$. Pick any bundle y that is optimal under the new price within the set $\{x \in \mathcal{X} : x_k \geq d\}$. Then $y_k > d$, because p_k has only been slightly increased and any bundle with d units of good k was strictly dominated by D_k^+. At the new price, y is dominated but cannot be strictly improved upon with reducing the amount of good k by at least two units, which violates the single-improvement property. ∎

The multi-unit single-improvement property alone is not equivalent to strong substitutes. For example, in an economy with two goods available in two units, consider the valuation v defined by $v(x) = \|x\|_1 - .1r(x)$, where $r(x)$ equals 1 if x contains exactly one unit of each good, and 0 otherwise. The valuation is not concave, and therefore cannot be a strong-substitute valuation. However, one can easily verify that v satisfies the multi-unit single-improvement property.

5 Aggregate Demand and Equilibrium Analysis

Our first theorem relates the set of equilibrium prices for the weak substitutes case to the solution of a certain dual minimization problem. For any (multi-unit) bundle x, let $\mathcal{P}(x)$ denote the set of price vectors such that $x \in D(p)$.

THEOREM 14 *If v is a weak-substitute valuation, then for all x, $\mathcal{P}(x)$ is either the empty set or the complete sublattice of $\mathcal{P}$ given by $\mathcal{P}(x) = \arg\min\{\pi(p) + px\}$.*

Proof. Fix $x \in \mathcal{X}$. From Theorem 5, $v(x) \leq \min_p\{\pi(p) + px\}$. Suppose that the inequality is strict. Then $v(x) - px < \pi(p)$ for all p, so $\mathcal{P}(x)$ is the empty set. Now suppose that

[9] See the proof of Proposition 1.

$v(x) = \min_p\{\pi(p) + px\}$. Then, for all $p \in \arg\min\{\pi(p) + px\}$, $v(x) - px = \pi(p)$, so $x \in D(p)$. Conversely, if $x \in D(\bar{p})$ for some price $\bar{p}$, then $\arg\min\{\pi(p) + px\} = v(x) = \pi(\bar{p}) + \bar{p}x$. Therefore, $\mathcal{P}(x) = \arg\min\{\pi(p) + px\}$. From Theorem 2, $\pi(p)$ is submodular. Therefore $\mathcal{P}(x)$ is the set of minimizers of a submodular function over a sublattice $\mathcal{P}$; hence, it is a sublattice of $\mathcal{P}$. Completeness is obtained by a standard limit argument. ■

In the binary formulation, all bundles can be achieved through nonlinear pricing, by setting some unit prices to zero and others to infinity. Therefore, Theorem 14 takes a simpler form. For any binary bundle $\tilde{x}$, let $\tilde{\mathcal{P}}(\tilde{x})$ denote the set of price vectors such that $\tilde{x} \in \tilde{D}(\tilde{p})$.

THEOREM 15 *If $\tilde{v}$ is a binary valuation satisfying the strong substitutes, then $\tilde{\mathcal{P}}(\tilde{x})$ is a complete, non-empty lattice for all $\tilde{x} \in \tilde{\mathcal{X}}$.*

Proof. For any bundle $\tilde{x}$, there exists a price $\tilde{p}$ such that $\tilde{x} \in \tilde{D}(\tilde{p})$. Therefore, $\tilde{\mathcal{P}}(\tilde{x})$ is nonempty. The rest of the proof is similar to the proof of Theorem 14. ■

Existing results by Gul and Stacchetti and by Milgrom assert *necessary* conditions for the existence of Walrasian equilibrium in the binary formulation. These results assume that individual valuations are drawn from a set that includes all unit-demand valuations (Gul and Stacchetti), which are defined next, or all additive valuations (Milgrom).[10] They establish that if the set of valuations includes any that are not strong substitutes, then there is a profile of valuations to be drawn from the set such that no competitive equilibrium exists.

These results are inapplicable in our multi-unit context, because they allow preferences to vary among identical items and the constructions used in those papers hinge on that freedom. The next theorem extends the earlier results by showing that the conclusions remain true when one imposes the restriction that units of the same good are interchangeable.

DEFINITION 10 *A unit-demand valuation is such that for all price p and $x \in D(p)$, $\|x\|_1 \leq 1$.*

Let $N = \sum_k N_k$ denote the total number of units in the economy.

THEOREM 16 *Consider a multi-unit endowment $\mathcal{X}$ and a firm having a concave, weak-substitute valuation v_1 on $\mathcal{X}$ that is not a strong-substitute valuation. Then there exist I firms, $I \leq N$, with unit-demand valuations $\{v_i\}_{i \in I}$, such that the economy $E = (\mathcal{X}, v_1, \ldots, v_{I+1})$ has no Walrasian equilibrium.*

Proof. See the Appendix.

Since preferences are assumed to be quasi-linear, one can conveniently analyze equilibrium prices and allocations in terms of the solutions to certain optimization problems.

[10] An *additive* valuation is a valuation with the property that the value of any set is equal to the sum of the values of the singletons in the set.

With that objective in mind, consider an economy consisting of n firms with valuations $\{v_i\}_{1\leq i\leq n}$. The valuations v_i are defined for $\{x \in \mathbb{N}^K : x_k \leq N_k \ \forall k \in \mathcal{K}\}$. It is convenient to extend the domain of v_i by setting $v(x) = v(x \wedge (N_1, \ldots, N_K))$ for all x in $\mathbb{N}^K$, where $\wedge$ denotes the "meet" operator for the lattice structure induced by the usual order. We now define the market-valuation v of the economy by

$$v(x) = \max\left\{\sum v_i(x_i) : \sum x_i = x \text{ and } x_i \in \mathbb{N}^K\right\}.$$

and the market dual profit function of the economy by $\pi(p) = \max_{x\in\mathbb{N}^K}\{v(x) - px\}$. The function π is convex, as can be checked easily.

THEOREM 17 *For all* $p \in \mathcal{P}$, $\pi(p) = \sum_{1\leq i\leq n} \pi_i(p)$.

Proof.

$$\begin{aligned}\pi(p) &= \max_x \left\{\max\{\textstyle\sum_i v_i(x_i) : \sum_i x_i = x\} - px\right\}\\ &= \max_{x_1,\ldots,x_n} \textstyle\sum_i \{v_i(x_i) - px_i\}\\ &= \textstyle\sum_i \pi_i(x_i),\end{aligned}$$

which concludes the proof. ■

Theorem 17 cannot be extended to nonlinear prices. To see this we observe, for example, that the cheapest unit of a given good can only be allocated to a single firm when computing the market dual profit function, whereas it is included in all individual dual profit functions involving at least one unit of this good. It is thus easy to construct examples where the market dual profit function is strictly lower than the sum of individual dual profit functions, the latter allowing each firm to use the cheapest units.

COROLLARY 1 *If all firms have weak-substitute valuations, then the market valuation v is also a weak-substitute valuation.*

Proof. If individual firms have weak substitute valuations, Theorem 2 implies that individual profit functions are submodular. By Theorem 17, the market dual profit function is therefore a sum of submodular functions, and so itself submodular. Theorem 2 then allows us to conclude that v is a substitute valuation. ■

DEFINITION 11 *A price vector p is a* pseudo-equilibrium price *of the economy with endowment $\bar{x}$ if $p \in \arg\min\{\pi(p) + p\bar{x}\}$.*

Section 6 uses the following characterization of pseudo-equilibrium prices.

PROPOSITION 2 *p is a pseudo-equilibrium price if and only if the supply vector $\bar{x}$ is in the convex hull $Co(D(p))$ of $D(p)$.*

Proof. By definition p minimizes the convex function $f : p \rightarrow \pi(p) + p\bar{x}$. Therefore, 0 is in the subdifferential of f at p.[11] That is, $0 \in \partial\pi(p) + \bar{x}$. The extreme points of

[11] See for example Rockafellar [15].

$-\partial\pi(p)$ are bundles that are demanded at price p. Moreover, $-D(p) \subset \partial\pi(p)$. Therefore $-Co(D(p)) = \partial\pi(p)$. Combining these results yields $\bar{x} \in Co(D(p))$. ■

Let $\mathcal{P}(\bar{x})$ denote the set of pseudo-equilibrium prices.

PROPOSITION 3 *If all firms have weak-substitute valuations, then $\mathcal{P}(\bar{x})$ is a complete sublattice of $\mathcal{P}$.*

Proof. Individual weak-substitute valuations imply that π_i is submodular for all i by Theorem 2. Therefore, π is submodular. The proof is then identical to the proof of Theorem 14. ■

THEOREM 18 *The economy with endowment $\bar{x}$ has a Walrasian equilibrium if and only if $v(\bar{x}) = \min_p\{\pi(p) + p\bar{x}\}$. Moreover, if the economy with endowment $\bar{x}$ has a Walrasian equilibrium, then the set of Walrasian equilibrium prices is exactly the set $P(\bar{x})$ of pseudo-equilibrium prices.*

Proof. Theorem 1 implies that $v(\bar{x}) \leq \min_p\{\pi(x)+p\bar{x}\}$. Suppose that $v(\bar{x}) = \pi(p)+px$ for some p. Let $\bar{x}_i$ denote the bundle received by firm i for some fixed allocation maximizing the objective in the definition of v. For all i, $v_i(\bar{x}_i) - p\bar{x}_i \leq \pi_i(p)$. Summing these inequalities yields $v(\bar{x}) \leq \pi(p) - p\bar{x}$. By assumption, the last inequality holds as an equality, which can only occur if $v_i(\bar{x}_i) - p\bar{x}_i = \pi(p)$ for all i, implying that $(p, \bar{x}_1, ..., \bar{x}_n)$ is a Walrasian equilibrium. To prove the second claim, suppose that $(\{\bar{x}_i\}_{1\leq i\leq n}, p)$ is a Walrasian equilibrium. Then, $v_i(\bar{x}_i) = \pi_i(p) + p\bar{x}_i$ for all i. Summing these equalities yields $v(\bar{x}) = \pi(p) + p\bar{x}$, which implies that $v(\bar{x}) = \min_p\{\pi(p) + p\bar{x}\}$ (since the minimum is always above $v(\bar{x})$). It is clear from the first part of the proof that if the economy has a Walrasian equilibrium, the set of Walrasian prices is exactly the set of pseudo-equilibrium prices. ■

Theorem 18 shows that whenever a Walrasian equilibrium exists, the concepts of pseudo-equilibrium and equilibrium coincide. In binary economies, where nonlinear pricing is available, the question of the existence of a Walrasian equilibrium have been solved by Gul and Stacchetti [4] and Milgrom [12], who both show that equilibrium exists in the binary formulation when goods are strong substitutes and establish the two partial converses described above.

