

Homework 1 - Solution

Problem 1

- (a) Let $\mathcal{G} = \bigcap_{\alpha} \mathcal{F}_{\alpha}$, with each \mathcal{F}_{α} a σ -algebra. Since \mathcal{F}_{α} a σ -algebra, we have that $\Omega \in \mathcal{F}_{\alpha}$, and as this applies for all α , it follows that $\Omega \in \mathcal{G}$. Suppose now that $A \in \mathcal{G}$. That is, $A \in \mathcal{F}_{\alpha}$ for all α . Since each \mathcal{F}_{α} is a σ -algebra, it follows that $A^c \in \mathcal{F}_{\alpha}$ for all α , and hence $A^c \in \mathcal{G}$. Similarly, let $A = \bigcup_i A_i$ for some countable collection A_1, A_2, \dots of elements of \mathcal{G} . By definition of \mathcal{G} , necessarily $A_i \in \mathcal{F}_{\alpha}$ for all i and every α . Since \mathcal{F}_{α} is a σ -algebra, we deduce that $A \in \mathcal{F}_{\alpha}$, and as this applies for all α , it follows that $A \in \mathcal{G}$.
- (b) We verify the conditions for σ -algebra.
- (a) $\Omega \in \mathcal{G}$ and $\Omega \cap H = H \in \mathcal{H}$. Hence $\Omega \in \mathcal{H}^H$.
- (b) Suppose $A \in \mathcal{H}^H$. Since \mathcal{G} is a σ -algebra and $A \in \mathcal{G}$, we have $A^c \in \mathcal{G}$. Note that $A^c \cap H = (A \cap H)^c \cap H$. Since by definition $A \cap H \in \mathcal{H}$, we have $A^c \cap H \in \mathcal{H}$ as well. Hence $A^c \in \mathcal{H}^H$.
- (c) Suppose $A_i \in \mathcal{H}^H$ for $i \in \mathbb{N}$. Since $A_i \in \mathcal{G}$, $\bigcup_i A_i \in \mathcal{G}$. Also, $(\bigcup_i A_i) \cap H = \bigcup_i (A_i \cap H) \in \mathcal{H}$ since each component $A_i \cap H \in \mathcal{H}$. Thus, $\bigcup_i A_i \in \mathcal{H}^H$.

Therefore, \mathcal{H}^H as defined is a σ -algebra.

- (c) Suppose we have $H_1 \subseteq H_2$. We want to show that $\mathcal{H}^{H_2} \subseteq \mathcal{H}^{H_1}$. In fact, given any $A \in \mathcal{H}^{H_2}$, since $H_1 \subseteq H_2$, we have $A \cap H_1 = (A \cap H_2) \cap H_1$. $A \cap H_2 \in \mathcal{H}$ by definition and we also know $H_1 \in \mathcal{H}$. This implies $A \cap H_1 \in \mathcal{H}$. Also, $A \in \mathcal{G}$ by definition. Thus, $A \in \mathcal{H}^{H_1}$. Since the choice of A is arbitrary, we conclude $\mathcal{H}^{H_2} \subseteq \mathcal{H}^{H_1}$.

$\mathcal{H}^{\Omega} = \{A \in \mathcal{G} : A \cap \Omega \in \mathcal{H}\} = \{A \in \mathcal{G} : A \in \mathcal{H}\} = \mathcal{H}$. On the other hand, $\mathcal{H}^{\emptyset} = \{A \in \mathcal{G} : A \cap \emptyset \in \mathcal{H}\} = \{A \in \mathcal{G} : \emptyset \in \mathcal{H}\} = \mathcal{G}$ due to the fact that whichever A is chosen in \mathcal{G} , \emptyset is always in \mathcal{H} .

First note $H \subseteq H \cup H'$. By the monotonicity derived above, $\mathcal{H}^{H \cup H'} \subseteq \mathcal{H}^H$. For the same reason, $\mathcal{H}^{H \cup H'} \subseteq \mathcal{H}^{H'}$. This results in one direction, $\mathcal{H}^{H \cup H'} \subseteq \mathcal{H}^H \cap \mathcal{H}^{H'}$. We are left to prove the other direction. In fact, if $A \in \mathcal{H}^H \cap \mathcal{H}^{H'}$, we have $A \cap H \in \mathcal{H}$ and $A \cap H' \in \mathcal{H}$, and thus $A \cap (H \cup H') = (A \cap H) \cup (A \cap H') \in \mathcal{H}$. By definition, we know $A \in \mathcal{H}^{H \cup H'}$. Therefore, $\mathcal{H}^H \cap \mathcal{H}^{H'} \subseteq \mathcal{H}^{H \cup H'}$. We conclude $\mathcal{H}^H \cap \mathcal{H}^{H'} = \mathcal{H}^{H \cup H'}$.

