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Coalition-Proofness and Correlation with Arbitrary Communication Possibilities*

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The ability of the members of a coalition to communicate secretly determines whether the coalition can coordinate to deviate from a proposed strategy and thus affects which strategies are "coalition proof." We show that the existence of a Pareto-best element in the set of strategies that survive iterated elimination of dominated strategies implies the existence of a coalition-proof correlated equilibrium for any specification of coalitional communication possibilities that always permits individual deviations. Such an element exists in games with strategic complementarities if either (1) there is a unique Nash equilibrium or (2) each player's payoff is nondecreasing in the others' strategies. *Journal of Economic Literature* classification number: C72. © 1996 Academic Press, Inc.

I. INTRODUCTION

A common justification for focusing on Nash equilibria as solutions of games is based on interpreting each equilibrium as a potential self-enforcing agreement or contract: An equilibrium strategy profile has the property that, once it has been agreed upon, no player has an incentive to deviate unilaterally. This contractual interpretation has been criticized on various grounds (see, e.g., Aumann, 1990). We are concerned here with

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two of these objections. First, if the coalition of the whole can communicate to agree on a particular strategy choice, then smaller coalitions may also be able to communicate and coordinate their actions. According to this line of argument, one ought not to be satisfied with strategy profiles that are immune only to individual deviations, but instead should insist also on immunity to deviations by those groups of players who can coordinate their actions. Second, if players can communicate, they may be able to achieve correlation among their strategic choices. This means that the set of potential initial contracts should be expanded to include not just pure and mixed strategies but also correlated ones. Moreover, deviating coalitions might also have access to correlating devices, so their deviations might involve correlated behavior as well.

These considerations have led to the companion papers to this one by Moreno and Wooders (1996) and by Ray (1996), both of which offer definitions of coalition-proof correlated equilibria. The analyses in these papers, however, implicitly assume that any coalition can plan a deviation and any subcoalition can plan a further deviation.

In this paper we first provide an explicit model of coalition communication possibilities. We then provide a sufficient condition on the game for existence of a coalition-proof correlated equilibrium for any "admissible" specification of coalition communication possibilities. (Any specification that always permits individual deviations is "admissible.") The sufficient condition, existence of a Pareto-best element in the set of pure strategies surviving iterated elimination of dominated strategies, was suggested by an equivalent condition first identified by Moreno and Wooders (1996). In addition, we show that the sufficient condition, which applies even in games with infinite strategy spaces, is met in games with strategic complementarities that satisfy either of two properties: (1) there is a unique Nash equilibrium, or (2) every player's payoff is nondecreasing (or nonincreasing) in every other player's strategy. In both these cases, the coalition-proof correlated equilibrium is in fact in pure strategies.

The two cited objections to the idea that the Nash equilibria are the set of possible agreements about how to play the game have both been studied previously. Requiring immunity to all possible coalitional deviations (while ignoring the possibility of correlation) leads to the notion of *strong equilibrium* (Aumann, 1959, 1964). This is a very demanding concept of equilibrium, and most games that have interested economists have no strong equilibria.

Bernheim, Peleg, and Whinston (1987) have offered an alternative formulation that restricts attention to a limited class of "self-enforcing" coalitional deviations, that is, ones that are themselves robust against further "self-enforcing" deviations by subcoalitions. They call their solution concept *coalition-proof Nash equilibrium.*

In an earlier version of this paper (Milgrom and Roberts, 1994b), we investigated a model that allows a larger set of coalitional deviations than Bernheim, Peleg, and Whinston, namely, all coalitional deviations that are robust against further individual deviations. We called the corresponding Nash equilibria *"strongly coalition proof."* (We later learned that the same concept had been proposed by Kaplan (1992), who called the equilibria *"semistrong."*) Note that any strongly coalition-proof equilibrium is coalition-proof in the sense of Bernheim, Peleg, and Whinston, because the former is robust against at least as many first-round deviations from initial proposals. In our paper we showed existence and uniqueness of strongly coalition proof equilibria in two classes of games with strategic complementarities: those with unique equilibria and those where each player's payoff is nondecreasing in the others' strategies, in which case the largest Nash equilibrium is the one in question. (The original proofs are reproduced here in the Appendix).

While the well-known concept of *correlated equilibrium* (Aumann, 1974) responds to the second criticism of Nash equilibrium as a self-enforcing agreement, the task of developing a concept of equilibrium in correlated strategies that is immune to self-enforcing coalitional deviations has only recently been addressed, notably in the companion papers to this one by Moreno and Wooders (1996) and by Ray (1996) (see also the papers referenced in these two).[1] In particular, both papers offer definitions of *coalition-proof correlated equilibria* based on logic similar to that of Bernheim, Peleg, and Whinston, in which all potential coalitional deviations are treated symmetrically. While this symmetric treatment is mathematically elegant, there is no reason to suppose it is descriptive of the real possibilities for coalitional deviations.

