Math 205A Real Analysis

Homework 2 - Solution

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- Return your assignments via Gradescope
- Solutions should be complete and concisely written. You can reference results/statements in either of the textbooks. Any other non-elementary fact must be proven.
- You are welcome to discuss problems with your colleagues, but should write and submit your own solution.
- Solutions are due on Thu, by 11:59PM.

Problem 1

(a) For $\varepsilon > 0$, let $\Gamma_\varepsilon(A) \equiv \{rx \,:\, r \in (\varepsilon, 1],\, x \in A\}$. Then $\Gamma_\varepsilon(A) = f_\varepsilon^{-1}(A)$, for the continuous mapping $f_\varepsilon : \{x \in \mathbb{R}^d : \varepsilon \le ||x|| \le 1\} \to S^{d-1}$, $x \mapsto x/||x||$. Since counterimages of Borel sets under continuous mappings are Borel, we have $\Gamma_\varepsilon(A) \in \mathcal{B}(\mathbb{R}^d)$. The thesis follows since

$$\Gamma(A) = \bigcup_{n=1}^{\infty} \Gamma_{1/n}(A) \cup \{0\}\,. \tag{1}$$

(b) Obviously μ is a non-negative set function, with $\mu(\emptyset) = d\lambda_d(\emptyset) = 0$. If $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{B}(S^{d-1})$ is a disjoint collection than $\{B_i\}_{i\in\mathbb{N}} \subseteq \mathcal{B}(\mathbb{R}^d)$ are also disjoint with $B_i = \Gamma(A_i)\setminus\{0\}$. Further $\Gamma(\cup_i A_i) = \cup_i \Gamma(A_i)$. Therefore, since $\lambda_d(\{0\}) = 0$, we have

$$\mu(\cup_{i\ge1} A_i) = d\lambda_d(\cup_{i\ge1}\Gamma(A_i)) = d\lambda_d(\cup_{i\ge1} B_i) = \sum_{i\ge1} d\lambda_d(B_i) = \sum_{i\ge1} d\lambda_d(\Gamma(A_i)) = \sum_{i\ge1}\mu(A_i)\,, \tag{2}$$

i.e. μ is countably additive, hence a measure.

Finally $\mu(S^{d-1}) = d\lambda_d(\{x : ||x|| \le 1\}) \le d\lambda_d(\{x : \max_i |x_i| \le 1\}) = d2^d$. Therefore μ is finite.

(c) First consider the case $b = 1$, $a/b = \alpha < 1$. Using the definition of $\Gamma_\varepsilon(A)$ in point (a), we have $\Gamma_0(A) = \cup_{i=0}^{\infty} C_{\alpha^{i+1},\alpha^i}(A)$. Since the union is disjoint, and $\lambda_d(\{0\}) = 0$, we have

$$\mu(A) = d\lambda_d(\Gamma_0(A)) = \sum_{i=0}^{\infty} d\,\lambda_d(C_{\alpha^{i+1},\alpha^i}(A)) = \sum_{i=0}^{\infty} d\,\alpha^{id}\lambda_d(C_{\alpha,1}(A)) = \frac{1}{1-\alpha^d}\,d\lambda_d(C_{\alpha,1}(A))\,. \tag{3}$$

For $b \neq 1$, it is sufficient to use $\lambda_d(C_{a,b}(A)) = b^d\lambda_d(C_{\alpha,1}(A))$ for $\alpha = a/b$.

Problem 2

(a) Throughout the solution we will use the fact that $\lambda_2 = \lambda_1 \times \lambda_1$ whence we obtain the action of λ_2 on rectangles: $\lambda_2(A_1 \times A_2) = \lambda_1(A_1)\lambda_1(A_2)$. Also, for $J_1, J_2 \subseteq \mathbb{R}$ two intervals, let T_{J_1,J_2} be any triangle with two sides equal to J_1 (parallel to the first axis) and J_2 (equal to the second axis). from the addittivity of λ_2 it follows immediately that $\lambda_2(T_{J_1,J_2}) = |J_1|\cdot|J_2|/2$. (We use here the fact that for a segment $S = \{x_0 + x_1\lambda : \lambda \in [a,b]\}$, $x_0, x_1 \in \mathbb{R}^2$, $\lambda_2(S) = 0$, which can be proved by covering S with squares.)