Problem 2

It suffices to show that if \mathcal{F} is the σ -algebra generated by $\{(a_1, b_1) \times \dots \times (a_d, b_d)\}$, then \mathcal{F} contains (a) the open sets and (b) all sets of the form $A_1 \times \dots \times A_d$ where $A_i \in \mathcal{B}$. For (a), note that if G is open and $x \in G$ then there is a set of the form $(a_1, b_1) \times \dots \times (a_d, b_d)$ with $a_i, b_i \in \mathbb{Q}$ that contains x and lies in G , so any open set is a countable union of these basic sets ($(a_1, b_1) \times \dots \times (a_d, b_d)$ with $a_i, b_i \in \mathbb{Q}$). In this argument we relied on the fact that there are only countably many such basic sets, hence we are not bothered by the fact that there are uncountably many points x in G .

For (b), fix A_2, \dots, A_d and let $\mathcal{G} = \{A : A \times A_2 \times \dots \times A_d \in \mathcal{F}\}$. Since \mathcal{F} is a σ -algebra it is easy to see that if $\mathbb{R} \in \mathcal{G}$ then \mathcal{G} is a σ -algebra so if $\mathcal{G} \supseteq \dot{A}$ then $\mathcal{G} \supseteq \sigma(\dot{A})$. Applying this for $A_i = (a_i, b_i)$, $i = 2, \dots, d$

it follows that if $A_1 \in \mathcal{B}$ then $A_1 \times (a_2, b_2) \times \cdots \times (a_d, b_d) \in \mathcal{F}$. Repeating now the preceding argument for $\mathcal{G} = \{A : A_1 \times A \times A_3 \cdots \times A_d \in \mathcal{F}\}$, $A_1 \in \mathcal{B}$ and $A_i = (a_i, b_i)$, $i = 3, \dots, d$, shows that if $A_1, A_2 \in \mathcal{B}$, then $A_1 \times A_2 \times (a_3, b_3) \times \cdots \times (a_d, b_d) \in \mathcal{F}$. Applying this type of argument $d-2$ more times, proves the assertion (b).

Problem 3

Omitted.

Problem 4

A1

For the first direction, let $f : A \rightarrow \mathbb{R}$ be an equidecomposition. and define $B_i = f(A_i) = B_i + t_i$ for $i \in \mathbb{N}$. Since $\{A_i\}$ is a countable partition of A , it is sufficient to show that $\{B_i\}$ is a countable partition of B . Indeed $B_i \cap B_j = \emptyset$ for $i \neq j$ follows from the injectivity of f (because otherwise there would be $y \in B_i \cap B_j$ whence $y = f(x_i)$, and $y = f(x_j)$ for some $x_i \in A_i$, $x_j \in A_j$ distinct). Further, $f(A) = \bigcup_i f(A_i) = \bigcup_i B_i$, and since f is surjective, $f(A) = B$.

To prove the converse, assume $\{A_i\}$ and $\{B_i\}$ to be partitions (respectively) of A and B , and let $\{t_i\}$ be the such that $B_i = A_i + t_i$. Define f by letting $f|_{A_i} = R_{t_i}|_{A_i}$. This map is clearly bijective (with $f^{-1}|_{B_i} = R_{-t_i}|_{A_i}$).

A2

Let $A' \subseteq A$, $B' \subseteq B$, and consider the bijective equidecompositions $f : A \rightarrow B'$ and $g : B \rightarrow A'$.