Each of the several formulations of coalition-proofness evaluates a proposed coalitional deviation by holding fixed the strategy of the complementary coalition. This is presumably justified by an implicit assumption that the deviating coalition can keep its plans secret from nonmembers. Yet there is no general reason to suppose that all coalitions are equally able to communicate and coordinate on deviations, let alone to do so secretly. One might imagine situations in which the players can meet only in a public place, so that secret agreements among subcoalitions are impossible even if the coalition of the whole can communicate and coordinate. It may also happen that only some coalitions have an opportunity to meet secretly, or that communications between individuals A and B can be kept secret from C but not from D. There might even be time limits

[1] We became aware of this work only after preparing the earlier draft (Milgrom and Roberts 1994b) of this paper.

or other capacity limits that prevent players from engaging in a long sequence of planned deviations.

Different possibilities for secret communications imply different possibilities for coalitional deviations and different opportunities to correlate strategies, and these need to be recognized in the model and solution concept. We address this matter in the next section of the paper. There we offer a model of coalition communication possibilities in terms of a collection of finite, decreasing sequences of coalitions, where the interpretation is that the first element of one of these sequences specifies a coalition that can communicate to coordinate on a deviation from an initial proposal, the second element is a subcoalition that can plan a further deviation from that planned by the first coalition, and so on. We then extend the Moreno−Wooders definition of coalition-proof correlated equilibrium to games with such *coalition communication structures.*

Our treatment of the existence question also builds on the results of Moreno and Wooders, who study a finite-player, finite-strategy game with unrestricted communication possibilities for coalitions. They examine the set of strategies that survive the iterated elimination of dominated strategies (the *dominance solution*) and show that if the collection of correlated strategies with support in that set has a Pareto-best element (i.e., one that simultaneously maximizes the payoff of every player over that set of correlated strategies) then that strategy is a coalition-proof correlated equilibrium. In particular, if a game is dominance solvable, then the unique Nash equilibrium is a coalition-proof correlated equilibrium with the unrestricted coalition communication structure that they implicitly impose.

We add to these implications in several ways. First, we show that any pure strategy that is Pareto-best in the set of strategies surviving iterated elimination of dominated strategies is coalition-proof for *any* admissible coalition communication structure. The same is true of any correlated strategy that is Pareto-best in the set of correlated strategies with support in the dominance solution. In particular, such a strategy is coalition-proof under the unrestricted communication possibilities in Moreno and Wooders (1996). In fact, such a strategy is (up to payoff equivalence) the unique coalition-proof correlated equilibrium under any admissible coalition communication structure. Moreover, our result dispenses with the restriction to finite strategy spaces imposed by Moreno and Wooders. (This is important for the many economic applications that specify continuous strategy spaces.)

Extending our results from the earlier version of this paper, we also show that the sufficient condition holds in games with strategic complementarities if either (1) the game has a unique Nash equilibrium or (2) the players' payoffs are all nondecreasing (or all nonincreasing) functions of the competitors' strategy choices. In either of these cases, the essentially unique coalition-proof equilibrium is the same under any admissible coalition communication structure and is also the essentially unique coalition-proof Nash equilibrium in the sense of Bernheim, Peleg, and Whinston and semistrong equilibrium in the sense of Kaplan.

There are surprisingly many games of economic interest that satisfy these conditions. Games with strategic complementarities and a unique equilibrium include the Bertrand pricing game with various demand specifications and the Arms Race game (Milgrom and Roberts, 1990) as well as the Hart−Moore (1990) investment game. With richer specifications, these games continue to have strategic complementarities and have payoffs that are increasing in the other player's strategies (Milgrom and Roberts, 1994a; Milgrom and Shannon, 1994). Many other such games have been studied in the literature (see, e.g., Bernheim and Whinston, 1987; Bulow, Geanakoplos, and Klemperer, 1985; Diamond and Dybvig, 1983).

II. COALITION-PROOFNESS WITH COALITION COMMUNICATION STRUCTURES

Let $\Gamma = (N, A, \pi)$ be a normal form game, with n denoting a typical player and $x_n, y_n \in A_n$ denoting typical strategies for n. The player set N is finite, but the strategy spaces may be infinite. We assume that the set of strategy profiles A is a compact metric space and, for each player n, the payoff function $\pi_n(x_n, x_{-n})$ is upper semicontinuous and, for all x_n, continuous in x_{-n}. Considering the mixed extension of Γ in the usual fashion, we note that the continuity and compactness properties carry over under the weak topology.

DEFINITION. A strategy x_n for player n is *dominated* if there is another (possibly mixed) strategy y_n such that, for all x_{-n}, $\pi(y_n, x_{-n}) > \pi(x_n, x_{-n})$. A strategy that is not dominated is said to be *undominated*.

