Multipliers and the LeChatelier Principle

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1. Introduction

Those studying modern economies often puzzle about how small causes are amplified to cause disproportionately large effects. A leading example that emerged even before Samuelson began his professional career is the Keynesian multiplier, according to which a small increase in government spending can have a much larger effect on economic output. Before Samuelson's LeChatelier principle, however, and the subsequent research that it inspired, the ways that multipliers arise in the economy had remained obscure.

In Samuelson's original formulation, the LeChatelier principle is a theorem of demand theory. It holds that, under certain conditions, fixing a consumer's consumption of a good X reduces the elasticity of the consumer's compensated demand for any other good Y. If there are multiple other goods, X_1 through X_N, then fixing each additional good further reduces the elasticity. When this conclusion applies, it can be significant both for economic policy and for guiding empirical work. On the policy side, for example, the principle tells us that in a wartime economy, with some goods rationed, the compensated demand for other goods will become less responsive to price changes. That changes the balance between the distributive and efficiency consequences of price changes, possibly favoring the choice of non-price instruments to manage wartime demand. For empirical researchers, the same principle suggests caution in interpreting certain demand studies. For example, empirical studies of consumers' short-run responses to a gasoline price increase may underestimate their long response, since over the long run more consumers will be free to change choices about other economic decisions, such as the car models they drive, commute-sharing arrangements, uses of public transportation, and so on. However, the principle tells us those things only when its assumptions are satisfied, so Samuelson made repeated efforts during his career to weaken the assumptions needed for the principle to apply.[1]

Newer treatments of the LeChatelier principle differ in several important ways from Samuelson's original. First, while the original conclusion applies solely to the choices of an optimizing agent, the newer extensions apply also to many other equilibrium systems. Second, the original conclusion was a local principle that applied only to small parameter changes, while the modern extension is a global principle that applies to all parameter changes, large and small. Finally, the original principle gives at least the appearance of great generality, because it applies locally for any differentiable demand system, while the modern extension depends on a restriction. However, because the restriction always holds locally for differentiable demand systems, the modern principle actually subsumes the original.

All versions of the LeChatelier principle explain how the direct effect of a parameter change can be amplified by feedbacks in the systems in which they are embedded. Thus, the principles provide a foundation for understanding economic multipliers and, more generally, how it may be that small causes can have large effects.

2. A Local LeChatelier Principle for Optimization Problems

To explain Samuelson's original LeChatelier principle and set a context for the modern extensions, we restrict attention to the simplest form of the principle—one

[1] Samuelson (1947, 1949, 1960a, 1960b, 1972).

governing the choices of a profit-maximizing firm with just two inputs. Define the firm's unrestricted and restricted choice functions as follows:

$$x^U(w) \text{ solves } \max_x f(x) - w \bullet x \tag{1}$$

$$x^R(w, \bar{x}_2) \text{ solves } \max_x f(x) - w \bullet x \text{ subject to } x_2 = \bar{x}_2 \tag{2}$$

In the unrestricted problem (1), the firm maximizes profits over a set such as $\mathbb{R}^2_+$, choosing quantities of both inputs. In the restricted problem (2), the firm maximizes profits subject to the additional constraint that its "choice" for input 2 is exogenously given. Clearly, if the maximum is unique at the prices $\bar{w}$ and $\bar{x}_2 = x_2^U(\bar{w})$, then $x^U(\bar{w}) = x^R(\bar{w}, \bar{x}_2)$. Then, the traditional LeChatelier principle is the following result.

<u>Theorem 1</u>. Suppose that the functions $x^U(w)$ and $x^R(w, \bar{x}_2)$ are well defined and continuously differentiable in w_1 in a neighborhood of $w = \bar{w}$ and that $\bar{x}_2 = x_2^U(\bar{w})$.

Then, $\frac{\partial x_1^U}{\partial w_1}(\bar{w}) \le \frac{\partial x_1^R}{\partial w_1}(\bar{w}, \bar{x}_2) \le 0$.

