## Sherrington Kirkpatrick model

$H_N : \{\pm 1\}^N \to \mathbb{R}$

$H_N(\sigma) = \frac{1}{2}\langle \sigma, W\sigma\rangle \quad ; \quad W \sim GOE(N)$

$\left[\, W = \frac{1}{\sqrt{2N}}\,(G + G^T) \qquad (G_{ij})_{ij\le N} \sim_{iid} N(0,1) \quad \right]$

$[H_N(\sigma)]_{\sigma\in\{\pm 1\}^N}$ centered Gaussian process

$\mathbb{E}\{H_N(\sigma)H_N(\tau)\} = \frac{1}{2N}\langle \sigma,\tau\rangle^2$

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## General p-spin model

$\{H_N(\sigma)\}_{\sigma\in\{\pm1\}^N}$ centered Gaussian with

$\mathbb{E}\{H_N(\sigma)H_N(\tau)\} = N\xi\left(\frac{\langle \sigma,\tau\rangle}{N}\right)$ $\in [-1,1]$

$\xi(x) = \sum_{k\ge 2} c_k^2 x^k \qquad (SK : \xi(x) = x^2/2 \,)$

$H_N(\sigma) = \sum_{k\ge2} \frac{c_k}{k!}\langle W^{(k)}, \sigma^{\otimes k}\rangle$

$\mathbb{R}^{\otimes k} \ni W^{(k)}$ indep gaussian tensors

- non convex
- many local minima

$\begin{cases} \text{maximize} & H_N(\sigma) \\ \text{subj to} & \sigma\in\{\pm 1\}^N \end{cases}$

Q: Can we solve this approx in poly time?

Input $(W^{(k)})_{k\ge2} \to$ Output $\sigma^{alg}\in\{+1,-1\}^N$ st

$\mathbb{P}\left( H_N(\sigma^{alg}) \ge (1-\epsilon)\max_{\sigma} H_N(\sigma)\right) \xrightarrow{N\to\infty} 1$

$\sigma\in\{\pm1\}^N$

Is this possible $\forall\, \epsilon>0$ ?

If not $\forall\, \epsilon > \epsilon_* > 0$ ?

- Exact optimization.

$$\mathbb{P}\left(H_N(\sigma^{alg}) = \max_\sigma H_N(\sigma)\right) \to 1 \quad ?$$

- Refutation / Upper bd

$\mathrm{ALG}(W) \in \mathbb{R}$ st:

(1) $\mathrm{ALG}(W) \geq \max_\sigma H_N(\sigma)$

(2) $\mathbb{P}\left(\mathrm{ALG}(W) \leq (1+\epsilon)\max_\sigma H_N(\sigma)\right) \to 1$.

<u>Worst case</u> Unless P = NP no algorithm can do

$$\langle \sigma^{alg}, A\sigma^{alg}\rangle \geq \frac{1}{(\log N)^c} \max_{\sigma\in\{\pm1\}^N} \langle \sigma, A\sigma\rangle.$$

Typical value (Parisi's formula)

$$U := \left\{ \gamma : [0,1) \to \mathbb{R}_{\geq 0} \text{ nondecr. } \int_0^1 \gamma(t)\,dt < \infty \right\}$$

$$\partial_t \Phi(t,x) + \frac{1}{2}\xi''(t)\left[\partial_x^2 \Phi(t,x) + \gamma(t)(\partial_x\Phi(t,x))^2\right] = 0$$

$[0,1]\times\mathbb{R}$ $\quad \Phi(1,x) = |x|$ , $\Phi_\gamma$

$$P(\gamma) = \Phi_\gamma(0,0) - \frac{1}{2}\int_0^1 t\,\xi''(t)\,\gamma(t)\,dt \quad ; \quad \mathbb{E} H(\sigma)H(\tau) = N\,\xi\!\left(\frac{\tau\cdot\sigma}{N}\right)$$

<u>Thm</u> $\quad \mathrm{OPT}_N = \frac{1}{N}\max_\sigma H_N(\sigma)$

$$\lim_{N\to\infty} \mathrm{OPT}_N = \inf_{\gamma\in U} P(\gamma)$$

<u>Thm</u> $\gamma \mapsto P(\gamma)$ is strictly convex, $\inf_{\gamma\in U} P(\gamma)$ achieved at unique $\gamma_* \in U$.

# Interpretation of $\gamma_*$

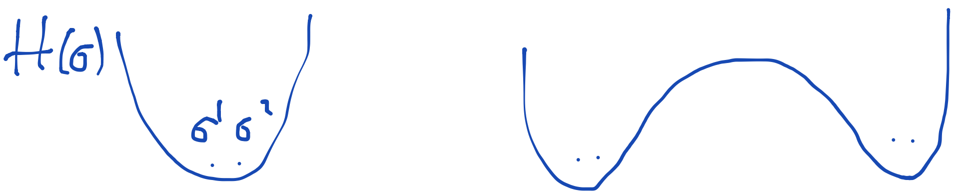
Consider Gibbs measure

$$\mu_{N,\beta}(\sigma) = \frac{1}{Z_{N,\beta}} e^{\beta H_N(\sigma)}$$

$$S_\epsilon := \{\sigma \in \{\pm 1\}^N : H_N(\sigma) \gtrsim (1-\epsilon) \max_\sigma H_N(\sigma)\}$$

$$\mu_{N,\beta} \approx \mathrm{Unif}(S_\epsilon) \qquad \epsilon = \epsilon_*(\beta)$$

$$(\sigma^1, \sigma^2)\big|_W \sim \mu_{N,\beta} \otimes \mu_{N,\beta}$$




$P_{\beta,N}$ law of $\frac{|\langle \sigma^1, \sigma^2 \rangle|}{N}$

$$P_{\beta,N} \overset{N\to\infty}{\Longrightarrow} \nu_\beta \quad \text{on } [0,1]$$