In games with finite strategy sets, each player has at least one undominated strategy, but this need not be so in infinite games. For example, in a one-player game in which the strategy space is $A = (0, 1)$ and the payoff $\pi(x) = x$, every strategy is dominated. Even when undominated strategies exist, it can happen that some strategies are dominated only by other dominated strategies. In Lemma 1, we exploit our continuity and compactness assumptions to rule out these possibilities, establishing that the infinite games we study display properties in this regard like finite games.

LEMMA 1. *For any dominated strategy x^0 of player n in Γ, there exists some undominated strategy y such that y dominates x^0.*

Proof. Fix a dominated strategy $x^0 \in A_n$ and consider the function f: $A_n \to R$ defined below:

$$f(z) \equiv \operatorname{Min}\left\{\pi_n(z, x_{-n}) - \pi_n\left(x^0, x_{-n}\right)\middle| x_{-n} \in A_{-n}\right\}.$$

By the continuity and compactness assumptions, the "Min" in the above expression exists and the function f is upper semicontinuous. Because x^0 is dominated, there is some $z \in A_n$ for which $f(z)$ is strictly positive. Let y be a maximizer of f over $z \in A_n$ (one exists because f is upper semicontinuous and A_n is compact). Then, $f(y) > 0$, so y dominates x^0. Any y' that dominates y would satisfy $f(y') > f(y)$, contradicting the definition of y. Hence, y is not itself dominated. ∎

The motivation for the finite-strategy analogs of the following definitions is found in Moreno and Wooders (1996).

DEFINITION. A *correlated strategy* is a probability distribution on A. Given a correlated strategy μ and a nonempty coalition S, a distribution ν on A is a *feasible deviation* for S if there exists a distribution η on $A_s \times A_s$ (where $A_s \equiv \prod_{n \in s} A_n$) such that, for all $B_s \subset A_s$,

$$\eta(A_s \times B_s) = \mu(B_s \times A_{-s})$$

and, for all $B \subset A$,

$$\nu(B) = \int_A \int_{\{x_s \mid (x_s, y_{-s}) \in B\}} \eta(dx_s \mid y_s)\, \mu(dy).$$

A correlated strategy μ is a *correlated equilibrium* if no player n has a feasible deviation that yields him a strictly higher expected payoff.

Remark. In interpreting the distribution η, we think of $\eta(x_s \mid y_s)$ as the probability that the deviating coalition S plays x_s when a mediator (for the coalition of the whole) has instructed it to play y_s, and we think of the marginal distribution $\eta_{y_s}(\cdot) = \eta(A_s \times \cdot\,)$ as specifying the probability that y_s is recommended by the mediator. The first displayed condition requires that $\eta_{y_s} = \mu_{y_s}$, that is, that the probability specified by η that the members of S receive a recommendation from the mediator to play any particular y_s agrees with that specified by μ. The second condition gives the probability of a strategy profile in B being played, given that $N \setminus S$ follows μ and S deviates in accordance with the recommendations given by η. The inner itegral is the probability that play lies in B given that a recommendation y has been made, and the outer integral is the probability weighted sum over possible recommendations. In this construction we are following Moreno and Wooders (1996) in limiting ourselves to what Ray (1996) calls "direct" correlated strategies and equilibria, where the message space used to communicate recommended actions to player n is itself the strategy space A_n.

For any set X, let ΔX denote the set of probability measures on X. If S is a coalition and μ is a correlated strategy, denote the set of feasible deviations for S from μ by $D(S, \mu)$.

We will be concerned with the set of stategies that survive iterated elimination of dominated strategies. Let $A^0 = A$ be the initial set of strategy profiles and let $A^k \subseteq A^{k-1}$ be defined recursively by eliminating from A^{k-1} all the dominated strategies in game $\Gamma^{k-1} \equiv (N, A^{k-1}, \pi)$. Since A^0 is nonempty and compact and the set of dominated strategies is open in A^0, A^1 is nonempty and compact, and the same holds recursively for A^k for all finite k. Hence, $A^\infty \equiv \cap A^k$ is nonempty and compact as well. Also, the set of correlated strategies with support in A^k is compact (in the weak topology) for $k = 1, \ldots, \infty$.

LEMMA 2. *For all finite k, the correlated equilibria of Γ^k are the same as those of Γ^{k-1}. Further, the correlated equilibria of Γ^∞ are the same as those of Γ.*

Proof. Clearly, no strategy profile involving a dominated strategy for some player ever lies in the support of any correlated equilibrium. It then follows that eliminating dominated stategies cannot reduce the set of correlated equilibria, so the correlated equilibria of Γ^k include those of Γ^{k-1}. Still, in infinite games, eliminating dominated strategies can sometimes introduce additional correlated equilibria. This occurs, however, only when some player has an improving deviation from the purported equilibrium in Γ^k, but all such deviations put positive measure on stategies in $A^{k-1} \setminus A^k$. Lemma 1 rules out that possibility for games satisfying our assumptions. Thus, the first part of the lemma is established, and a routine limiting argument establishes the second part. ∎

We now turn to specifying the possibilities for coalitional communications to select among and coordinate on feasible deviations. As suggested earlier, we want to allow that not all coalitions can necessarily communicate secretly to plan deviations and also that the ability of a coalition to plan secretly may depend on how many rounds of deviations have already been planned. The following formalism attempts to capture these *desiderata*.