<u>Proof</u>. Let $\pi^U(w) = \max f(x) - w \bullet x$ and $\pi^R(w, \bar{x}_2) = \max_x f(x) - w \bullet x$ subject to $x_2 = \bar{x}_2$ be the corresponding unrestricted and restricted profit functions. Since the value is always higher in a problem with fewer constraints, $\pi^U(w) \ge \pi^R(w, \bar{x}_2)$ and, by construction, $\pi^U(\bar{w}) = \pi^R(\bar{w}, \bar{x}_2)$.

By the envelope theorem, the profit functions are differentiable at $\overline{w}$ and the derivatives satisfy $x_1^U(\overline{w}) = -\frac{\partial \pi^U}{\partial w_1}(\overline{w}) = -\frac{\partial \pi^R}{\partial w_1}(\overline{w}, \overline{x}_2) = x_1^R(\overline{w}, \overline{x}_2)$. Then, by the results of the previous paragraph, $\frac{\partial x_1^U}{\partial w_1}(\overline{w}) = -\frac{\partial^2 \pi^U}{(\partial w_1)^2}(\overline{w}) \le -\frac{\partial^2 \pi^R}{(\partial w_1)^2}(\overline{w}, \overline{x}_2) = \frac{\partial x_1^R}{\partial w_1}(\overline{w}, \overline{x}_2)$. □

This is a "local" principle, because it allows comparative conclusions only for infinitesimal price changes. It cannot be directly extended to a global principle without extra assumptions. The following simple example, adapted from Milgrom and Roberts (1996), illustrates the problem.

Example. Suppose that a firm can produce one unit of output using two workers or using one worker and one unit of capital, or it can shut down and produce zero. It can also do any convex combination of these three activities. We represent the three extreme points of the firm's production possibility set by triples consisting of labor inputs, capital inputs, and output, as follows: (0,0,0), (-1,-1,1), and (-2,0,1). At an initial price vector of (.7,.8,2), the firm maximizes profits by choosing (-2,0,1), that is, it demands two units of labor and earns a profit of 0.6. If a wage increase leads to the new price vector (1.1,.8,2), then the firm's new optimum is (-1,-1,1), that is, it demands one unit of labor and earns a profit of 0.1. If capital is fixed in the short-run, however, then the firm must choose between its using two units of labor, which now earns -0.2, or shutting down and earning zero. So, the firm's short run demand for labor is zero. The important point is that labor demand adjusts *more* when capital is held fixed, in contrast to the conclusion of the LeChatelier principle.

3. Positive Feedbacks

We now consider a much more general approach to the LeChatelier conclusion that is not founded in optimization theory at all, but treats of the principle as a *global property* of positive feedback systems. We will show below how this theory specializes to yield a global LeChatelier principle for optimization models and how it implies Theorem 1.

For comparability with the preceding results, let us limit attention to a simple system of two equations, as follows:

$$\begin{aligned} x_1 &= f_1(\theta, \overline{x}_2) \\ x_2 &= f_2(\theta) \end{aligned} \tag{3}$$

The variables x_1, x_2 and $\overline{x}_2$ and the parameter θ are all real numbers.

We need to assume that f_1 is monotonic in the parameter. Since our central example is one with an input price parameter and the corresponding input choice, let us assume that f_1 is non-increasing in θ. Then, this system exhibits positive feedbacks if either of the following two conditions holds globally: (i) f_2 is non-decreasing and f_1 is non-increasing in x_2 or (ii) f_2 is non-increasing and f_1 is non-decreasing in x_2. When (i) holds, let us say that "the choices are substitutes" and when (ii) holds, that "the choices are complements."[2] This corresponds exactly to the use of these terms in the theory of the firm, subsuming the insight that the relation that two inputs are substitutes (complements) is a symmetric one.

[2] If f_1 is non-decreasing in θ, then the conditions change. In that case, we need that either (i) f_2 is non-increasing and f_1 is non-increasing in x_2 ("decisions are substitutes") or (ii) f_2 is non-decreasing and f_1 is non-decreasing in x_2 ("decisions are complements").