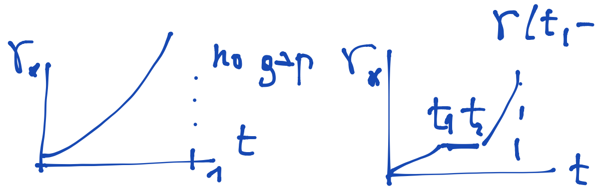
$$\gamma_*(t) = \lim_{\beta\to\infty} \beta \, \nu_\beta([0,t])$$

Structure of $\gamma_*$ ?

I : No overlap gap : $\gamma_*$ strictly incr on $[0,1)$

II : Overlap gap : $\exists (t_1, t_2) \subseteq [0,1)$ st

$$\gamma(t_1 - \epsilon) < \gamma(t_1 +) = \gamma(t_2 -) < \gamma(t_2 + \epsilon)$$




$\nu_\beta([0,t])$ strictly increasing for $t \in [0, q_*]$ constant above

$$\mathbb{P}\left(\exists\, \sigma^1, \sigma^2 \in S_\epsilon ; \frac{|\langle \sigma^1, \sigma^2 \rangle|}{N} \in (t-\delta, t+\delta)\right) \to 1$$

$\forall t \in (0, q_*)$

Algorithms

$$\mathcal{L} := \left\{ r:[0,1) \to \mathbb{R}_{\geq 0} : \|\xi'' r\|_{TV[0,t]} < \infty \ \forall t \in [0,1),\ \int_0^1 \xi'' r(t)\,dt < \infty \right\}$$

$\xi'' r(t) = \xi''(t) \cdot \gamma(t)$.

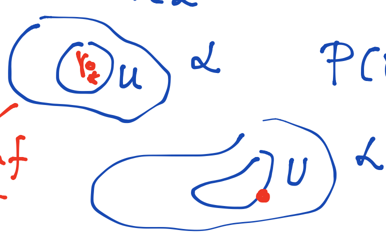
$\mathcal{L} \supseteq U = \{ r \text{ non decr. } \int_0^1 r(t)\,dt < \infty \}$

$U = \mathcal{L} \cap \{\text{non decr}\}$

Thm Assume $\inf_{r \in \mathcal{L}} P(\gamma)$ is achieved. Then $\forall \epsilon > 0\ \exists$ alg with linear complexity st

$$\mathbb{P}\left(\frac{1}{N} H_N(\sigma) \gtrsim \inf_{\gamma \in \mathcal{L}} P(\gamma) - \epsilon\right) \xrightarrow[N \to \infty]{} 1 \quad \square$$

Rmk $\inf_{r \in \mathcal{L}} P(\gamma) \lesssim \inf_{\gamma \in U} P(\gamma)$




$P(\gamma)$

$\inf_U = \inf_{\mathcal{L}}$ → near opt

$\inf_{\mathcal{L}} < \inf_U$ → const fact.

Lem If no overlap gap, then

$$\underbrace{\inf_{r \in \mathcal{L}} P'(\gamma)}_{\text{ALG}} = \underbrace{\inf_{\gamma \in U} P(\gamma)}_{\text{OPT}} \quad \#$$

Corollary If no gap $\nexists\ \forall \epsilon > 0\ \exists$ linear time alg

s.t. $\mathbb{P}\left(H_N(\sigma^{alg}) \gtrsim (1-\epsilon) \max_\sigma H_N(\sigma)\right) \xrightarrow{N \to \infty} 1$ $\square$

Conjecture For SK no overlap gap $\square$

Conjecture : If overlap gap, no poly time alg $\forall \epsilon >$

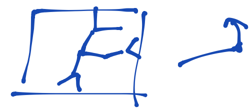
Unless $P = NP$ □

Proof * Construct AMP algo (SK)

$$x^{t+1} = W f_t(x^1,\dots x^t) - \sum_{s=1}^{t} d_{t,s} f_{s-1}(x^1,\dots x^{s-1})$$

* $z^t = f_t(x^1,\dots x^t)$

$\frac{1}{n}\langle z^{t+1} - z^t, z^t\rangle \approx 0$



* $t \in [0, \delta, 2\delta, \dots, 1-\delta, 1]$

$\delta \to 0$

* st. evolution ⇒ SDE (drift)
* Choose coefficients of SDE opt.

⇒ solving a stoch. opt. contr.

⇒ $\inf_{\gamma \in \mathcal{L}} P(\gamma)$

$\sum_k c_k^2 x^k$

$x^0 \approx N(0, \delta I_d)$.