Suppose for some game $\Gamma = (N, A, \pi)$ we have determined which coalitions can communicate secretly to plan first-level deviations from proposed correlated strategies, which subsets of each of these coalitions can communicate secretly to plan deviations from the first-level deviations, and so on. We can represent this by a *coalition communication structure*, which is a collection Σ of sequences σ of subsets of N, each of which is decreasing with respect to set inclusion (and thus finite). Here, $\sigma = (S_1, S_2, \ldots, S_T) \in \Sigma$ means that S_1 can communicate to deviate from an initial plan, that once S_1 has deviated then the members of $S_2 \subset S_1$ can communicate to plan a further deviation from that planned by S_1, and so on. To simplify notation, we will write (S, n) for the sequence $(S, \{n\})$.

Given a sequence $\sigma = (S_1, \ldots, S_T)$, consider the sequence consisting of the first t elements of σ. We shall refer to any such sequence as an *initial segment* of σ and use the notation $\underline{\sigma}$ to indicate a typical initial segment. If $\underline{\sigma}$ is an initial segment of σ for some $\sigma \in \Sigma$, then we say that $\underline{\sigma}$ is *initial* in Σ. Given a game Γ, a coalition communication structure Σ and an initial segment $\underline{\sigma} = (S_1, \ldots, S_t)$, suppose the coalitions in $\underline{\sigma}$ have deviated successively. Then the coalitions S that are able to plan deviations at the next stage are precisely those for which $(\underline{\sigma}, S)$ is initial in Σ.

We impose two conditions on coalition communication structures to ensure that the most relevant *individual* deviations are never ruled out.

Assumption 1. For each $n \in N$, $(n) \in \Sigma$.

Assumption 2. If (S) is initial in Σ and $n \in S$ then (S, n) is in Σ.

Coalition communications structures satisfying assumptions 1 and 2 will be called *admissible* and we henceforth limit attention to admissible structures. One example of such a structure is that containing every decreasing sequence of subsets of N. This structure, which is obviously admissible, is implicit in the analyses of Bernheim, Peleg, and Whinston; Moreno and Wooders; and Ray, all of which treat all coalitional deviations symmetrically. The admissible coalition communication structure corresponding to Nash or correlated equilibria is just that in which only the singleton coalitions can plan deviations: $\Sigma = \{\sigma = (n), n \in N\}$. Another example is that in which any coalition (including singletons) can plan first-level deviations, whereas the singleton coalitions alone can deviate from first-level deviations. We denote this coalition communication structure by Σ^*. Then $\sigma \in \Sigma^*$ if and only if either $\sigma = (n)$ for some $n \in N$ or $\sigma = (S, n)$ for some $s \subseteq N$ and $n \in S$. This is the coalition communication structure corresponding to Kaplan's semistrong equilibria and the strongly coalition-proof equilibria from the earlier version of this paper.

Given a coalition communication structure Σ and an initial segment $\underline{\sigma}$, define the *induced coalition communication structure* $\Sigma(\underline{\sigma})$ as the set of all sequences $\underline{\underline{\sigma}}$ such that $(\underline{\sigma}, \underline{\underline{\sigma}}) \in \Sigma$. Once the coalitions in $\underline{\sigma}$ have deviated, the opportunities for further coalitions to plan deviations are described by $\Sigma(\underline{\sigma})$. We refer to the pair $(\Gamma, \Sigma(\underline{\sigma}))$ as the game with the induced coalition structure. In this context, it will be convenient to allow that $\underline{\sigma}$ is trivial (that is, that no coalition has yet deviated), in which case $\Sigma(\underline{\sigma}) = \Sigma$.

With this notation in place, we can define coalition-proof correlated equilibria for games with coalition communication structures.

DEFINITION. A feasible deviation ν from a correlated strategy μ by a coalition S is *payoff-improving* if the expected utility of each member of S is at least as high under ν as under μ, with a strict inequality for some member of S. A feasible deviation ν from a correlated strategy μ by a coalition S is *self-enforcing* in $(\Gamma, \Sigma(\underline{\sigma}))$ if (S) is initial in $\underline{\sigma}$ and there is no S' and ξ with ξ a self-enforcing, payoff-improving deviation from ν for S' in $(\Gamma, \Sigma(\underline{\sigma}, S))$. A correlated strategy is a *coalition-proof correlated equilibrium* for (Γ, Σ) if no coalition S such that (S) is initial in Σ has a payoff-improving, self-enforcing deviation.