For the multi-unit formulation, we have already established the partial converse in Theorem 16. We now consider the other direction: we prove that strong substitutes implies the existence of a Walrasian equilibrium with *linear* pricing. This result is then used to prove the stronger theorem that strong-substitute valuations are closed under aggregation: if all valuations satisfy strong-substitutes, then so does the market valuation.

THEOREM 19 (LINEAR-PRICING WALRASIAN EQUILIBRIUM) *In a multi-unit exchange economy with individual strong-substitute valuations, there exists a Walrasian equilibrium with linear prices.*

Proof. Considering the binary form of the economy, Gul and Stacchetti [4, Corollary 1] have shown that the set of (nonlinear pricing) Walrasian equilibria is a complete lattice. In particular, it has smallest and largest elements. We now prove that these two elements consist of linear prices, which proves the result. Suppose by contradiction that the largest element $\tilde{p}$ is such that $\tilde{p}_{ki} \neq \tilde{p}_{kj}$ for some units i, j of some good k. Then the price vector $\tilde{p}'$ equal to $\tilde{p}$ except for units i and j of good k, where $\tilde{p}_{ki}$ and $\tilde{p}_{kj}$ are swapped, is also a Walrasian equilibrium. Therefore $\tilde{p} \vee \tilde{p}' > \tilde{p}$ is also a Walrasian equilibrium, which contradicts maximality of $\tilde{p}$. Linearity of the smallest element is proved similarly. ∎

Corollary 2 (Concavity of Aggregate Demand) *In a multi-unit exchange economy with individual strong-substitute valuations, the market valuation is concave.*

Proof. Denote by x the total endowment of the economy, and n the number of firms. We show that for all y such that $0 \leq y \leq x$, there exists a linear price vector p such that y is in the demand set of the market valuation. From Theorem 19, we already know that the result is true if $y = x$. Thus suppose that $y < x$. Consider an additional firm with valuation $v_{n+1}(z) = Kz \wedge (x - y)$, where K is a large constant, greater than the total value of other firms for the whole endowment x. One can easily check that v_{n+1} is an assignment valuation, and therefore a strong-substitute valuation (see Hatfield and Milgrom [6]). Applying Theorem 19 to the economy with $(n+1)$ firms, there exists a Walrasian equilibrium with linear price vector p. At this price, the additional firm obtains the bundle $x-y$ since its marginal utility dominates all other firms' for any unit up to this bundle, and vanishes beyond this bundle. This implies that the remaining firms ask for y at price p, or equivalently, that y belongs to the demand set of n firms' market valuation at price p. Concavity is then obtained as in the proof of Theorem 9. ∎

Theorem 20 (Aggregation) *If individual firms have strong-substitutes valuations, then the market valuation v is a strong-substitute valuation.*

Proof. Let $\{v_i\}_{1\leq i\leq n}$ denote the family of individual valuations and v denote the market valuation, defined by $v(x) = \max\{\sum_i v_i(x_i) : \sum x_i = x, x_i \in \mathbb{N}\}$. From Theorem 12, we will prove the result if we show that v is a concave weak-substitute valuation that satisfies the consecutive-integer property. Corollary 2 states that v is concave. From Corollary 1, v is a weak-substitute valuation. It thus remains to show that v satisfies the consecutive-integer property. For any price p, the demand set of v is the solution of

$$\max_x\{v(x) - px\} = \max_x \left\{ \max\left\{ \sum_i v_i(x_i) : \sum_i x_i = x \right\} - px \right\} = \sum_i \max_{x_i} v_i(x_i) - px_i.$$

Therefore, $D(p) = \sum_i D_i(p)$. In particular, the projection of D on the k^{th} coordinate satisfies $D_k = \sum_i D_{i,k}$. The sets $D_{i,k}$ consist of consecutive integers by Theorem 9, implying that D_k also consists of consecutive integers. ∎

Finally, we examine the connections between strong-substitute valuations and the structure of the core of the associated cooperative game. The setting considered in this section is the same as Ausubel and Milgrom [2], but with the multi-unit formulation replacing their binary formulation. We first recall the definitions of coalitional value functions, the core, and Vickrey payoffs.

Suppose that, in addition to bidders, there exists a single owner (labeled "0") of all units of all goods, who has zero utility for her endowment.

Definition 12 *The coalitional value function of a set S of bidders is defined by $w(S) = \max\left\{\sum_{i \in S} v_i(x_i) : \sum x_i \in \mathcal{X}\right\}$ if $0 \in S$, and $w(S) = 0$ otherwise.*

Denote L the set consisting of all bidders and the owner of the goods.

Definition 13 *The core of the economy is the set*

$$Core(L, w) = \left\{\pi \in \mathbb{R}_+^L : w(L) = \sum_{l \in L} \pi_l, w(S) \leq \sum_{l \in S} \pi_l \text{ for all } S \subset L\right\}.$$

Definition 14 *The Vickrey payoff vector is given by $\bar{\pi}_l = w(L) - w(L \setminus l)$ for $l \in L \setminus 0$, and $\bar{\pi}_0 = w(L) - \sum_{l \in L \setminus 0} \bar{\pi}_l$.*

Ausubel and Milgrom [2] show that this is the payoff at the dominant-strategy solution of the generalized Vickrey auction.

Definition 15 *The coalitional value function w is bidder-submodular if for all $l \in L \setminus 0$ and sets S and S' such that $0 \in S \subset S'$, $w(S) - w(S \setminus l) \geq w(S') - w(S' \setminus l)$.*

Theorem 21 *Suppose that there are at least $2 + \max_k N_k$ bidders. If any bidder has a concave, weak-substitute valuation that is not a strong-substitute valuation, then there exist linear or unit-demand valuations for remaining bidders such that the coalitional value function is not bidder-submodular and the Vickrey payoff vector is not in the core.*

Proof. See the Appendix.

Although related to a theorem by Ausubel and Milgrom [2], Theorem 21 is different because it requires a counter-example where bidders treat all units of any given good identically (counter examples where two units of the same good have different values for a bidder are not allowed). The converse result, which states that strong substitutes are sufficient for the Vickrey payoffs to be in the core, is true. For that direction, the arguments provided by Ausubel and Milgrom go through because the arguments do not hinge on intra-good symmetry, as checked by the following proof.

Theorem 22 *If all bidders have strong-substitute valuations, then the coalitional value function is bidder-submodular and the vector of Vickrey payoffs is in the core.*

Proof. From Ausubel and Milgrom [2, Theorem 7], it is enough to show that the coalitional value function is bidder-submodular. By assumption, the binary form $\tilde{v}_i$ of each bidder satisfies the substitutes property. Theorem 11 in Ausubel and Milgrom is valid for the binary formulation and implies that the coalitional value function is bidder-submodular. This property is independent of the formulation (binary or multi-unit). ■

6 Walrasian Tâtonnement and Clock Auctions

This section analyzes auctions where goods are available in multiple units and prices are linear. We propose a class of algorithms guaranteeing monotonic convergence of the auction to a pseudo-equilibrium whenever bidders have weak-substitute valuations. This result extends existing analyses where bidders have the more restrictive strong-substitute valuations, as in Gul and Stacchetti [5] and Milgrom [12]. Moreover, we show that these algorithms generically work under the natural assumption, used in practical designs, that bidders only submit demand singletons, as opposed to their entire demand set.[12] Combining these results with those of Section 5 shows that clock auctions *always* converge to a Walrasian equilibrium whenever there exists one. For the present analysis, we define a clock auction as a price adjustment process in which the path of prices is monotonic—either increasing or decreasing. In practice this monotonicity and other features, especially activity rules for bidders (see Milgrom [12]), differentiate clock auctions from a Walrasian tâtonnement. In order to understand the relation between substitute valuations and clock auctions, it is useful to start the analysis with Walrasian tâtonnement and only later to impose monotonicity on the process.

6.1 Continuous time and price

Our goal is to construct algorithms where i) prices increase over time, and ii) converge to the smallest pseudo-equilibrium price $\underline{p}$ whenever bidders have weak-substitute valuations. Reverse algorithms, where price decreases and converges to the largest pseudo-equilibrium price can similarly be constructed. By definition, pseudo-equilibrium prices minimize the convex function $f : p \to \pi(p) + \bar{x}p$ where π is the market dual profit function and $\bar{x} \in \mathbb{N}^K$ is the supply vector. Among the general algorithms to find such minimizers are steepest-descent algorithms. At any time, price changes are determined by the gradient of f whenever f is differentiable, and by the vector of smallest norm of its subdifferential otherwise.[13] Such algorithms amount to a particular Walrasian tâtonnement, as they adjust prices to eliminate excess demand. For any price vector p, we denote by $z(p)$ the point of smallest norm in the opposite of the differential of f at p. When f is differentiable, z corresponds to the excess (aggregate) demand $D(p) - \bar{x}$. In general, z is the vector of smallest norm in the convex hull of the set of excess demand.

Definition 16 *A correspondence-based Walrasian tâtonnement (or simply "correspondence tâtonnement") is a price-setting algorithm defined by some equation*[14] *of motion*

$$\dot{p}_r(t) = \alpha(t, p(t))z(p(t)), \tag{3}$$

[12] In contrast, the Gul-Stacchetti algorithm requires that bidders report their entire demand set at each time. When interpreted as an auction, the requirement that demand sets be reported makes their procedure different from any auction process in current use.

[13] By definition, the subdifferential $\partial f(p)$ at p of a convex function f is the set of vectors x such that $f(q) - f(p) \geq x(q-p)$ for all q. The subdifferential is always a nonempty convex set, and coincides with f's gradient whenever it is differentiable.

[14] The subscript "r" denotes right derivatives. Right derivatives are necessary to allow for discontinuities in the smallest-norm gradient $z(p)$.

where α is a continuous function taking values in $[\underline{\alpha}, \bar{\alpha}]$ for some constants[15] $0 < \underline{\alpha} < \bar{\alpha}$.

Let $\mathcal{L} = \{p : p \leq \underline{p} \text{ and } z(p) \geq 0\}$ denote the set of price vectors where all goods are in excess demand. The following theorem states that, starting any price in $\mathcal{L}$, correspondence tâtonnements are *well defined* (i.e. from any initial price, it generates a unique trajectory in the price space), monotonic, and converge to the lowest pseudo-equilibrium price, $\underline{p}$. In practice, the assumption that $p(0)$ is in $\mathcal{L}$ is satisfied when the clock auction starts at zero price, but also any "reasonably low" prices.

