Math 205A Real Analysis

Practice Final

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Solutions should be complete and concisely written. Please, mark clearly the beginning and end of each problem.

You have 3 hours but you are not required to solve all the problems!

Just solve those that you can solve within the time limit. Points assigned to each problem are indicated in parenthesis. I recommend to look at all problems before starting.

For any clarification on the text, the TA's will be outside the room.

You can consult textbooks (Billinglsely, Cohn) and your notes. You cannot use computers, and in particular you cannot use the web. You can cite theorems (propositions, corollaries, lemmas, etc.) from textbooks by number, and exercises you have done as homework by number as well. Any other non-elementary statement must be proven!

Problem 1 (30 points)

- Let $\mathbb{X}_1$ denote the one-point compactification of $\mathbb{R}$ (i.e. $\mathbb{X}_1 := \mathbb{R} \cup \{\infty\}$ with open sets being $\mathcal{O}_1 := \{U \subseteq \mathbb{R} : U \text{ open}\} \cup \{K^c \cup \{\infty\} : K \subseteq \mathbb{R}, K \text{ compact}\}$).
- Let $\mathbb{X}_2$ denote the two-points compactification of $\mathbb{R}$ (i.e. $\mathbb{X}_2 = \mathbb{R} \cup \{+\infty, -\infty\}$ with open sets being $\mathcal{O} = \{U \subseteq \mathbb{R} : U \text{ open}\} \cup \{K^c \cup \{+\infty\} : K \subseteq \mathbb{R} \text{ closed upper bounded}\} \cup \{K^c \cup \{-\infty\} : K \subseteq \mathbb{R} \text{ closed lower bounded}\} \cup \{K^c \cup \{+\infty, -\infty\} : K \subseteq \mathbb{R} \text{ compact}\}$).

With the above definitions:

(*a*) Describe the spaces of functions $C_c(\mathbb{X}_1)$, $C_c(\mathbb{X}_2)$ (continuous compactly supported functions) in terms of spaces of functions on $\mathbb{R}$.

(*b*) Define the functional $L(f) := \lim_{x\to+\infty} f(x)$. Prove that $L(f)$ is a linear functional either on $C_c(\mathbb{X}_1)$, $C_c(\mathbb{X}_2)$. Apply Riesz-Markov-Kakutani's Theorem. What are the corresponding measures μ_1, μ_2 representing L, respectively, on $C_c(\mathbb{X}_1)$, $C_c(\mathbb{X}_2)$?

(*c*) Consider now $T(f) := \lim_{x\to+\infty} f(x) - \lim_{x\to-\infty} f(x)$. Show that for each $i \in \{1, 2\}$ there exists a signed measure ν_i such that

$$f \in C_c(\mathbb{X}_i) \quad \Rightarrow \quad T(f) = \int f(x)\, \nu_i(\mathrm{d}x)\,, \tag{1}$$

and characterize the measures ν_i.

Problem 2 (30 points)

Let $\mathcal{C} \subseteq [0, 1]$ be the Cantor set (i.e. $\mathcal{C} = \cap_{n=0}^{\infty} \mathcal{C}_n$, where $\mathcal{C}_0 = [0, 1]$ and $\mathcal{C}_{n+1} = (\mathcal{C}_n/3) \cap ((2 + \mathcal{C}_n)/3)$) Let $F : [0, 1] \to [0, 1]$ be the Cantor function (i.e. $F(t) = \lim_{n\to\infty} F_n(t)$, where $F_n(t) = \mu_n([0, t])$ for μ_n the uniform measure over $\mathcal{C}_n$).

(*a*) Prove that F is differentiable at any $x \in [0, 1] \setminus \mathcal{C}$.

(*b*) Prove that F is not differentiable at any $x \in \mathcal{C}$.

(*c*) What does Lebesgue's differentiation theorem imply about the differentiability of F?

Problem 3 (30 points)

Let μ_1, μ_2 be two σ-finite measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, with $\mu_1(\mathbb{R}), \mu_2(\mathbb{R}) > 0$. Denote by $\mu_1 \otimes \mu_2$ their product (which is a measure on $(\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2})$). For $\varepsilon \geq 0$, define $L_\varepsilon := \{(x, y) \in \mathbb{R}^2 : |x - y| \leq \varepsilon\}$ and assume that

$$(\mu_1 \otimes \mu_2)(\mathbb{R}^2 \setminus L_\varepsilon) = 0 \, . \tag{2}$$

(a) Prove that, for μ_1-almost all x, we have $\mu_2([x - \varepsilon, x + \varepsilon]^c) = 0$.

(b) Deduce that there exists x_i such that $\mu_i([x_i - \varepsilon, x_i + \varepsilon]^c) = 0$ for all $i \in \{1, 2\}$.

(c) Can you strengthen the above conclusion? Namely, let $a_i \leq b_i$ be defined by $a_i := \sup\{x : \mu_i((-\infty, x)) = 0\}$, $b_i := \inf\{x : \mu_i((x, +\infty)) = 0\}$. What is the best upper bound on $\max(|a_1 - b_1|, |a_2 - b_2|)$ under Assumption (2)?

Problem 4 (20 points)

Let $(\mathbb{X}, \mathcal{F}, \mu)$ be a σ-finite measure space, $\mathcal{G} \subseteq \mathcal{F}$ a sub σ-algebra and $f \in L^1(\mathbb{X}, \mathcal{F}, \mu)$.

(a) Prove that there exists a function $g \in L^1(\mathbb{X}, \mathcal{G}, \mu)$ such that

$$\int f \, \mathbf{1}_A \mathrm{d}\mu = \int g \, \mathbf{1}_A \mathrm{d}\mu \, , \quad \forall A \in \mathcal{G} \, . \tag{3}$$

(b) Prove that any two functions g_1, g_2 satisfying the above condition are a.e. equal.