Theorem 2. Suppose that (i) or (ii) is satisfied (so the choices are substitutes or complements). If $\theta \geq \bar{\theta}$, then $f_1(\theta, f_2(\theta)) \leq f_1(\theta, f_2(\bar{\theta})) \leq f_1(\bar{\theta}, f_2(\bar{\theta}))$ and if $\theta \leq \bar{\theta}$, then $f_1(\theta, f_2(\theta)) \geq f_1(\theta, f_2(\bar{\theta})) \geq f_1(\bar{\theta}, f_2(\bar{\theta}))$.

According to the theorem, the unrestricted change is in the same direction as the restricted change and larger in magnitude, and this holds globally for any change in the parameter. The proof is quite trivial; it uses the fact that the composition of two non-increasing functions (or of two non-decreasing functions) is non-decreasing.

To apply this theorem to the model of a firm's input choices analyzed above, fix the price w_2 of input 2 and treat the parameter as being the price of input 1: $\theta = w_1$. Let f_1 and f_2 denote the restricted and unrestricted demands for inputs 1 and 2, respectively. In symbols, this means that $x_1^R(w, \bar{x}_2) = f_1(\theta, \bar{x}_2)$ and $x_2^U(w) = f_2(\theta)$. The unrestricted choice for input 1 is the same as the restricted choice when input 2 is chosen at its unrestricted level, so $x_1^U(w) = f_1(f_2(\theta), \theta)$. With these specifications, the theorem says that, provided inputs are (globally) either substitutes or complements and the price of input 1 increases ($w_1 \geq \bar{w}_1$), demand will fall by more in the unrestricted case than in the restricted case: $x_1^U(w) \leq x_1^R(w, x_2^U(\bar{w})) \leq x_1^U(\bar{w})$. The inequalities are all reversed for the case of a price decrease, so in that case demand rises by more in the unrestricted case. In both cases, unrestricted responses are larger.

The counterexample presented earlier, in which the conclusion of the LeChatelier principle fails, is a case where the positive feedbacks condition does not apply globally. In that example, the two inputs (capital and labor) are sometimes complements and sometimes substitutes. When the output price is 2 and capital costs .8 per unit, an increase in the wage rate from .7 to 1.1 causes the profit-maximizing firm to substitute capital for labor, switching from the production plan (-2,0,1) to the plan (-1,-1,1). For that range of prices, inputs are substitutes. When the wage further increases beyond 1.3, the firm switches to the plan (0,0,0), reducing its use of capital and revealing the inputs to be complements on that portion of the price domain. The pattern displayed in this example is not pathological and represents an economically significant restriction on the scope of the LeChatelier principle.

How can one check whether the complements or substitutes conditions are satisfied? Recall that a smooth function $f(x_1, x_2)$ is *supermodular* if the mixed partial derivative $\partial^2 f / \partial x_1 \partial x_2 \geq 0$ everywhere and is *submodular* if $-f$ is supermodular.

Theorem 3. Suppose there are *just two* choice variables. If $f(x_1, x_2)$ is supermodular and the optimal choices are unique, then the choices are complements. If $f(x_1, x_2)$ is submodular and the optimal choices are unique, then the choices are substitutes.

Theorem 3 also lends insight into the original Samuelson-LeChatelier principle. In a differentiable demand system, the production function *f* is twice differentiable. There are three cases, according to whether $\frac{\partial^2 f}{\partial x_1 \partial x_2}(\bar{x})$ is positive, negative, or zero. In the mixed partial derivative is positive, it is positive in a neighborhood of $\bar{x}$. In that case, inputs are complements in a neighborhood and, restricting attention to choices in the neighborhood, theorem 2 applies. Similarly, if the mixed partial derivative is negative, then the inputs are substitutes and theorem 2 applies. By continuity, the theorem also applies when the mixed partial derivative is zero (although in that case, the inequality of theorem 1 holds as an equality: $\frac{\partial x_1^U}{\partial w_1}(\bar{w}) = \frac{\partial x_1^R}{\partial w_1}(\bar{w}, \bar{x}_2)$).