Theorem 23 *Correspondence tâtonnements are well defined. Suppose that bidders have weak-substitutes valuations and that $p(0) \in \mathcal{L}$. Then, for any correspondence tâtonnement, the following holds: i) $p(t) \in \mathcal{L}$ for all t, ii) $p(t)$ is increasing, and iii) $p(t)$ converges to $\underline{p}$ in finite time.*

The proof is in the Appendix. Theorem 23 implies that, when bidders have weak-substitute valuations, *any correspondence tâtonnement is an ascending clock auction and converges to the smallest pseudo-equilibrium price.* This result is of major importance for clock auctions, because it ensures that they always converge even with the weaker notion of substitutes. Formally, we define a correspondence (ascending) clock auction as a correspondence tâtonnement where prices can only increase: (3) is replaced by $\dot{p}_r(t) = \alpha(t, p(t))z_+(p(t))$, where $z_+ = \max\{z, 0\}$ is the componentwise maximum between the gradient $z(p(t))$ and 0.

Corollary 3 *If bidders have weak-substitute valuations, any correspondence clock auction starting from a price in $\mathcal{L}$ converges to the smallest pseudo-equilibrium price.*

In particular, if goods are weak substitutes, ascending clock auctions will find an equilibrium whenever there exists one. By contrast, it is easy to build examples of valuations violating weak-substitutes such that a Walrasian equilibrium exists but ascending clock auctions fail to find it.

Our result extends Ausubel [1] in three ways. First, it searches on the space of linear prices, while Ausubel's algorithm specifies separate prices for each unit of the good. Second and more importantly, it relies only on the assumption of weak substitutes, where Ausubel's analysis requires on the stronger assumption of strong substitutes. Third, it shows that the process converges monotonically to a pseudo-equilibrium price, which always exist in this setting and which are equilibrium prices whenever an equilibrium exists.

6.2 Discrete time and price

In actual auctions, prices are often restricted to lie on some grid. In such setting, time can be decomposed in rounds between which prices are adjusted from one element of the grid

[15] The lower bound $\underline{\alpha}$ ensures that the algorithm does not stall at a suboptimal price, and the upper bound ensures that that the equation is integrable.

to another. We now show that previous results are approximately true for fine enough price grids and that clock auctions still work if bidders only announce one desired bundle at each round, rather than their entire demand set. We emphasize again the importance of this last result, since bidders do not submit their entire demand set in current practice.

A price grid is a lattice $\mathcal{P}_\eta = (\eta\mathbb{N})^K$, where η is a small positive constant. A discrete algorithm generates a sequence of prices $\{p_t : t = 0, 1, \ldots\}$ in $\mathcal{P}_\eta$, whose evolution is determined by excess demand at any period. Even though prices are restricted to a grid, they can follow the exact direction of the gradient $z(p)$, which has rational coordinates, provided the price step is large enough. Moreover, the thinner the price grid, and the smaller the price step required to move along that direction (Lemma 6 in the Appendix analyzes this point in detail). More importantly, the price sequence generated by a discrete algorithm may in principle diverge from the price path generated by its continuous equivalent, because demand gradients are discontinuous functions of price. For example, excess demand for one good can turn into excess supply if the price for that good is slightly increased. Fortunately, Lemma 7 in the Appendix shows that such gradient discontinuities bear no significant consequences. Formally, it shows that trajectories of any discrete steepest-descent algorithm that start close to each other stay close to each other. This implies that any discrete algorithm closely follows its continuous equivalent (i.e. following the same gradient rule, but where prices evolve in continuous time). For any price p_0, denote by $T(p_0) = \{p(t) : t \in \mathbb{R}_+, p(0) = p_0\}$ the trajectory generated by the continuous correspondence tâtonnement of the previous section, and let $T(p_0, \varepsilon) = \cup_{p\in T(p_0)} B(p, \varepsilon)$ denote the tube[16] of radius ε around $T(p_0)$.

Theorem 24 (Discrete Steepest-Descent Algorithm) *For any $\varepsilon > 0$, there exist $\eta > 0$ and $\bar{\alpha} > 0$ such that for any grid finer than η, step size less than $\bar{\alpha}$, and initial price p_0, the trajectory generated by the discrete steepest descent algorithm is contained in $T(p_0, \varepsilon)$.*

The proof of Theorem 24 is in the Appendix. Finally, the above analysis allows us get rid of the assumption that bidders submit their entire demand set. Bidder valuations can be seen as vectors of the finite-dimensional space $\mathcal{V} = \mathbb{R}^{\bar{x}}$. Say that a property of an algorithm holds *for almost all economies* if it holds for all bidder valuations except for a subset of Lebesgue measure zero in $\mathcal{V}^n$, where n is the number of bidders. Let us also define singleton-based algorithms in the same way as the correspondence-based algorithms, except that bidders ask only one bundle at each period. Concretely, this means that instead of using the vector of smallest norm in the excess demand set, the algorithm may follow any vector of that set. The following result shows that this information loss does not affect Theorem 24 except possibly on a set of economies with Lebesgue measure zero. As Proposition 4 below makes clear, this set only depends on the price grid chosen and is otherwise independent of the algorithm.

Theorem 25 (Singleton-Based Algorithm) *Under the assumptions of Theorem 24, let p_0 be any initial price of the algorithm. The trajectory of a singleton-based steepest-descent algorithm is contained in $T(p_0, \varepsilon)$ for almost all economies.*

[16] $B(p, \varepsilon)$ is the open ball centered at p and radius ε.

The proof is based on the following proposition.

PROPOSITION 4 *For all $(v_1, \ldots, v_n) \in \mathcal{V}^n$, the demand correspondence $p \to D(p)$ is single-valued almost everywhere in $\mathcal{P}$ with respect to the Lebesgue measure on this set.*

Proof. Consider first one bidder with valuation v. For any two bundles x and x', the subset $P(x, x')$ of $\mathcal{P}$ defined by $P(x, x') = \{p : p(x - x') = v(x) - v(x')\}$, is the intersection of a hyperplane with the positive orthant $\mathcal{P}$, and has therefore zero Lebesgue measure. Since the number of possible bundles is finite, the set

$$Q = \bigcup_{x \neq x'} P(x, x'),$$

which contains all prices at which the bidder's demand is multi-valued, also has zero Lebesgue measure. For a countable (in particular, finite) number of bidders, the set of prices where aggregate demand is multi-valued is contained in $Q^a = \cup Q_i$, which has zero Lebesgue measure. ■

Proposition 4 implies that the set of economies such that $Q^a \cap \mathcal{P}_\eta \neq \emptyset$ has Lebesgue measure zero. Therefore, singleton-based and correspondence-based algorithms are identical for almost all bidder valuations and price grids.

In practice, the auctioneer does not know bidder valuations. Theorem 25 implies that for any belief that is absolutely continuous with respect to the Lebesgue measure, the algorithm is arbitrarily close to the continuous, correspondence-based steepest descent algorithm of the ideal economy. In particular, the algorithm completely ignores bidders' indifference sets. This feature contrasts with Gul and Stacchetti [5], whose algorithm gives much importance to indifference sets.

7 Divisible Goods

For discrete goods, we showed that strong substitutes were sufficient and in some sense necessary for several major results, such as i) the existence of a Walrasian equilibrium, ii) the fact that the Vickrey outcome is in the core, iii) submodularity of the dual profit function over nonlinear prices, and iv) the law of demand. For divisible goods we define a notion of substitutes based on nonlinear pricing[17] guaranteeing that the first three properties above hold, but failing the law of demand and a natural generalization thereof. We also show that this notion is *equivalent* to weak substitutes (i.e. based on linear pricing), provided that valuations are concave. En route, we also show that concave substitute valuations are characterized by submodularity of the dual profit function over nonlinear prices and are robust with respect to additive concave perturbations, which extends a related notion of robustness established for complements, as in Milgrom [11].

[17] Using nonlinear prices is not identical to treating goods individually, because treating goods as distinct expands both the domain and the range of the demand correspondence, while only the domain is changed by the switch to nonlinear pricing.

7.1 Definitions

Let

$$\mathcal{X} = \Pi_{k \in \mathcal{K}}[0, X_k]$$

represent the space of possible bundles of the exchange economy. Throughout the paper, we assume that agents have quasi-linear utilities: the utility of an agent with valuation v acquiring a bundle x at price $p(x)$ is $u(x,p) = v(x) - p(x)$. The demand function of any agent is defined by $D(p) = \arg\max_{x \in \mathcal{X}}\{v(x) - p(x)\}$. To emphasize the difference between the two notions of substitutes considered for divisible goods, we call them "nonlinear substitutes" and "linear substitutes", according to whether nonlinear pricing is allowed or not. Linear substitutes is identical to the weak notion of substitutes used for discrete goods, while nonlinear substitutes is closer to the stronger notion of substitutes (but see footnote 17).

Definition 17 *v is a linear-substitute valuation if whenever $p_j \leq p'_j$, $p_k = p'_k$ for all $k \neq j$, and $x \in D(p)$, there exists $x' \in D(p')$ such that $x'_k \geq x_k$ for all $k \neq j$.*

In the discrete case with individual item pricing, a rational consumer who buys k units of some type of good always buys the cheapest k units. Therefore, one way to describe individual item pricing is to specify that the cost of acquiring goods is a convex function of the number of goods acquired from each class and is additive across classes of goods. Higher prices mean that the marginal cost of acquiring additional units is higher. This characterization of the cost of acquiring goods and the corresponding representation of higher prices can be applied directly to the continuous case. That is the approach we adopt in this section.

Following the above argument, let $\mathcal{C}$ denote the space of componentwise convex functions from $\mathbb{R}^K_+$ to $\mathbb{R}$ that vanish at 0. An element of $\mathcal{C}$ contains K convex cost functions, one for each good. We endow $\mathcal{C}$ with the following partial order: $C \preceq \hat{C}$ if for all k, $C'_k \leq \hat{C}'_k$, where C'_k and $\hat{C}'_k$ are the derivatives of C_k and $\hat{C}_k$, respectively.[18] With this order, $\mathcal{C}$ is a lattice, where for any C and $\hat{C}$, the meet and the join satisfy, for all k and $x_k \geq 0$, $(C \vee \hat{C})'_k(x_k) = \max\{C'_k(x_k), \hat{C}'_k(x_k\}$ and $(C \wedge \hat{C})'_k(x_k) = \min\{C'_k(x_k), \hat{C}'_k(x_k)\}$, respectively.[19] We extend the domain of any dual profit function π from linear prices to $\mathcal{C}$ and denote $\bar{\pi}$ its extension:

$$\bar{\pi}(C) = \max_x\{v(x) - C(x)\},$$

where $C(x) = \sum_k C_k(x_k)$.