We remark that the definition of payoff-improving is unusual in that it does not require a strict increase in the payoffs of all members of the coalition. All of the results to follow, except Corollary 2, would continue to hold with the more usual definition that requires each member of the coalition to benefit strictly form a deviation for it to be payoff-improving.

Recall that Σ^* was defined above by $\sigma \in \Sigma^*$ if and only if either $\sigma = (n)$ for some $n \in N$ or $\sigma = (S, n)$ for some $S \subseteq N$ and $n \in S$.

LEMMA 3. *The coalition-proof correlated equilibria of* (Γ^k, Σ^*) (*denoted* C^k) *are the same as those of* (Γ^{k+1}, Σ^*) (*that is,* $C^k = C^{k+1}$). *Further, the coalition-proof correlated equilibria of* $(\Gamma^\infty, \Sigma^*)$ *are the same as those of* (Γ, Σ^*) (that is, $C^\infty = C^1$).

Note. In the arguments that follow, μ will denote the correlated strategy that is the initial proposal, ν will denote a "first-level" proposed deviation form μ by some coalition, and ξ will denote a further deviation by a subcoalition from ν.

Proof. Let $D^k(S, \mu) \subset \Delta A^k$ denote the set of feasible deviations for S from μ in Γ^k. Note that if $\mu \in \Delta A^{k+1}$ then $D^{k+1}(S, \mu) \subset D^k(S, \mu)$ for all coalitions S because $A^{k+1} \subset A^k$. Also, if $\nu \in \Delta A^\infty$, then $D^k(n, \nu)$ is compact for each n and $k = 1, \ldots, \infty$ (because A^k is compact and the constraint fixing the marginal distribution is closed).

We first argue that $C^k \subset C^{k+1}$. Suppose $\mu \in C^k$. Since μ is correlated equilibrium in Γ^k, it does not put positive probability on dominated strategies (Lemma 2), so $\mu \in \Delta A^{k+1}$. If $\mu \notin C^{k+1}$, then there exists a coalition S and a feasible, payoff-improving, self-enforcing deviation $\nu \in D^{k+1}(S, \mu)$ for S in (Γ^{k+1}, Σ^*). Since $D^{k+1}(S, \mu) \subset D^k(S, \mu)$, the deviation ν is feasible for S in (Γ^k, Σ^*). So, $\mu \in C^k$ implies that there is some $n \in S$ and some payoff-improving deviation ξ for n in $D^k(n, \nu)$. Then by Lemma 1, we can replace any dominated strategy for n in the support of ξ by another, undominated strategy (that is, one in A_n^{k+1}) that dominates it and increases n's payoff from the deviation. By construction, the revised strategy lies in $D^{k+1}(n, \nu)$, which contradicts the hypothesis that ν is self-enforcing for S in (Γ^{k+1}, Σ^*). Thus, the hypotheses that $\mu \in C^k$ and $\mu \notin C^{k+1}$ are inconsistent, proving that $C^k \subset C^{k+1}$.

To show $C^{k+1} \subset C^k$, we establish the contrapositive. Suppose $\mu \notin C^k$. If $\mu \notin \Delta A^{k+1}$ then it is immediate that $\mu \notin \Delta C^{k+1}$, so assume that $\mu \in \Delta A^{k+1}$. Then, there exists a coalition S and a $\nu \in \Delta A^k$ such that $\nu \in D^k(S, \mu)$ is payoff-improving for S and there is no player $n \in S$ with a payoff-improving feasible deviation ξ from ν in $(\Gamma^k, \Sigma^*(S))$. So, by the logic of Lemma 2, $\nu \in D^{k+1}(S, \mu)$. Hence, $\mu \notin C^{k+1}$ as was to be proved.

To establish the second part of the lemma, we first assume the $\mu \in C^k$ for all k and show that it is also in C^∞. First, since $\mu \in \bigcap C^k$, $\mu \in \Delta A^\infty$. If such a μ is not in C^∞ then there is a coalition S with a payoff-improving, self-enforcing deviation $\nu \in D^\infty(S, \mu) \subset \Delta A^\infty$. We will establish a contradiction to this. Note that ν was feasible and payoff-improving for S in each Γ^k and, in particular in $\Gamma^0 = \Gamma$, so there exists n in S such that n has a payoff-improving deviation $\xi \in D^0(n, \nu) = D(n, \nu)$. Define $\xi^0 \equiv \xi$, and if $\xi^k \in \Delta A^{k+1}$, define $\xi^{k+1} \equiv \xi^k$. If $\xi^k \notin \Delta A^{k+1}$, it must be the case that ξ^k includes in its support some strategies for n that are dominated in Γ^k and so are not available in Γ^{k+1}. But, in this case, replace each such strategy in the support of ξ^k by an undominated one that dominates it (Lemma 1), and then let ξ^{k+1} be the resulting deviation. Then $\xi^{k+1} \in D^{k+1}(n, \nu)$, and further, the utility gain to n from deviating to ξ^k is strictly positive and nondecreasing in k for all k. Now, consider any accumulation point of $\{\xi^k\}$. Because the $D^k(n, \nu)$ form a decreasing sequence of compact sets, such a point exists in $D^\infty(n, \nu)$. By construction, it is payoff-improving for n, which contradicts the assumption that ν is self-enforcing. Thus, if μ is a coalition-proof correlated equilibrium of (Γ^k, Σ^*) for each k, it is a coalition-proof correlated equilibrium in $(\Gamma^\infty, \Sigma^*)$.