Consider next a rectangle $A = [0, a) \times [0, b)$, and let $A' = \mathsf{R}(\alpha)A$. Using again addittivity it follows that, for $\beta = \pi/2 - \alpha$:

$$\begin{aligned}\lambda_2(A') &= (a\cos\alpha + b\cos\beta)(a\sin\alpha + b\sin\beta) - a^2\sin\alpha\cos\alpha - b^2\sin\beta\cos\beta \\ &= ab(\cos\alpha\sin\beta + \cos\beta\sin\alpha) = ab\sin(\alpha+\beta) = ab\,.\end{aligned}$$

Hence $\lambda_2(A) = \lambda_2(\mathsf{R}(\alpha)A)$ and by translation invariance this holds for any $A = [a_1, a_2) \times [b_1, b_2)$ (not necessarily with a corner at the origin).

Since the π-system $\mathcal{P} = \{A = [a_1, a_2) \times [b_1, b_2) :\ a_1 < a_2, b_1 < b_2\}$ generates the Borel σ algebra, and recalling that λ_2 is σ-finite, this proves the claim by Caratheodory uniqueness theorem.

(b) The proof is analogous to the previous one. Let μ be the measure defined by $\mu(B) \equiv s^{-2}\lambda_2(sB)$. For $A = [a_1, a_2) \times [b_1, b_2)$, $a_1 < a_2, b_1 < b_2$, we have $sA = [sa_1, sa_2) \times [sb_1, sb_2)$, whence

$$\mu(A) = \frac{1}{s^2}\lambda_2(sA) = \frac{1}{s^2}(sa_2 - sa_1)(sb_2 - sb_1) = (a_2 - a_1)(b_2 - b_1) = \lambda_2(A)\,.$$

The claim follows by Caratheodory uniqueness theorem.

(c) Notice that $C_{r,\alpha,\beta} = rC_{1,\alpha,\beta}$. Therefore, by point (b) above, it is sufficient to prove the claim for $r = 1$. Further by invariance under rotation (point (a)), $\lambda_2(C_{1,\alpha,\beta}) = \lambda_2(C_{1,0,\beta-\alpha})$. It is therefore sufficient to show that $F(\theta) \equiv \lambda_2(C_{1,0,\theta}) = \theta/2$.

By covering $C_{1,0,\theta}$ with a triangle and inscribing a triangle in it we have

$$\frac{1}{2}\sin\theta\cos\theta \le F(\theta) \le \frac{1}{2}\,\tan\theta\,.$$

From these we have $F(\theta) = \theta/2 + O(\theta^2)$ as $\theta \to 0$. By addittivity of λ_2, and splitting $C_{1,0,\theta} = C_{1,0,\theta/n} \cup C_{1,\theta/n,2\theta/n} \cup \cdots \cup C_{1,\theta-\theta/n,\theta}$, we get

$$F(\theta) = nF(\theta/n) = \lim_{n\to\infty} nF(\theta/n) = \lim_{n\to\infty} n\Big[\frac{\theta}{2n} + O(\theta^2/n^2)\Big] = \frac{\theta}{2}\,.$$

This finishes the proof.

