

## Homework 1

- Return your assignments via Gradescope
- Solutions should be complete and concisely written. You can reference results/statements in either of the textbooks. Any other non-elementary fact must be proven.
- You are welcome to discuss problems with your colleagues, but should write and submit your own solution.
- Solutions are due on Thu, by 11:59PM.

**Problem 1**

- (a) Prove that the intersection of (possibly uncountably many)  $\sigma$ -algebras is also a  $\sigma$ -algebra.
- (b) Verify that for any  $\sigma$ -algebras  $\mathcal{H} \subseteq \mathcal{G}$ , and any  $H \in \mathcal{H}$ , the collection  $\mathcal{H}^H := \{A \in \mathcal{G} : A \cap H \in \mathcal{H}\}$  is a  $\sigma$ -algebra.
- (c) Show that  $H \mapsto \mathcal{H}^H$  is non-increasing with respect to set inclusions, with  $\mathcal{H}^\Omega = \mathcal{H}$  and  $\mathcal{H}^\emptyset = \mathcal{G}$ . Deduce that  $\mathcal{H}^{H \cup H'} = \mathcal{H}^H \cap \mathcal{H}^{H'}$  for any pair  $H, H' \in \mathcal{H}$ .

**Problem 2**

Define the following  $\sigma$  algebras (for  $d < \infty$ ):

- $\mathcal{B}_{\mathbb{R}^d}$ , the Borel  $\sigma$ -algebra on  $\mathbb{R}^d$  (endowed with the usual topology).
- $\mathcal{F}_{\mathbb{R}^d} = \mathcal{B}_{\mathbb{R}} \times \cdots \times \mathcal{B}_{\mathbb{R}} := \sigma(\{A_1 \times \cdots \times A_d : A_i \in \mathcal{B}_{\mathbb{R}} \forall i\})$ .
- $\mathcal{G}_{\mathbb{R}^d} := \sigma(\{(a_1, b_1) \times \cdots \times (a_d, b_d) : a_i < b_i \in \mathbb{R}, \forall i\})$ .

Prove that

$$\mathcal{B}_{\mathbb{R}^d} = \mathcal{F}_{\mathbb{R}^d} = \mathcal{G}_{\mathbb{R}^d}. \quad (1)$$

**Problem 3**

Suppose that  $\Omega$  is an uncountable set. Define functions  $\mu^*$  and  $\nu^*$  on  $2^\Omega$  by:

$$\mu^*(S) = \begin{cases} 0 & \text{if } S \text{ is countable,} \\ 1 & \text{if } S \text{ is uncountable,} \end{cases} \quad (2)$$

$$\nu^*(S) = \begin{cases} 0 & \text{if } S \text{ is countable,} \\ \infty & \text{if } S \text{ is uncountable.} \end{cases} \quad (3)$$

- (a) Prove that  $\mu^*, \nu^*$  are outer measures.
- (b) Find all sets that are  $\mu^*$ -measurable. (Prove your answer)
- (b) Find all sets that are  $\nu^*$ -measurable. (Prove your answer)

### Problem 4: The Banach-Tarski paradox in one dimension

The objective of this homework is to prove a simplified version of the Banach-Tarski paradox, in the case of the real line. The definition of equidecomposable subsets of  $\mathbb{R}$  is provided below.

**Definition 1.** *The sets  $A, B \in \mathbb{R}$  are (countably) equidecomposable if there exist countable partitions*

$$A = \cup_{i=1}^{\infty} A_i, \quad B = \cup_{i=1}^{\infty} B_i, \quad (4)$$

and real numbers  $\{t_1, t_2, t_3 \dots\}$  such that for every  $i \in \mathbb{N}$   $A_i = B_i + t_i$  ( $+t_i$  here indicates translation by  $t_i$ ).

It might also be useful to recall the Axiom of Choice.

**Axiom 2.** *Let  $\Omega$  be a set and  $\mathcal{C} = \{A_\alpha\}_{\alpha \in \Gamma}$  be a collection of nonempty subsets  $A_\alpha \subseteq \Omega$ . Then there exists at least one choice function, i.e. a function  $f : \mathcal{C} \rightarrow \Omega$  such that*

$$f(A) \in A, \quad (5)$$

for each  $A \in \mathcal{C}$ .

First, we start with some useful remarks. Here  $A, B, A_i, B_i$  are subsets of  $\mathbb{R}$ . Further, for  $t \in \mathbb{R}$ ,  $R_t : \mathbb{R} \rightarrow \mathbb{R}$  is the translation by  $t$ :  $R_t(x) = x + t$ .

A1 We will say that a function  $f : A \rightarrow \mathbb{R}$  is a (countable) *equidecomposition* if there exists a countable partition  $A = \cup_{i=1}^{\infty} A_i$ , and reals  $\{t_i\}_{i \in \mathbb{N}}$  such that  $f|_{A_i} = R_{t_i}|_{A_i}$  for each  $i$ .

Show that  $A$  is equidecomposable with  $B$  if and only if there exists an equidecomposition  $f : A \rightarrow B$  which is bijective.

A2 Let  $A' \subseteq A$  and  $B' \subseteq B$ , and assume there exist bijective equidecompositions  $f : A \rightarrow B'$  and  $g : B' \rightarrow A'$ .

Construct an equidecomposition  $h : A \rightarrow B$ , and prove that it is bijective.

[Hint: Let  $A^{(0)} \equiv A \setminus g(B)$ , and  $A^{(*)} \equiv \cup_{n=0}^{\infty} (g \circ f)^n(A^{(0)})$ . Define  $h(x) = f(x)$  if  $x \in A^{(*)}$  and  $h(x) = g^{-1}(x)$  if  $x \in A \setminus A^{(*)}$ .]

Next to the actual problem:

B1 Use the axiom of choice to show that there exists  $C \subseteq [0, 1/2]$  such that the following is a partition

$$\mathbb{R} = \cup_{x \in C} \{x + \mathbb{Q}\}. \quad (6)$$

B2 Show that  $\mathbb{Q} \cap [0, 1/2]$  is equidecomposable with  $\mathbb{Q}$ .

B3 Deduce that

$$A \equiv \cup_{x \in C} \{x + (\mathbb{Q} \cap [0, 1/2])\} \subseteq [0, 1] \quad (7)$$

is equidecomposable with  $\mathbb{R}$ .

B4 Use A1, A2 above to show that this implies that  $[0, 1]$  is equidecomposable with  $\mathbb{R}$ . What does this result imply for measures on  $\mathbb{R}$ ?