Finally, suppose that $\mu \in C^\infty \setminus C^0$. Recall from the first part of the lemma that $C^0 = \bigcap C^k = \bigcup C^k$. Note that $\mu \in \Delta A^k$ for $k = 1, \ldots, \infty$. Since $\mu \notin C^k$ for each finite k, there is some S with a payoff-improving, self-enforcing deviation $\nu \in D^k(S, \mu)$. Note that $\nu \in \Delta A^{k+1}$: the players outside S are using only strategies in A^∞ and therefore in A^{k+1}, while if ν assigns positive probability to strategy profiles in which a member n of S plays a strategy in $A_n^k \setminus A_n^{k+1}$, then by Lemma 1 that player has a payoff-improving deviation form ν that consists in replacing the dominated strategies by dominating ones in A_n^{k+1}. In that case, ν would not be self-enforcing. Hence $\nu \in D^{k+1}(S, \mu)$ and, iteratively, $\nu \in D^k(S, \mu)$ for all finite k. Thus, in fact, $\nu \in \Delta A^\infty$ and so $\mu \notin C^\infty$, contradicting the hypothesis that $\mu \in C^\infty \setminus C^0$. ■

With this result, our first theorem follows trivially.

THEOREM 1. *If there is a Pareto-best element in* A^∞, *then the correlated strategy placing probability one on that pure strategy is a coalition-proof correlated equilibrium of* (Γ, Σ) *for any admissible coalition communication structure* Σ.

Proof. Note that the coalition-proof correlated equilibria of (Γ, Σ^*) are contained in those of (Γ, Σ) for any Σ satisfying Assumptions 1 and 2. This is because Σ^* allows the maximal set of possible self-enforcing deviations at the first round: it permits all coalitions to plan first-round deviations and allows only individuals to deviate from these. Now, let a be the Pareto-best element of A^∞. Then it defines a coalition-proof correlated equilibrium in $(\Gamma^\infty, \Sigma^*)$, because no coalition can have a payoff-improving deviation in ΔA^∞. The result then follows by Lemma 3. ■

COROLLARY 1. *If* μ *is a Pareto-best correlated strategy among those with support in* A^∞, *then it is a coalition-proof correlated equilibrium of* (Γ, Σ) *for any admissible* Σ.

Proof. If μ is Pareto-best from among those strategies with support in ΔA^∞, then all the pure strategies in its support must be payoff equivalent for all $n \in N$; otherwise, we could easily construct another correlated strategy that gave at least one player a higher payoff. Now apply the argument from Theorem 1. ■

COROLLARY 2. *Suppose* (N) *is initial in* Σ *and that* Σ *is admissible. If there is a Pareto-best element in* A^∞, *then it is (up to payoff equivalence for all* $n \in N$*) the unique coalition-proof correlated equilibrium of* (Γ, Σ).

Proof. Let a be the Pareto-best element in $A\infty$ and suppose that μ is a coalition-proof correlated equilibrium of (Γ, Σ) that is not payoff-equivilent to a. Then there is a coalition S all of whose members receive lower expected payoff at μ, while the members of $N \setminus S$ do no better at a. Then N has a payoff-improving, self-enforcing deviation to a. ■

Thus, the condition identified by Moreno and Wooders (1996) as sufficient for existence of coalition-proof correlated equilibrium with unlimited communication is also sufficient for the existence of coalition-proof correlated equilibrium for any arbitrary specification of coalitional opportunities for communication and coordination that meets Assumptions 1 amd 2. The condition, however, might seem quite demanding, so it is of special interest to identify a class of games in which it holds.

III. COALITION PROOFNESS AND STRATEGIC COMPLEMENTARITY

Recall that a lattice is a partially ordered set S with the property that for each x and y in S there exists a greatest lower bound for the set

$\{x, y\}$, denoted $x \wedge y$, and a least upper bound for $\{x, y\}$, denoted $x \vee y$. For example, R^k with the usual partial order is a lattice, where $x \wedge y = (\min\{x_1, y_1\}, \dots, \min\{x_k, y_y\})$ and $x \vee y = (\max\{x_1, y_1\}, \dots, \max\{x_k, y_k\})$. The following definitions are due to Milgrom and Shannon (1994).