Definition 18 *v is a nonlinear-substitute valuation if whenever $C_j \leq \hat{C}_j$, $C_k = \hat{C}_k$ for all $k \neq j$, and $x \in D(C)$, there exists $x' \in D(\hat{C})$ such that $x'_k \geq x_k$ for all $k \neq j$.*

[18]These derivatives exist almost everywhere and are increasing by convexity of the cost functions (by the Rademacher theorem, see Magaril-Ilyaev and Tikhomirov [10, p. 160]. Marginal cost can be extended over the entire domain, for example by imposing right-continuity.

[19]As can be easily checked, the marginal costs of $(C \wedge \hat{C})'_k$ and $(C \vee \hat{C})'_k$ are nondecreasing for all k, and constructed cost functions both vanish at 0, so that $C \wedge \hat{C}$ and $C \vee \hat{C}$ belong to $\mathcal{C}$.

For the discrete case, we have shown that there are several properties distinguishing weak substitutes and strong substitutes, so there is scope for judgment in creating the analogue of strong substitutes in the continuous case. In place of strong substitutes, we study the concept of *concave, nonlinear-substitute* valuations. This definition preserves properties distinguishing strong substitutes from weak substitutes in the discrete setting, including robustness of the substitutes property with respect to nonlinear price changes and existence of Walrasian equilibria.[20] Moreover, we find below that these valuations are characterized by dual submodularity on the domain of nonlinear prices, which was also the characterization of strong substitutes in the discrete case. We find further that, given concavity, the linear-substitute and nonlinear-substitute properties are equivalent. Therefore, our divisible-good extensions of the two concepts coincide in the case of concave valuations. The next two theorems develop all of these relationships.

THEOREM 26 (DUAL SUBMODULARITY) *If v is a concave linear-substitute valuation, then $\bar{\pi}$ is submodular on $\mathcal{C}$.*

Proof. Convexity of π and C implies that the function $(p, x) \mapsto \pi(p) + px - C(x)$ is concave in x and convex in p. The minmax theorem (see e.g. Stoer and Witzgall [16]) implies that

$$\max_x \min_p \{\pi(p) + p \cdot x - C(x)\} = \min_p \max_x \{\pi(p) + px - C(x)\}.$$

Concavity of v implies that $v(x) = \min_p(\pi(p) + px)$ for all x. This, along with $\bar{\pi}(C)$'s definition, implies that

$$\bar{\pi}(C) = \max_x \{\min_p(\pi(p) + px) - C(x)\}.$$

This and the minmax theorem then imply that

$$\bar{\pi}(C) = \min_p \left\{\pi(p) + \max_x \{px - C(x)\}\right\}.$$

The inner maximum equals

$$\sum_k \int_0^\infty (p_k - c_k(z_k))_+ dz_k.$$

Therefore,

$$\bar{\pi}(C) = \min_p \left\{\pi(p) + \sum_k \int_0^\infty (p_k - c(z_k))_+ dz_k\right\}.$$

We now show that the function $h : (p, C) \to h(p, C) = \int_0^\infty (p - C'(z))_+ dz$ is submodular on $\mathbb{R}_+ \times \mathcal{C}_1$. For $q < r$, $h(r, C) - h(q, C)$ is the area of the region $\{(z, p) : p \in [q, r] \text{ and } C(z) \leq p\}$, which is also equal to $\int_q^r z(p, C) dp$, where $z(p, C) = \sup\{z : C'(z) \leq p\}$. Since $z(p, C)$

[20] Another possible extension of strong substitutes would impose the law of aggregate demand. That is equivalent to requiring that a dominant diagonal property hold for the matrix $[\partial x_i / \partial p_j]$ of partial derivatives of demand. Since all important properties (except of course for the law of demand itself) hold without this quantitative restriction, we do not follow this approach.

is nonincreasing in C for all p, so is $h(r,C)-h(q,C)$, which proves submodularity of h in (p,C). Moreover, for any numbers p, c and d, $(p-c)_+ + (p-d)_+ = (p-\max(c,d))_+ + (p-\min(c,d))_+$, which implies that h is modular in C. Linear substitutes implies that π is submodular in p. Therefore, $\bar{\pi}$ is the minimum over $p \in \mathcal{P} = \mathbb{R}_+^K$ of an objective function that is submodular on $\mathcal{P} \times \mathcal{C}$. This implies (Topkis [18]) that π is submodular on $\mathcal{C}$. ■

Theorem 26 allows us to prove the equivalence of three candidate definitions for the divisible-good extension of strong substitutes.

Theorem 27 *Suppose that v is concave. Then the three following statements are equivalent.*

(i) v is a linear-substitute valuation.

(ii) v is a nonlinear-substitute valuation.

(iii) π is submodular on $\mathcal{C}$.

Proof. Clearly, (ii) implies (i). From Theorem 26, (i) implies (iii). To conclude the proof, we show that (iii) implies (ii). We fix a direction of price increase for some good, and show that along this direction, the demand for any other good is nondecreasing. Fix goods $j \neq k$ and a direction of increase δ (i.e. δ is nondecreasing, vanishes at 0, and is such that $C+\delta$ is convex) for good j. Consider the restriction

$$\tilde{\pi}(\lambda,\mu) = \max_x \{v(x) - C(x) - \lambda x_k - \mu\delta(x_j)\}$$

of π, defined[21] on $\mathbb{R}_+ \times [0,1]$. Since π is submodular, so is $\tilde{\pi}$. $\tilde{\pi}$ is convex as the pointwise maximum of a family of functions that are affine in (λ,μ). In particular, $\partial\tilde{\pi}/\partial\lambda$ exists almost everywhere. By an envelope theorem[22] $\partial\tilde{\pi}/\partial\lambda$ exists everywhere that demand for good k $x_k(\lambda,\mu)$ is a singleton and at those prices, $\partial\tilde{\pi}/\partial\lambda = -x_k(\lambda,\mu)$. Submodularity of $\tilde{\pi}$ implies that $\partial\tilde{\pi}/\partial\lambda(\lambda,\mu)$ is nonincreasing in μ or, equivalently, that x_k is nondecreasing in μ. ■

Theorems 26 and 27 have an important consequence: concave nonlinear-substitute valuations are stable under perturbation by any concave modular function. Thus comparative statics results are robust with respect to such perturbations, as stated in the following theorem.

Theorem 28 *If v is a concave nonlinear-substitute valuation, then $v+f$ is a concave nonlinear-substitute valuation for all f modular and concave.*

Proof. Suppose that v is a concave nonlinear-substitute valuation. Then, $v+f$ is concave whenever f is concave. By Theorem 27, it remains to show that $v+f$ is a linear-substitute valuation. Let

$$x^f(p) = \arg\max_x \{v(x) + f(x) - px\}.$$

[21]The function $C(x) + \lambda x_k + \mu\delta(x_j)$ is convex for $\mu \in [0,1]$, as can be easily checked.
[22]Milgrom and Segal[14, Corollary 4].

Without loss of generality, we can assume that $f_i(0) = 0$ for all terms of f. Let $C(x,p) = px - f(x)$. Since f is modular and concave, C is modular and for each i, C_i is convex and vanishes at 0. Therefore, C belongs to $\mathcal{C}$. Moreover, increasing p_k implies increasing C_k. Since v is a nonlinear-substitute valuation and

$$x^f(p) = \arg\max_x \{v(x) - C(x,p)\},$$

$x_j^f(p)$ is nondecreasing in C_k, thus in p_k. ■

We now turn to the consequences of the substitutes properties in settings with multiple firms.

7.2 Multiple firms: Aggregation and Core

We first show that both linear and concave nonlinear substitute valuations are closed under aggregation.

THEOREM 29 *The class of linear-substitute valuations is closed under aggregation*

Proof. Suppose that individual firms have linear substitute valuations. Theorem 2, which also holds for divisible goods, then implies that individual profit functions are submodular. By Theorem 17, the market dual profit function is therefore a sum of submodular functions, and so itself submodular. Applying Theorem 2 once again, we conclude that v is a linear substitute valuation. ■

With divisible goods, concavity is also closed under aggregation: the maximization

$$v(x) = \max_x \sum_i v_i(x_i)$$

subject to $\sum x_i \leq x$ has a concave objective function and a convex constraint function, so v is concave[23] in the constraint bound x. This shows the following result.

THEOREM 30 *Concave nonlinear-substitute valuations are closed under aggregation.*

Proof. The above results show that concave linear-substitute valuations are closed under aggregation. This, along with Theorem 27, implies that the same is true of concave nonlinear-substitute valuations. ■

With divisible goods, concavity suffices for the existence of a Walrasian equilibrium, since each bundle is supported by a price vector. We show that if, in addition, firms have nonlinear-substitute valuations, then the Vickrey outcome is in the core. The setting and definitions of coalitional value function, core, Vickrey payoff, and bidder submodularity are identical to those of Section 5.

[23] See for example Luenberger [9, p.216].

Theorem 31 *If all bidders have concave nonlinear-substitute valuations, the Vickrey outcome is in the core.*

Proof. From Theorem 7 of Ausubel and Milgrom [2], the Vickrey outcome is in the core if the coalitional value function is bidder submodular. Therefore, we need to show that $w(S \cup \{l\}) - w(S)$ is nonincreasing in S. Let $x \in \mathbb{R}^K$ denote the supply vector. Then,

$$w(S \cup \{l\}) - w(S) = \max_{y \leq x} \{v_l(x-y) + v_S(y) - v_S(x)\},$$

where $v_T(z)$ denote the optimal value of bundle z for coalition T (so that $v_T(x) = w(T)$ for all T). Therefore, it is enough to show that $v_S(y) - v_S(x)$ is non increasing in S or simply that $v_S(z)$ is submodular in $(-z, S)$. Concavity[24] of v_S implies that

$$v_S(z) = \min_p \{\pi_S(p) + pz\}, \tag{4}$$

where π_S is the dual profit function of v_S and is equal to $\sum_{i \in S} \pi_i(p)$. Since π is submodular in p, the objective in (4) is submodular in $(-z, S, p)$. Hence, $v_S(z)$ is submodular in $(-z, S)$, as required. ■

7.3 Law of Aggregate Demand

We have focused so far on monotone comparative statics of the demand function. In the discrete case, strong substitutes not only implies that x_j is nondecreasing in the price of other goods, but also that $\sum_j x_j(\tilde{p})$ is nonincreasing in $\tilde{p}$, which is the discrete law of aggregate demand. In this section, we show that this property no longer holds, and thus is not necessary for Theorems 26 and 31 to hold.