As suggested, we define $A^{(0)} \equiv A \setminus g(B)$, and $A^{(*)} \equiv \bigcup_{n=0}^{\infty} (g \circ f)^n(A^{(0)})$. Let $h : A \rightarrow B$ be defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A^{(*)}, \\ g^{-1}(x) & \text{if } x \in A \setminus A^{(*)}. \end{cases} \quad (1)$$

Notice that h is well defined because $A \setminus A^{(*)} \subseteq A \setminus A^{(0)} = g(B)$. Further, it is a countable equidecomposition. To prove this, consider the partitions $A = \bigcup_{i=1}^{\infty} A_i$ and $B = \bigcup_{i=1}^{\infty} B_i$, with respect to which f and g are (respectively) equidecompositions with translation parameters $\{t_i\}$ and $\{s_i\}$. Then

$$A = \left\{ \bigcup_{i=1}^{\infty} (A_i \cap A^{(*)}) \right\} \cup \left\{ \bigcup_{i=1}^{\infty} (g(B_i) \cap (A \setminus A^{(*)})) \right\} \quad (2)$$

is a countable partition of A and it is easy to check that h is an equidecomposition with respect to this partition. Indeed $h|_{A_i \cap A^{(*)}} = f|_{A_i \cap A^{(*)}} = R_{t_i}|_{A_i \cap A^{(*)}}$ and $h|_{g(B_i) \cap (A \setminus A^{(*)})} = g^{-1}|_{g(B_i) \cap (A \setminus A^{(*)})} = R_{-s_i}|_{g(B_i) \cap (A \setminus A^{(*)})}$.

It remains to prove that h is bijective. To this end, define the mapping $l : B \rightarrow A$ by

$$l(y) = \begin{cases} f^{-1}(y) & \text{if } g(y) \in A^{(*)}, \\ g(y) & \text{otherwise.} \end{cases} \quad (3)$$

It is not hard to prove that l is the inverse of h . Indeed, if $x \equiv g(y) \notin A^{(*)}$, then $h(x) = g^{-1}(x) = y$. On the other hand, if $g(y) \in A^{(*)}$, then $g(y) = (g \circ f)^k(A^{(0)})$ for some $k \geq 1$ (because $A^{(0)} \cap g(B) = \emptyset$). By injectivity of g , $y = f((g \circ f)^{k-1}(x_0))$ for some $x_0 \in A^{(0)}$. If we let $x = l(y) = f^{-1}(y)$, then $x \in A^{(*)}$ as well, whence $h(x) = f(x) = y$.

B1

Consider the equivalence relation $x \sim y$ if $(x - y) \in \mathbb{Q}$, and let \mathcal{EE} denote the collection of its equivalence classes. For every $E \in \mathcal{EE}$, $E \cap [0, 1/2]$ is non-empty and by the axiom of choice, there exist a choice function $E \mapsto x_E$ such that $x_E \in E \cap [0, 1/2]$ for each E . Obviously $E = x_E + \mathbb{Q}$. Therefore

$$\mathbb{R} = \cup_{E \in \mathcal{EE}} E = \cup_{E \in \mathcal{EE}} \{x_E + \mathbb{Q}\} = \cup_{x \in C} \{x + \mathbb{Q}\}, \quad (4)$$

where $C = \{x_E\}_{E \in \mathcal{EE}}$.

B2

Since the rationals are countable, there exist partitions in isolated points

$$\mathbb{Q} \cap [0, 1/2] = \cup_{i=1}^{\infty} \{q_i\}, \quad \mathbb{Q} = \cup_{i=1}^{\infty} \{p_i\}. \quad (5)$$

Of course, for any i , $\{p_i\} = R_{t_i}(\{q_i\})$ if we set $t_i = p_i - q_i$.

B3

Using the enumerations of $\mathbb{Q} \cap [0, 1/2]$ and of \mathbb{Q} at the previous point, we get

$$A \equiv \cup_{x \in C} \{x + (\mathbb{Q} \cap [0, 1/2])\} = \cup_{i=1}^{\infty} \{q_i + C\}, \quad \mathbb{R} = \cup_{x \in C} \{x + \mathbb{Q}\} = \cup_{i=1}^{\infty} \{p_i + C\}. \quad (6)$$

We have $\{p_i + C\} = R_{t_i}(\{q_i + C\})$ if we set $t_i = p_i - q_i$ as above.

B4

Clearly $[0, 1]$ is equidecomposable with $[0, 1] \subseteq \mathbb{R}$ (via the identity mapping). On the other hand \mathbb{R} is equidecomposable with $A \subseteq [0, 1]$. By points A1, A2, this implies that $[0, 1]$ is equidecomposable with \mathbb{R} .

This implies that there exists no measure on \mathbb{R} satisfying the following requirements

1. The measure is countably additive (As it should be).
2. $\mu([0, 1]) \notin \{0, \infty\}$.
3. Any set is measurable.
4. $\mu(S) = \mu(R_t S)$ for any measurable set S .