DEFINITION. A function $f\colon X \to R$, where X is a lattice, is *quasisupermodular* if for all x and y in X, $f(x) \geq f(x \wedge y)$ implies $f(x \vee y) \geq f(y)$ and $f(x) > f(x \wedge y)$ implies $f(x \vee y) > f(y)$.

DEFINITION. A function $f\colon X \times T \to R$, where X and T are partially ordered sets, satisfies the *single crossing property* in $(x; t)$ if for all $x' > x''$ and $t' > t''$, $f(x', t'') > f(x'', t'')$ implies $f(x', t') > f(x'', t')$ and $f(x', t'') \geq f(x'', t'')$ implies $f(x', t') \geq f(x'', t')$.

DEFINITION. A game (N, A, π) is a *game with strategic complementarities* if

(i) each A_n is a compact lattice,

(ii) each π_n is upper semi-continuous in x_n and continuous in x_{-n}, and

(iii) each π_n is quasisupermodular in x_n and has the single crossing property in $(x_n; x_{-n})$.

If $A_n = R^k$, then condition (iii) may be replaced by the following equivalent condition:

(iii′) for all $x_n > y_n$, the sets $\{x_{-n} | \pi_n(x_n, x_{-n}) \geq \pi_n(y_n, x_{-n})\}$ and $\{x_{-n} | \pi_n(x_n, x_{-n}) > \pi_n(y_n, x_{-n})\}$ are both comprehensive upward.

This implies that the best-response correspondences are increasing. If $A_n = R$ and each π_n is twice differentiable with all the mixed partials nonnegative, then the game has strategic complementarities in the above sense. Thus, the class of games considered here includes those having strategic complementarities in the sense of Bulow, Geanakoplos, and Klemperer (1985). As well, it includes the class of supermodular games as defined in Milgrom and Roberts (1990). On the other hand, not all games with upward-sloping reaction curves are games with strategic complementarities.

THEOREM 2. *Let $\Gamma = (N, A, \pi)$ be a game with strategic complementarities. Then,*

(1) *if the strategy profile x' is the unique Nash equilibrium in pure strategies, or*

(2) *each π_n is nondecreasing in x_{-n} and x' is the largest element of A^∞,*

or

(3) *each* π_n *is nonincreasing in* x_{-n} *and* x' *is the smallest element of* A^∞,

then x' *is the unique coalition-proof correlated equilibrium of* (Γ, Σ) *for any admissible coalition communication structure* Σ *with* (N) *initial in* Σ. *Moreover,* x' *is the unique coalition-proof Nash equilibrium and semi-strong equilibrium.*

Proof. Milgrom and Shannon (1994) show that games with strategic complementarities have largest and smallest elements in the serially undominated set, that is, the set of (pure) strategies surviving iterated elimination of strategies that are dominated by other pure strategies. Further, these are the largest and smallest Nash equilibria. If there is a unique Nash equilibrium in pure strategies, then these coincide, so there is a unique element in A^∞, and so there is a Pareto-best element. Milgrom and Shannon also show that if the payoffs to each player are always nondecreasing (nonincreasing) in the strategies of the others, then the largest (smallest) element of the serially undominated set is the Pareto-best element in the set. Because this profile is a Nash equilibrium, it is also in A^∞ and is the Pareto-best element in A^∞. In either case, Corollary 2 applies. ■

IV. CONCLUSION

The symmetric treatment of all coalitions that is built into the usual definitions of coalition-proof Nash and correlated equilibrium carries an implicit assumption that all coalitions are fully and equally able to communicate secretly in planning deviations. This is not always a natural or appropriate assumption. In this paper we offered a model of coalitional communication possibilities and have extended the definition of coalition-proof correlated equilibrium to games with a coalition communication structure. We suggested that it was natural to limit consideration to coalition communication structures that allow any individual to deviate from an initial proposed correlated strategy or a first-level deviation by a coalition to which the individual player belongs. We then showed that the sufficient condition identified by Moreno and Wooders for the existence of a coalition-proof correlated equilibrium when coalitional communication is unrestricted also implies the existence of a coalition-proof correlated equilibrium with any communication structure meeting these conditions, and it does so even without the restriction to the finite strategies that they adopted. We also showed this condition is met in two classes of games with strategic complementarities that arise in economic applications.

APPENDIX

This appendix contains the proofs we originally offered in Milgrom and Roberts (1994b) for existence of semistrong (strongly coalition-proof) equilibria. These arguments may be of independent interest because of the way they utilize the structure of games with strategic complementarities.

Recall that in this section we are not allowing correlated strategies. Given a game $\Gamma = (N, A, \pi)$, a strategy z, and a coalition M, the *induced game* is the game $\Gamma(M, z) = (M, S_M, \hat{\pi})$ where for all $m \in M$ and $x_M \in S_M$, $\hat{\pi}_m(x_M) \equiv \pi_m(x_M, z_{N-M})$.