What is the analogue of the law of aggregate demand for divisible substitute goods? One problem is to determine the units in which such a law might be expressed. For example, suppose that one unit of good i represents a 10-ride train pass between two cities, while one unit of good j is a one-way bus ticket between the same cities. One expects that, starting from prices where a consumer chooses the train pass, a large price increase in the train pass results in the consumer buying several bus tickets to replace the train pass, implying that the sum $x_i + x_j$ *increases* as p_i increases, which violates the law of aggregate demand. One way to pose the problem without units is to ask whether there exist constants a_i such that $\sum_i a_i x_i$ be nonincreasing in prices? In the previous example, a natural choice would be $a_i = 10$ and $a_j = 1$, given the relative similarity of a train trip and a bus trip. More generally, we say that a valuation v satisfies the *generalized law of aggregate demand* (GLoAD) if there exist increasing functions f_i such that

$$\sum_i f_i(x_i(C))$$

is nonincreasing in C. It satisfies the *law of aggregate demand* if one can take $f_i(x_i) = x_i$ for all i. The GLoAD seems so much more flexible than the law of aggregate demand

[24] See discussion preceding Theorem 30.

that one is led to wonder whether it is satisfied by linear-substitute valuations, or at least concave nonlinear-substitute valuations. However, the following theorem and its corollary show that the GLoAD is equivalent to the law of aggregate demand up to a mere convex re-scaling of goods. For the remaining of this section we assume that the cost functions are nondecreasing.[25] To simplify the exposition, let $(f \circ g)(x) = f(g_1(x_1), \ldots, g_K(x_K))$, for any function f and modular function g. Clearly, $f \circ g$ is modular if f is also modular. Restricted to the class of increasing modular functions, let f^{-1} denote the sum of component-wise inverse functions: $f^{-1}(x) = \sum_k f_k^{-1}(x_k)$. For functions of one variable, these definitions coincide with the usual ones. For the next two theorems, we assume that valuations are nondecreasing.

Theorem 32 *Let v be a nondecreasing concave nonlinear-substitute valuation satisfying the generalized law of aggregate demand for some function f, and g be an increasing, concave, modular function. Then $\tilde{v} = v \circ g$ is a nondecreasing, concave nonlinear-substitute valuation satisfying the generalized law of aggregate demand for the modular function $\tilde{f} = f \circ g$.*

Proof. Since v and g are nondecreasing concave, so is $\tilde{v}$. Let $\tilde{C}$ be a convex price schedule, and $y(\tilde{C}) = \arg\max \tilde{v}(y) - \tilde{C}(y)$. We wish to show that y_j is nondecreasing in $\tilde{C}_k$ for $j \neq k$, and that there exists an increasing modular function $\tilde{f}$ such that $\tilde{f}(y(\tilde{C}))$ is nonincreasing in $\tilde{C}$. The function $\gamma = g^{-1}$ is increasing, convex, and modular. By assumption, there exists a modular function f such that $f(x(C))$ is nondecreasing in C, where $x(C)$ is the demand of v at the convex price schedule C. Let $C = \tilde{C} \circ \gamma$. Since all components of γ and $\tilde{C}$ are nondecreasing convex, so are the components of C. Increasing $\tilde{C}_k$ to $\tilde{C}'_k$ is equivalent to increasing C_k to $C'_k = \tilde{C}'_k \circ \gamma_k$. Therefore, if $j \neq k$, $y_j(\tilde{C}) = \gamma_j(x_j(C))$ is nondecreasing when $\tilde{C}_k$ increases. Moreover, letting $\tilde{f} = f \circ g$, we have $\tilde{f}(y(\tilde{C})) = f(x(\tilde{C} \circ \gamma))$, which is nonincreasing in $\tilde{C}$. ■

Corollary 4 *Suppose that v is a nondecreasing, concave nonlinear-substitute valuation satisfying the generalized law of aggregate demand for some convex function f. Then, $\tilde{v} = v \circ f^{-1}$ satisfies the law of aggregate demand.*

Thus, the generalized law of aggregate demand corresponds to a quantitative rather than a qualitative relaxation of the law of aggregate demand. In fact, it is possible to construct a concave nonlinear-substitute valuation that does not satisfy *any* generalized law of aggregate demand. We construct a counter-example in the Appendix, which establishes a fundamental difference between the cases of discrete and divisible goods.

8 Conclusion

The substitutes concepts play a critical role in equilibrium theory. For discrete economies, strong substitutes is necessary for the robust existence of equilibrium and weak substi-

[25] This assumption is used in the proof of Theorem 32. We did not make this assumption earlier in order to prove Theorem 28, where we consider $C(x) = px - f(x)$ and f may be increasing.

tutes drive the monotonicity that is exploited by most clock auction algorithms. Strong substitutes is also the condition that determines whether the Vickrey outcome is in the core. A related concept–the law of aggregate demand–has been the informal justification for the wide adoption of activity rules in practical auctions. Among our findings is that the law of aggregate demand is precisely the additional property that converts a concave weak substitute valuation to a strong-substitute valuation when goods are discrete.

Section 7 extends the analysis to divisible goods and finds some significant differences from the discrete case. For divisible goods and concave valuations, defining strong substitutes by allowing non-linear pricing (where the discrete case allowed separate prices for each unit), we find no difference between weak and strong substitutes. Unlike the discrete case, the law of aggregate demand and its unit-free extensions generally fail for the continuous case, but the other implications of strong substitutes survive: there exists a Walrasian equilibrium, the Vickrey outcome is in the core, and the dual profit function is submodular over the space of convex price schedules. Thus, for concave valuations, the law of aggregate demand characterizes the difference between the cases of discrete goods and divisible goods.

9 Appendix: Proofs

9.1 Section 4

Proof of Lemma 1. Consider a bundle x such that $x_k \leq N_k - 1$ and $x_j \leq N_j - 2$. Take any binary representant $\tilde{x}$ of x, and call l and m two units of good j not in $\tilde{x}$, and s a unit of good k not in $\tilde{x}$. Since $\tilde{v}$ satisfies the gross-substitute property, the triple

$$\{\tilde{v}(\tilde{x} + e_l + e_m) - \tilde{v}(x + e_l) - \tilde{v}(\tilde{x} + e_m), \tilde{v}(\tilde{x} + e_l + e_s) - \tilde{v}(\tilde{x} + e_l) - \tilde{v}(\tilde{x} + e_s),$$
$$\tilde{v}(\tilde{x} + e_m + e_s) - \tilde{v}(\tilde{x} + e_m) - \tilde{v}(\tilde{x} + e_s)\} \quad (5)$$

has at least two maximizers. Symmetry of $\tilde{v}$ implies that the last two arguments of that quantity are equal, and therefore greater than or equal to the first one. That is, written in multi-unit form $v(x + e_k + e_j) - v(x + e_k) - v(x + e_j) \geq v(x + 2e_j) - 2v(x + ej)$ which concludes the proof after simplification. ∎

Proof of Theorem 10. Suppose by contradiction that the law of aggregate demand is violated: there exist k, p and x such that for all ε small enough, we have (i) $x \in D(p - \varepsilon e_k)$, and (ii) for all $y \in D(p + \varepsilon e_k)$, $\|y\|_1 > \|x\|_1$. Clearly, for any such y, we have $y_k < x_k$. Let $D_k = D_k(p)$, $\underline{d} = \min D_k$ and $\bar{d} = x_k = \max D_k$. By continuity, we have (i) $x \in D(p)$, (ii) there exists some $y \in D(p)$ such that $y_k < x_k$, and (iii) for all $y \in D$ such that $y_k = \underline{d}$, $\|y\|_1 > \|x\|_1$.

For each $d \in D_k$, define $g(d) = \min\{\|y_{-k}\|_1 : y_k = d \text{ and } y \in D(p)\}$. Let $\gamma : [\underline{d}, \bar{d}] \to \mathbb{R}$ denote the largest convex function such that $\gamma(d) \leq g(d)$ for all $d \in D_k$. The function γ is well defined and piecewise affine: there exists a partition $\Delta = \{\delta_l\}_{l \in \Lambda}$ of $[\underline{d}, \bar{d}]$ such that

γ is affine on $[\delta_l, \delta_{l+1}]$. Moreover, $\bar{d}$ and $\underline{d}$ are elements of Δ: there exist $\underline{l}$ and $\bar{l}$ such that $\underline{d} = \delta_{\underline{l}}$ and $\bar{d} = \delta_{\bar{l}}$. For $l \in \{\underline{l}+1, \bar{l}\}$, denote $H(l)$ the hyperplane containing the two $(K-2)$-dimensional affine varieties

$$\{z \in \mathbb{R}^K : \|z_{-k}\|_1 = \gamma(\delta_l) \text{ and } z_k = \delta_l\}$$

and

$$\{z \in \mathbb{R}^K : \|z_{-k}\|_1 = \gamma(\delta_{l-1}) \text{ and } z_k = \delta_{l-1}\}.$$

There exists a unique hyperplane containing these two affine varieties, so $H(l)$ is well defined. Moreover, $H(l)$ lies below $D(p)$ and contains at least two elements z and y of $D(p)$ such that $z_k = \delta_l$ and $y_k = \delta_{l-1}$.

We claim that there exists $l \in \{\underline{l}+1, \bar{l}\}$ such that $\gamma(\delta_{l-1}) - \gamma(\delta_l) > \delta_l - \delta_{l-1}$. Suppose that the contrary holds. Then, $\gamma(\underline{d}) - \gamma(\bar{d}) \leq \bar{d} - \underline{d} = x_k - \underline{d}$. But then, there exists y in $D(p)$ such that $y_k = \underline{d}$ and $\|y_{-k}\|_1 = \gamma(\underline{d})$, implying that $\|x\|_1 = x_k + \gamma(\bar{d}) \geq \underline{d} + \gamma(\underline{d}) = \|y\|_1$, which contradicts the hypothesized violation of the law of aggregate demand.