The positive feedbacks approach to the LeChatelier principle can be extended to a much richer array of problems. Within optimization models, one can drop the assumption that optimal choices are unique at the cost of a slightly subtler statement about how the *set* of optima changes. One can also drop the assumptions that the objective is smooth and or that there are just two choice variables. Milgrom and Roberts (1996) develop these generalizations and others.

The LeChatelier conclusion, however, is not limited to optimization problems. One can also apply the positive feedbacks approach to study the behavior of fixed points of systems such as the following one:

$$x_1 = f_1(x_1, x_2, \theta) \tag{4}$$

$$x_2 = f_2(x_1, x_2, \theta) \tag{5}$$

Suppose that the relevant domain is some product set, say $f : [0,1]^3 \to [0,1]^2$. If *f* is non-decreasing in all its arguments then, by Tarski's theorem, there exist a maximum fixed point and a minimum fixed point and those are given by $x^{\max}(\theta) = \max\{x \mid f(x,\theta) \geq x\}$ and $x^{\min}(\theta) = \min\{x \mid f(x,\theta) \leq x\}$, and these are non-decreasing functions of θ.[3]

Positive feedback systems like (4) arise frequently in economic analysis and game theory (see Milgrom and Roberts (1990)). To simplify our study the LeChatelier effect in such systems, we focus on the *largest* fixed points of the system (a similar analysis

[3] For a more complete analysis, see Milgrom and Roberts (1994) and references therein.

applies to the *smallest* fixed points of the system). Thus, let $\bar{x}_2 = x_2^{\max}(\bar{\theta})$. Our goal is to compare changes in $x_1^{\max}(\theta)$ when the parameter changes with the corresponding changes in x_1 in the *restricted* system in which (5) is replaced by $x_2 = \bar{x}_2$. By the logic of the preceding paragraph, in the restricted system, the maximum fixed point for x_1 is $g_1(\theta,\bar{x}_2) \equiv \max\{x_1 \mid f(x_1,\bar{x}_2,\theta) \geq (x_1,\bar{x}_2)\}$, which is a non-decreasing function of both arguments. Let us define $g_2(\theta) \equiv x_2^{\max}(\theta)$. By a direct application of Theorem 2 to the pair of functions (g_1, g_2), we again get the LeChatelier conclusion, as follows:

Theorem 4. Suppose that f_1 and f_2 are nondecreasing, $\theta > \bar{\theta}$, and $x_1^{\max}$, g_1 and g_2 are defined as above. Then, $x_1^{\max}(\theta) \geq g_1(\theta, g_2(\bar{\theta})) \geq x_1^{\max}(\bar{\theta})$.

The conclusion, again, is that the change in the endogenous variable x_1 is larger when x_2 is free to change than when x_2 is restricted. The key is the positive feedback: the change in x_1 pushes x_2 up, and that in turn pushes x_1 up further.

4. Conclusion

In modern theory, Samuelson's LeChatelier principle has evolved into a principle for understanding multipliers. The original principle was limited to demand theory applications and reflects the symmetry of the substitution matrix, which implies that the relations of being substitutes or complements are symmetric relations. That symmetry creates a positive feedback system. For example, if capital and labor are complements, then an increase in the wage not only directly reduces the hiring of labor but also reduces the use of capital which leads to a further reduction the hiring of labor. Alternatively, if capital and labor are substitutes, then an increase in the wage not only directly reduces the hiring of labor but also increases the use of capital which leads to a further reduction in the hiring of labor. Including capital in the model can attenuate the direct effect of the wage increase only when capital is sometimes a substitute and sometimes a complement for labor.

To a modeler who finds that the direct effect of a parameter change cannot explain an observed effect, the LeChatelier principle analysis suggests a line of further analysis. It may be that the variable in question is part of a positive feedback system. Such systems amplify the direct effect of parameter changes. This reasoning is not limited to demand systems, nor to small parameter changes, nor to models with divisible choice variables. This knowledge is helpful not only for new applications, but also for thinking about the policy and empirical consequences ascribed, only sometimes correctly, to the original LeChatelier principle.

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