THEOREM A1. *Suppose that* $\Gamma = (N, A, \pi)$ *is a game with strategic complementarities and suppose that* Γ *has a unique Nash equilibrium* x. *Then for any coalition* $M \subset N$, *the induced game* $\Gamma(M, x)$ *has* x_M *as its unique Nash equilibrium.*

REMARK. In particular, it follows trivially that for any coalition $M \subset N$ and any Nash equilibrium y_M of $\Gamma(M, x)$, $\pi_M(y_M, x_{-M}) \leq \pi_M(x)$; that is, the members of M unanimously prefer the initial equilibrium x. The equilibrium x, being strongly coalition-proof, is automatically also coalition-proof. And, since x is the unique Nash equilibrium, it is the unique strongly coalition-proof equilibrium and the unique coalition-proof equilibrium.

Proof. Let x be the unique Nash equilibrium of Γ and let M be an arbitrary coalition. It follows directly from the definitions that the induced game $\Gamma(M, x)$ is a game with strategic complementarities and that x_M is a Nash equilibrium of the induced game. By a theorem of Milgrom and Shannon (1994), every game with strategic complementarities has a highest equilibrium y_M and a lowest equilibrium y_M'. To show that the Nash equilibrium of $\Gamma(M, x)$ is unique, we assume that there are multiple equilibria in $\Gamma(M, x)$ and use this to construct a second equilibrium in the original game Γ. Indeed, if there are multiple equilibria, then at least one of y_M and y_M' does not coincide with x_M. Assume $x_M \neq y_M$: the argument for $x_M \neq y_M'$ is essentially the same. Since y_M is the largest equilibrium componentwise, it follows that $y_M > x_M$.

Let $B : S \to S$ be the "largest best reply" function, mapping strategy profiles into the profile of largest best replies for each player. Since y_M is a Nash equilibrium of $\Gamma(M, x)$, $B_M(y_M, x_{-M}) \geq y_M$. Since x is a Nash equilibrium of Γ, $B_{-M}(x) \geq x_{-M}$ and since B is nondecreasing (Milgrom and Shannon, 1994), $B_{-M}(y_M, x_{-M}) \geq B_{-M}(x) \geq x_{-M}$. Hence, $B(y_M, x_{-M}) \geq (y_M, x_{-M})$. Then, since B is nondecreasing, it maps the set $T = \{z \in A \mid z \geq (y_M, x_{-M})\}$ into itself. By Tarski's fixed point theorem, B has a fixed point z^* on T and, by construction, z^* is a Nash equilibrium of

Γ. Also, $z^* \geq (y_M, x_{N-M}) > x$, contradicting the uniqueness of the Nash equilibrium x. ■

THEOREM A2. *Let (N, A, π) be a game with strategic complementarities and assume, in addition, that each individual payoff function $\pi_n(x_n, x_{-n})$ is monotonically nondecreasing in x_{-n}. Let x be the highest Nash equilibrium of the game. Let $M \subset N$ be a coalition and let y_M be a Nash equilibrium profile of the induced game $\Gamma(M, x)$. Then $y_M \leq x_M$ and $\pi_M(y_M, x_{-M}) \leq \pi_M(x)$.*

REMARK. The theorem asserts that the maximum equilibrium is unanimously preferred to any possible "secret agreement" that the players in any coalition M might make, even when that agreement is allowed to be any Nash equilibrium of the induced game. Hence, it is strongly coalition-proof and coalition-proof. Also, any other equilibrium is vulnerable to a deviation by the coalition of the whole to x, so x is the unique strongly coalition-proof equilibrium and the unique coalition-proof equilibrium.

Proof. Observe again that for any coalition M, the induced game $\Gamma(M, x)$ has strategic complementarities and has the property that for all $m \in M$, $\pi_m(x_m, x_{-m})$ is monotonically nondecreasing in x_{-m}. By a theorem of Milgrom and Shannon (1994), the induced game has a Pareto best equilibrium y_M, which is also the game's largest equilibrium. Since x_M is a Nash equilibrium profile for the induced game and y_M is the largest such profile, $y_M \geq x_M$. Hence, if the two are not equal, then $y_M > x_M$.

This allows us to argue exactly as in the proof of Theorem A1 that B has a fixed point z^* in the set $T = \{z \in A | z \geq (y_M, x_{-M})\}$ and, hence, that $z^* > x$, contradicting the hypothesis that x is the largest equilibrium of Γ. ■

REMARK. The assumption of strategic complementarities plays two roles in these proofs: ensuring, first, that there exists a largest Nash equilibrium in the original game and in each induced game and, second, that the largest best reply function B is nondecreasing. For the class of games with nondecreasing largest and smallest best reply functions there are largest and smallest *pure strategy* Nash equilibria. Thus, if attention is limited throughout to pure strategies, both Theorems A1 and A2 can be extended to this larger class of games.

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