Consider an index l as in the previous paragraph, and modify p slightly so that the demand set becomes $D(p) \cap H(l)$. The price vector can be further modified so that the remaining bundles in the demand set are aligned on a unique straight line and, for the new price $\bar{p}$, there still exist z and y in $D(\bar{p})$ such that $z_k > y_k$ and $\|z\|_1 < \|y\|_1$. There are two cases: either there are two indices i and j such that $y_i > z_i$ and $y_j > z_j$, or there exists an index i such that $y_i - x_i > x_k - y_k$. Since optimal bundles are aligned, the same properties hold for the extremities bundles of the segment containing $D(\bar{p})$, so we assume without loss of generality that z and y are these extreme bundles. In the first case, increasing p_i slightly violates the weak-substitute property, as the optimal quantity of good j also decreases. In the second case, the convex-demand property is violated: the set $D_i(\bar{p})$ contains a hole between z_i and y_i. ■

Proof of Proposition 1 Trivially,

$$\max_x \min_p \{\pi(p) + px - (\tilde{p}, x)\} \leq \min_p \max_x \{\pi(p) + px - (\tilde{p}, x)\}. \tag{6}$$

We need to prove that the reverse inequality also holds. We fix $\tilde{p}$ throughout the proof. Consider a price p solving $\min_p \max_x\{\pi(p)+px-(\tilde{p}, x)\}$. Let $N(p) = \arg\max_x\{px-(\tilde{p}, x)\}$. $N(p)$ is a hyper-rectangle: there exist two bundles r and R with $r \leq R$ such that $N(p) = \{z \in \mathbb{Z}^K : r \leq z \leq R\}$.

Suppose that there exists a bundle x in $N(p) \cap D(p)$. Then, the right-hand side of (6) equals $\pi(p) + px - (\tilde{p}, x) = v(x) - (\tilde{p}, x)$, where the last equality comes from the fact that x belongs to $D(p)$. Now consider any linear price vector q. We have $\pi(q) + qx - (\tilde{p}, x) \geq v(x) - (\tilde{p}, x)$, by definition of $\pi(q)$. This last inequality implies that the left-hand side of (6) is actually greater than or equal to its right-hand side. Therefore, we will have concluded the proof if we show that $N(p) \cap D(p)$ is nonempty, which we now turn to.

Let $Co(D(p))$ and $Co(N(p))$ denote the convex hulls of $D(p)$ and $N(p)$. We first show that $Co(D(p)) \cap Co(N(p))$ has a nonempty intersection. Suppose by contradiction that $Co(D(p)) \cap Co(N(p)) = \emptyset$. Then, since these two sets are closed and convex, the separating-hyperplane theorem implies that there exists a direction δ and a number a such that $y\delta < a$ for $y \in N(p)$ and $x\delta > a$ for $x \in D(p)$. Now modify p by an infinitesimal amount along the direction δ, yielding a new level $q = p + \varepsilon\delta$. The objective function $\pi(p) + \max_z\{pz - (\tilde{p}, z)\}$ is affected by this change in two ways. First, through the sensitivity of π with respect to p. Taking any $x \in D(q) \subset D(p)$, we have $\pi(p) = v(x) - px$ and $\pi(q) = v(x) - qx$. Therefore, the change of π is $-\varepsilon x\delta$. Second, through the sensitivity of $\max_z\{pz - (\tilde{p}, z)\}$ with respect to p. There exists $y \in N(p)$ such that $\max_z\{pz - (\tilde{p}, z)\} = py - (\tilde{p}, y)$ throughout the price change. Therefore, the effect on this term equals $\varepsilon y\delta$. The overall change of the objective function is then $\varepsilon(y - x)\delta < 0$, implying that q leads to a strictly lower objective function than p, which contradicts optimality of p.

We have proved that the sets $Co(D(p))$ and $Co(N(p))$ have a non empty intersection. We now prove that this intersection contains a point with integer coordinates. Consider any polytope of $\mathbb{R}^K$. We say that an edge (i.e. a segment joining two vertices of the polytope) is *simply oriented* if either (i) it is parallel to one coordinate axis $\{\lambda e_i : \lambda \in \mathbb{R}\}$ of the space or (ii) there exist two coordinates i and j such that the edge is parallel to $e_i - e_j$. We say that a polytope is simply oriented if all its edges are simply oriented. Last, we recall that a polytope all of whose vertices have integer coordinates is called a *lattice polytope*.

LEMMA 2 *If a lattice polytope P is simply oriented, and H is the half space $\{x : x_k \geq q\}$, where $k \in \{1, \ldots, K\}$ and q is an integer, then $P \cap H$ is either the empty set, or a simply oriented, lattice polytope.*

Proof. Suppose that $Q = P \cap H$ is nonempty. Its vertices are either vertices of P, in which case they are integral, or new vertices belonging to H. We prove that any such vertex also has integer coordinates. Any new vertex x is the intersection of H with an edge E of P that is not parallel to H. In particular, there exists an integral vertex y of P such that $x - y$ is parallel to E. Moreover, $y_k \neq q$, since the edge is not parallel to H. The edge is either parallel to e_k or to $e_k - e_i$ for some $i \neq k$. In the first case, we have $x_j = y_j \in \mathbb{Z}$ for all $j \neq k$ and $x_k = q \in \mathbb{Z}$, so x has integer coordinates. In the second case, $x_j = y_j \in \mathbb{Z}$ for all $j \notin \{i, k\}$, $x_k = q \in \mathbb{Z}$, and $x_i = y_i + (y_k - x_k) \in \mathbb{Z}$, so x also has integer coordinates. We now prove that the edges of Q are simply oriented. Thus consider an edge E of Q, joining vertices x and y. If either x or y are vertices of P, then E is either an edge of P, or the result of such an edge being cut by H. In either case, it is simply oriented because P is simply oriented. If both x and y are new vertices, E is the intersection of a two-dimensional face F of P with H, where F is not parallel to H. F is defined by two linearly independent edges E' and E'' of P which are simply oriented, and at least one of which contains e_k. Suppose first that either E' or E'', say E', is orthogonal to e_k. Then it is easy to show that E is parallel to E'' and therefore simply oriented. Now suppose that both E' and E'' have a nonzero k^{th} component. Because they are linearly independent, there exist i and j such that F is generated by $e_k - e_i$ and $e_k - e_j$ (where the signs come from the fact that P is simply oriented). In that case, as can be easily verified, E is parallel to $e_i - e_j$, and therefore simply oriented. ■

We observe that Lemma 2 still holds if the inequality sign is reversed in the definition of H.

$Co(D(p))$ is a lattice polytope since $D(p)$ consists of integral vectors. We now prove that $Co(D(p))$ is simply oriented. Thus consider any edge E of $Co(D(p))$. There exists a vector δ of $\mathbb{R}^K$ such that E is included in some straight line $\Delta = \{x_0 + \lambda\delta\}_{\lambda\in\mathbb{R}}$. We first show that δ has at most two nonzero components. Suppose on the contrary that δ has at least three components, say i, j, and k. Without loss of generality assume that δ_i and δ_j are positive. Since E is a face of $Co(D(p))$, there exists an infinitesimal modification of the price vector p, such that $D(p) = E$. Moreover, E contains two vectors x and y such that $x - y = \lambda\delta$ for some $\lambda > 0$. If we slightly increase p_i, x becomes suboptimal, so the optimal quantity of good j decreases, which violates the weak-substitute property. Thus, δ has at most two nonzero components. We now prove that E is simply oriented. If δ has only one nonzero component, the claim is trivial. Suppose that δ has two positive components, say i and j. We show that $\delta_i = -\delta_j$. Since E has integer vertices, we can assume that δ_i and δ_j are integers.[26] If $\delta_i\delta_j > 0$, slightly increasing p_i reduces the optimal quantity of good j which violates the weak-substitute property. Thus, δ_i and δ_j have opposite signs. Now suppose that $|\delta_i| < |\delta_j|$. This implies that for all integral vectors x and y in E, we have $|x_j - y_j| \geq 2$, which violates the consecutive-integer property. Thus, $\delta_i = -\delta_j$, which concludes the proof.

We have shown that $Co(D(p))$ is a simply oriented lattice polytope. Since $Co(N(p))$ is a hyperrectangle of the form $\{x \in \mathbb{R}^K : a \leq x \leq b\}$ for some integral vectors a and b, we have, denoting $H(k,q)_+ = \{x : x_k \geq q\}$ and $H(k,q)_- = \{x : x_k \leq q\}$,

$$Co(D(p)) \cap Co(N(p)) = Co(D(p)) \bigcap_{1\leq k\leq K} \left(H_+(k,a_k) \cap H_-(k,b_k)\right).$$

Iterating Lemma 2 $2K$ times implies that $Co(D(p)) \cap Co(N(p))$ is either the empty set or a lattice polytope. Since we have already shown that this intersection is nonempty, it must contain an integral point, which concludes the proof of Proposition 1. ■

9.2 Section 5

Proof of Theorem 16. We extend part of the proof of Theorem 2 in Gul and Stacchetti [4] to a multi-unit context. By assumption, there exist a price vector $\bar{p}$, a good k, and bundles x and x' such that (i) $\{x, x'\} \in D(\bar{p})$, (ii) $x'_k - x_k \geq 2$, and (iii) for all z in $D(\bar{p})$, $z_k \notin (x_k, x'_k)$. This implies that at the price $p = \bar{p} - \eta e_k$, x is only dominated by bundles z such that $z_k \geq x_k + 2$. In particular, the single-improvement property is violated by x at price p. Therefore, any bundle y that solves $\min_z \sum_k |x_k - z_k|$ subject to $u_1(z,p) > u_1(x,p)$ satisfies $y_k \geq x_k + 2$.

Let $\rho = \sum_j (y_j - x_j)_+$. By hypothesis, $\rho \geq 2$. Let $\varepsilon = \frac{u_1(y,p) - u_1(x,p)}{2\rho}$. Let $I_+ = \{j : x_j < y_j\}$,

[26] See for example Korte and Vygen [8].

$I_- = \{j : x_j > y_j\}$, and $I_0 = \{j : x_j = y_j\}$. If $j \in I_+$, introduce $N_j - y_j$ firms, call them "C_j", with unit-demand valuation $v_1(\mathcal{X}) + 2$ for a single unit of good j. If $j \in I_+ \setminus \{k\}$, introduce $y_j - x_j$ firms, call them "c_j", with unit-demand valuation $p_j + \varepsilon$ for a single unit of good j. If $j = k$, introduce $y_k - x_k - 1$ firms ("c_k") with unit-demand valuation $p_k + \varepsilon$ for a single unit of good k. If $j \in I_-$, introduce $N_j - x_j$ firms (C_j) with unit-demand valuation $v_1(\mathcal{X}) + 1$ for a single unit of good j, and $x_j - y_j$ firms (c_j) with unit-demand valuation p_j for a single unit of good j. If $j \in I_0$, introduce $N_j - x_j$ firms with unit-demand $v_1(\mathcal{X}) + 1$. Last, introduce a special firm, "firm 2", with unit-demand $p_k + v_1(\mathcal{X}) + 1$ for a single unit of good k.

Now suppose that there exists a Walrasian equilibrium with price vector t, and let X_i denote the bundle of the equilibrium received by firm i. Necessarily, $(X_1)_j \geq \min\{x_j, y_j\}$ for all j, since even if all unit-demand firms get one unit, there remain $\min\{x_j, y_j\}$ units of good j. Define a new price vector as follows: $q_j = t_j$ for $j \notin I_-$ and $q_j = p_j$ for $j \in I_-$. For $j \in I_-$, $N_j - x_j$ units go to firms C_j. The remaining x_j units are shared between firm 1 and firms c_j, with at least y_j units for firm 1. Now, if firm 1 has none of the remaining $x_j - y_j$ units, it means that $t_j \leq p_j$, and this share remains optimal when t_j is increased to p_j. If firm 1 has all of the remaining units, it means that $t_j \geq p_j$, and this share remains optimal when t_j is decreased p_j. If firm 1 has only a part of these remaining units, it means that t_j is already equal to p_j. Thus (X, q) is also a Walrasian equilibrium, such that $X_1 \geq x \wedge y$. Moreover, all C_j get their units, so that $X_1 \leq x \vee y$. Therefore

$$x \wedge y \leq X_1 \leq x \vee y. \tag{7}$$

Firm 2 necessarily gets a unit of good $k \in I_+$. Therefore, $X_{1k} < y_k$. This, together with (7), implies that $\sum_k |x_k - X_{1k}| < \sum_k |x_k - y_k|$, and thus

$$u(X_1, p) \leq u(x, p). \tag{8}$$

Suppose that there exist some goods j in I_+ such that $X_{1j} > x_j$. This implies that $q_j \geq p_j + \varepsilon$, since firms c_j would otherwise want to get all the units. Combining these price inequalities with (8) yields $u_1(X_1, q) < u_1(x, q)$, which contradicts optimality of X_1 for firm 1.

Suppose instead that $X_{1j} \leq x_j$ for all j. Then, all units between x_j and y_j for $j \in I_+$ are consumed by firms c_j and by firm 2. For $j \neq k$, this implies that c_j have a positive value for the good: $q_j \leq p_j + \varepsilon$. For $j = k$, even though firm 2 takes one units of the $y_k - x_k$ available units of k, the fact that $y_k \geq x_k + 2$ implies that there is also a firm c_k taking one unit of good k, which implies that $q_k \leq p_k + \varepsilon$. Since $X_1 = x$ on I_+ and $p_j = q_j$ for $j \notin I_+$, (8) implies $u_1(X_1, q) \leq u_1(x, q)$. Since $q_j \leq p_j + \varepsilon$ for all $j \in I_+$, the value initially chosen for ε implies that $u_1(x, q) < u_1(y, q)$, and thus $u_1(X_1, q) < u_1(y, q)$, which contradicts optimality of the bundle X_1 for firm 1. ■

Proof of Theorem 21 From A&M Theorem 7 (which allows for multiple units of goods), the vector of Vickrey payoff vector is in the core if and only if the coalitional value function is bidder-submodular. We show that under the assumptions of Theorem 21, there always exist bidder valuations such that the coalitional value function is not bidder-submodular. Suppose that bidder 1's valuation violates the consecutive-integer property.

There exist $\hat{p}$ and k such that $D_k(\hat{p})$ does not consist of consecutive integers. Let $p = \hat{p} + \varepsilon e_k$ for ε small enough. Then there exists x and z such that $x_k \geq z_k + 2$, and

$$v(z) - pz > v(x) - px > v(y) - py \tag{9}$$

for all y such that $y_k \in (z_k, x_k)$. Introduce a second bidder with linear valuation $v_2(x) = p_{-k}x_{-k}$, and $x_k - z_k$ unit-demand bidders who only value good k. The total number of bidders is $x_k - z_k + 2 \leq N_k + 2 \leq \max_k N_k + 2$. From (9), we have

$$v(x) + p_{-k}(\bar{x} - x)_{-k} \geq v(y) + p_{-k}(\bar{x} - y)_{-k} + p_k(x_k - y_k)$$

whenever $x_k - y_k \leq x_k - z_k - 1$, and

$$v(z) + p_{-k}(\bar{x} - z)_{-k} p_k(x_k - z_k) > v(x) + p_{-k}(\bar{x} - x)_{-k}.$$

Therefore, denoting S the set consisting of bidders 1, 2 and the $x_k - z_k - 2$ unit-demand firms, and s and t the last two unit-demand bidders, we have $w(S \cup \{s\}) = w(S)$ and $w(S \cup \{s, t\}) > w(S \cup \{t\})$, showing that w is not bidder-submodular. ■

9.3 Section 6.1

Proof of Theorem 23. The proof is based on three lemmas, proving respectively well-definedness, monotonicity, and confinement in $\mathcal{L}$.

Lemma 3 (Well-definedness) *The continuous SDA algorithm is well defined.*

Proof. On any region of the price space where excess demand is constant, the algorithm defines a straight trajectory of direction z, and is thus well-defined.[27] The only possible problem, thus, is to rule out the possibility that there are infinitely many region changes in an arbitrarily small amount of time. With the steepest-descent algorithm, the norm of z is nondecreasing in time. Since z is constant over any region where aggregate demand is constant, and the norm of z strictly decreases each time it changes, any region that is left is never visited again. ■

Lemma 4 (Monotonicity) *When bidders have weak-substitute valuations and $z(0) \geq 0$, $p(\cdot)$ is nondecreasing.*

Proof. Suppose by contradiction that $z(t)$ fails to be nonnegative at some time t, and take the smallest such time. Since $z(0) \geq 0$, $t > 0$. By construction, $z(s) \geq 0$ on a left neighborhood of t. Let $m = z(t)$, $x = z(t_-)$, and P be the opposite of the subdifferential of f at $p(t)$. P is a convex polytope, whose vertices are elements of the excess demand at $p(t)$, and m is the element of P with smallest norm. By assumption, x is nonnegative. By continuity of demand, x must also belong to P. Let $J = \{k : m_k < 0\}$. By assumption, $J \neq \emptyset$. Let H be the affine hyperplane going through (the point) m

[27] The scalar function α is immaterial, as long as it is bounded away from 0 and $+\infty$.

and orthogonal to (the vector) m. By assumption, P is on one side of H and touches H at m. Let F be the largest face of P contained in H, y be any vertex of F, and $C_y = \{z : \sum_J m_k z_k \geq \|m\|^2 - \sum_{J^c} m_s y_s\}$. Since $y - m$ is orthogonal to m, C_y is a cone with vertex y. We will show that C_y contains P but not x, a contradiction.

Since $y-m$ is orthogonal to m, we have $\|m\|^2 - \sum_{J^c} m_s y_s = \sum_J m_k y_k = m_J y_J$, where the components of m_J are equal to those of m on J and vanish on J^c, and a similar definition for y_J. By convexity of F, $m = y + \sum_l \alpha_l E_l$, where $\{E_l\}$ is the family of direction vectors of the edges of F emanating from y. Taking the scalar product of the previous equality with m_J yields $m m_J = y_J m_J + \sum_l \alpha_l E_l m_J$. We now prove that $E_l m_J = 0$ for all l. By construction of F,

$$m.E_l = 0. \tag{10}$$

Moreover the weak substitute property implies that E_l has at most two nonzero components, and any two nonzero components are of opposite sign (see the proof of Proposition 1). If E_l has one nonzero component, it must be in J^c, otherwise it would violate (10). If it has two nonzero components, then either they are both in J or both in J^c, for otherwise (10) would be violated. In any case, this implies that $E_l.m_J = 0$. Thus, $m_J v_J = m_J^2 > 0$. In particular $C_y = \{z : \sum_J m_k z_k \geq m_J^2\}$. Since the components of x are nonnegative by construction, x cannot belong to C_y.

To conclude the proof, we show that C_y contains P. By convexity of P, it is enough to show that all edges of P emanating from y are going in the cone C_y. This will be the case if we show that for any such edge with direction δ (away from y), we have

$$\delta m_J \geq 0. \tag{11}$$

By definition of F, we have $\delta m \geq 0$ (i.e. any edge from y must point outwards from H). Since bidders have weak-substitute valuations, δ has at most two nonzero components. Suppose first that it has exactly two components, δ_i and δ_j. If i, j are in J, then (11) trivially holds. If i, j are in J^c, then (11) is an equality. If $i \in J$ and $j \in J^c$, then $\delta m \geq 0$ and the fact that $\delta_i \delta_j < 0$ (by weak-substitutes) implies that $\delta_i < 0$, and thus that (11) holds. If there is only one nonzero component, (11) holds trivially. ■

Lemma 5 (Confinement) *If bidders have weak-substitute valuations, $p(0) \leq \underline{p}$ and $z(0) \geq 0$, then $p(t) \leq \underline{p}$ for all $t \geq 0$.*

Proof. Suppose not: there exists a time t such that $p(t)$ crosses the hyperrectangle $R = \{z : z \leq \underline{p}\}$ from inside out. In particular, the index subset $I = \{j : p_j(t) = \underline{p}_j\}$ is nonempty, and we have $p_j(t) < \underline{p}_j$ for $j \notin I$. Moreover, $p(s) \not\leq \underline{p}$ for s in a right neighborhood of t: there exists a nonempty subset $J \subset I$ such that $p_{s,j} > \underline{p}_j$ for $j \in J$ and $s \in (t, t+\varepsilon)$. By construction of the algorithm, this means that the vector n of smallest norm in the opposite of the subdifferential of $p(t)$ satisfies $n_j > 0$ for $j \in J$. We will contradict this statement by showing that the vector m defined by $m_j = n_j$ for $j \notin J$ and $m_j = 0$ for $j \in J$ is in the opposite of the subdifferential. m's norm is strictly smaller than n's, contradicting the assumption that n is of smallest norm in the opposite of the subdifferential. By definition of the subdifferential, we need to show that, letting

$p = p(t)$,
$$m(q - p) \geq f(p) - f(q) \tag{12}$$
for all q. We first show this inequality in a neighborhood of p. By construction of n, $n(q-p) \geq f(p) - f(q)$ for all q. Therefore, (12) is automatically satisfied for q such that $q_j \leq p_j$ for $j \in J$. Now consider the case where $q_j > p_j$ for a subset $J(q)$ of J. Consider the vector q' such that $q'_j = q_j$ for $j \notin J(q)$ and $q'_j = p_j$ for $j \in J(q)$. Since we are in a neighborhood of p, $q_j \leq \underline{p}_j$ for all $j \notin J(q)$. This implies that $q' \leq \underline{p}$ and, therefore, that $q' = q \wedge \underline{p}$. Submodularity of f implies $f(\underline{p} \wedge q) + f(\underline{p} \vee q) \leq f(\underline{p}) + f(q)$. The inequality, combined with the fact that $\underline{p}$ is a minimum of f, implies that $f(q') \leq f(q)$. By construction of q',
$$m(q-p) = m(q'-p) \geq n(q'-p) \geq f(p) - f(q') \geq f(p) - f(q),$$
which concludes the proof on a neighborhood of p. To prove the result globally, consider any vector q and let $q_\lambda = \lambda q + (1-\lambda)p$ where $\lambda \in (0,1)$. From the previous analysis, we have for λ small enough $m(q_\lambda - p) \geq f(p) - f(q_\lambda)$. By convexity of f, $f(q_\lambda) \leq \lambda f(q) + (1-\lambda) f(p)$. Combining the previous two inequalities and dividing by λ yields the result. ■

We now conclude the proof of the theorem. Since $p(t)$ is nondecreasing and bounded, it must converge to some limit in $\mathcal{L}$. Since α is bounded away from zero, the rate of change of p is bounded away from zero on any closed subset of the price space that does not contain any pseudo-equilibrium price. Since the only pseudo-equilibrium price contained in $\mathcal{L}$ is $\underline{p}$, this has to be the limit.

9.4 Section 6.2

Let $z(p)$ denote the vector of smallest norm in the convex hull of the excess (aggregate) demand set $D(p) - \bar{x}$.

Lemma 6 (Feasible directions of descent) *Suppose that the number of bidders is less than some constant $N > 0$, and that no bidder can demand more than overall supply $\bar{x}$. Then, for any grid $\mathcal{P}_\eta$, there exists $\alpha(\eta) > 0$ such that $\alpha(\eta) z(p) \in \mathcal{P}_\eta$ for all p and all bidder valuations. Moreover, α can be chosen such that $\alpha(\eta) \to 0$ as $\eta \to 0$.*

Proof. By assumption, the excess demand set is an integer polytope of $\mathbb{R}^K$, bounded by the rectangle $[-\bar{x}, N\bar{x}]$. Therefore, z can only take finitely many values. Since any such z is the vector of minimum norm of an integral polytope, it has rational coordinates. Therefore, its direction can always be achieved on any regular lattice. That is, there exists a positive number $\alpha(z)$ such that $\alpha(z)z$ is the difference vector of two points of the lattice. Moreover, the smallest such $\alpha(z)$ gets arbitrarily small as the grid gets arbitrarily thin. Since there are finitely many values of z, $\max_z\{\alpha(z)\}$ goes to zero as the grid thinness η goes to zero. ■

Let $\{p(t)\}_{t\in\mathbb{N}}$ and $\{q(t)\}_{t\in\mathbb{N}}$ denote the trajectories generated by a given steepest-descent algorithm, starting from respective initial prices $p(0)$ and $q(0)$.

Lemma 7 (Nearness Lemma) *Suppose that the number of bidders is less than some constant $N > 0$, that no bidder can demand more than aggregate supply $\bar{x}$, and that there exists a vector $M \in \mathbb{R}_+^K$ such that bidders demand none of good i whenever $p_i > M_i$. Then, for any $\varepsilon > 0$, there exists $\bar{\eta} > 0$ and $\bar{\alpha} > 0$ such that for all $\eta < \bar{\eta}$ and step sizes less than $\bar{\alpha}$, $\|p(0) - q(0)\| < \varepsilon$ implies $\|p(t) - q(t)\| < \varepsilon$ for all periods and all bidder valuations.*

Proof. Without loss of generality, we can restrict attention to price vectors less than M. Since the number of bidders is finite, the function $f : p \to \pi(p) + \bar{x}p$ is piecewise affine, with finitely many regions. Moreover, directions of the hyperplanes supporting f are determined by excess demand vectors, which take finitely many values (cf. proof of Lemma 6). Since z is in the opposite of the differential of f, $f(q) - f(p) \geq z(p)(p - q)$ for all q, with strict inequality if p and q are in distinct regions. The fact that p is bounded by M and that there are finitely many possible slopes for f implies the existence of a constant $\rho > 0$ such that

$$f(q) - f(p) \geq \rho + z(p)(p - q) \tag{13}$$

whenever p and q are not in the same region. We now consider paths of the discrete steepest-descent algorithm starting from respective initial price vectors p_0 and q_0, with $\|p_0 - q_0\| < \varepsilon$. Trajectories are parallel until the two prices reach different regions, and thus leave the vector $p_t - q_t$ unchange until that time. Let $s \geq 0$ denote the first time that the two paths hit distinct regions. (13) implies $f(q_s) - f(p_s) \geq \rho + z(p_s)(p_s - q_s)$ and $f(p_s) - f(q_s) \geq \rho + z(q_s)(q_s) - p_s)$. Summing these inequalities yields[28] $(z(p_s) - z(q_s))(p_s - q_s) \leq -2\rho$. Let α be the step size[29] of the steepest-descent algorithm: $p_{s+1} = p_s + \alpha z(p_s)$, and $q_{s+1} = q_s + \alpha z(q_s)$

$$\|p_{s+1} - q_{s+1}\|^2 = \|p_s - q_s\|^2 + \|\alpha(z(p_s) - z(q_s))\|^2 + 2\alpha(z(p_s) - z(q_s)) \cdot (p_s - q_s).$$

Therefore,

$$\|p_{s+1} - q_{s+1}\|^2 - \|p_s - q_s\|^2 \leq -4\rho\alpha + O(\alpha^2),$$

which is negative for α small enough, which we impose by appropriately setting $\bar{\alpha}$. Thus, we have proved that $\|p_t - q_t\|$ remains constant when prices are in the same region, and decreases otherwise. ■

Proof of Theorem 24. Starting in the same region, trajectories of both algorithms are initially undistinguishable, since gradients are identical. Let t_0 denote the first time that the trajectory T of the discrete algorithm overshoots, causing the two paths to have distinct vectors. Let $\epsilon > 0$ be a positive constant (to be chosen later), and denote by p_{t_0} the price of the discrete algorithm, and by q_{t_0} a price on $T(p_0)$ such that $\|p_{t_0} - q_{t_0}\| < \epsilon$. Such a price exists if the step size $\bar{\alpha}(\epsilon)$, which gives an upper bound on the overshoot, is small enough. Let T_1 denote the trajectory that the discretized algorithm would generate if it were starting from q_{t_0}. By construction T_1 coincides with $T(p_0)$ until there is a second overshoot. By the nearness lemma, T and T_1 are within ϵ from each other. Therefore, when T_1 overshoots, at time t_1, there is a price q_{t_1} of $T(p_0)$ such that $\|p(t_1) - q_{t_1}\| < 2\epsilon$.

[28]This proof strategy introduces a strict version of the theory of maximally monotone mapping. See Rockafellar [15].

[29]The result holds if α depends on t and p, as long as it is continuous in p.

Iterating the process, we thus prove that, up to the k^{th} overshoot, we have $T \subset T(p_0, k\epsilon)$ when T is truncated at $t = t_k$. The number of overshoots is bounded above by the number R of regions (since any region is visited at most once by the continuous algorithm, see proof of Theorem 23). Therefore, the result obtains by setting $\epsilon = \varepsilon/R$. ■

9.5 Section 7

COUNTER-EXAMPLE 1 *There exist concave nonlinear-substitute valuations that do not satisfy the generalized law of aggregate demand.*

PROOF. [Sketch] The example only requires two goods. Let $x < x'$ and $y < y'$ be positive numbers, and consider the bundles $A = (x, y)$, $B = (x', y)$, $C = (x, y')$, and $D = (x', y')$ and let (p_i, q_i) be the supporting prices for $i = A, B, C, D$ (such prices will exist by concavity). If GLoAD held, there would exist some increasing functions f and g with $f(x(p, q)) + g(y(p, q))$ nonincreasing in (p, q), where (p, q) is the price vector of the two goods and $(x(p, q), y(p, q))$ is the demand at that price. Suppose that at (p_B, q_B) and (p_C, q_C), a small increase in price p reduces $x(p, q)$ by a very small amount and increases $y(p, q)$ by a very large amount (as in the ticket/pass example above). GLoAD can[30] only hold if $f'(x')$ is much larger than $g'(y)$ (looking at B), and if $f'(x)$ is much larger than $g'(y')$ (looking at C). Now suppose that at (p_A, q_A) and (p_D, q_D), a small increase in price q reduces $y(q)$ by a very small amount and increases $x(q)$ by a very large amount. GLoAD can only hold if $g'(y)$ is much larger than $f'(x)$ (looking at A) and if $g'(y')$ much larger than $f'(x')$ (looking at D). These two sets of conditions are incompatible proving that GLoAD cannot hold. To conclude the counter-example, it remains to show that there exist concave nonlinear-substitute valuations satisfying the demand behavior described at points A, B, C, and D. Demand variations are determined by the Hessian of the valuation at these points. As is easily checked, one can choose for each bundle in $\{A, B, C, D\}$ a Hessian matrix that is negative definite with negative cross derivatives and that satisfies the demand behavior specified at that point. It is also possible to extend these Hessian matrices over the entire consumption space, while keeping negative definiteness and negative cross derivatives. This construction is achieved by superposition of four concave submodular functions, one for each bundle, whose Hessian coincides with the specified Hessian at that bundle and vanishes around the three remaining bundles. For each of the four bundles, it is possible by Urysohn's lemma to construct a continuous, matrix-valued function that equals the desired Hessian at the bundle and vanishes outside an arbitrarily small neighborhood of that bundle (see e.g. Willard [19, p. 102]). Such superposition defines a valuation (up to an affine term) that is submodular and concave. In two dimensions, submodularity implies the linear-substitute property. By Theorem 27, the constructed valuation is therefore a concave nonlinear-substitute valuation. ■

[30] The proof is easily adapted if f and g are not differentiable at these points.

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