Problem 4

nsider the space $\Omega = C([0, 1])$ of continuous functions on the interval $[0, 1]$. For any two such functions, define the distance

$$d(\omega_1, \omega_2) = \sup_{t\in[0,1]} \big|\omega_1(t) - \omega_2(t)\big|\,.$$

We endow Ω with the uniform topology, i.e. the topology induced by this distance. We define the following σ-algebras

$$\mathcal{F}_1 \equiv \sigma(\{O \subseteq \Omega :\ O \text{ is open in the uniform topology }\})\,, \tag{4}$$

$$\mathcal{F}_2 \equiv \sigma(\{A(t_1, S_1; t_2, S_2; \dots; t_k; S_k) :\ t_i \in [0, 1], S_i \in \mathcal{B}_{\mathbb{R}},\ k \in \mathbb{N}\})\,, \tag{5}$$

$$\mathcal{F}_3 \equiv \sigma(\{A(t, (a, b]) :\ t \in [0, 1],\ a, b \in \mathbb{R}, a < b\})\,, \tag{6}$$

where for $t_1, \dots, t_k \in [0, 1]$, $S_1, \dots, S_k \subseteq \mathbb{R}$ we define

$$A(t_1, S_1; t_2, S_2; \dots; t_k; S_k) \equiv \Big\{\omega \in C([0, 1]) :\ \omega(t_1) \in S_1,\ \omega(t_2) \in S_2,\ \dots, \omega(t_k) \in S_k\Big\}\,. \tag{7}$$

(a)] Prove that $\mathcal{F}_1 = \mathcal{F}_2 \subseteq \mathcal{F}_3$.
(b) Prove that $\mathcal{F}_3 \subseteq \mathcal{F}_2$. Call this σ algebra $\mathcal{F} = \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3$.

[This is mostly a topology exercise, and you might want to keep it last. It might be useful to remember Weierstrass approximation theorem. This says that the (countable) set of polynomials with rational coefficients is dense in $C([0,1])$ with respect to the uniform topology.]

Solution: Obviously $\mathcal{F}_3 \subseteq \mathcal{F}_2$ because all the generators of $\mathcal{F}_3$ are also generators of cF_2. On the other hand, let

$$\mathcal{G}_t \equiv \{S \subseteq \mathbb{R} : \; A(t,S) \in \mathcal{F}_3\} . \tag{8}$$

It is easy to see that $\mathcal{G}_t$ is a σ-algebra, whence $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{G}_t$. i.e. $A(t,S) \in \mathcal{F}_3$ for all $S \in \mathcal{B}_{\mathbb{R}}$. Therefore, for each $k \in \mathbb{N}$, $t_1, \dots, t_k \in [0,1]$ and $S_1, \dots, S_k \in \mathcal{B}_{\mathbb{R}}$, we have

$$A(t_1, S_1; \dots ; t_k, S_k) = \cap_{i=1}^k A(t_i, S_i) \in \mathcal{F}_3 \, , \tag{9}$$

and therefore $\mathcal{F}_2 \subseteq \mathcal{F}_3$, which leads to $\mathcal{F}_2 = \mathcal{F}_3$.

Let $\mathcal{F}_3' = \sigma(\{A(t,(a,b)) : \; t \in [0,1], a, b \in \mathbb{R}\})$. We obviously have $\mathcal{F}_3' \subseteq \mathcal{F}_2 = \mathcal{F}_3$. Further, for each $t \in [0,1]$, and $a < b$, we have

$$A(t,(a,b]) = \cap_{k=1}^{\infty} A(t,(a,b+k^{-1})) \tag{10}$$

whence $A(t,(a,b]) \in \mathcal{F}_3'$ and therefore $\mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_3'$. Note that $A(t,(a,b))$ is open in the uniform topology and therefore $\mathcal{F}_2 = \mathcal{F}_3 \subseteq \mathcal{F}_1$.

Finally, in order to show that $\mathcal{F}_1 \subseteq \mathcal{F}_2$, let

$$\mathsf{B}(\omega_0, \epsilon) \equiv \big\{\omega \in C([0,1]) : \; d(\omega_0, \omega) < \epsilon\big\} . \tag{11}$$

Note that

$$\mathsf{B}(\omega_0, \epsilon) = \cap_{t \in [0,1] \cap \mathbb{Q}} A(t, (\omega_0(t) - \epsilon, \omega_0(t) + \epsilon)) \in \mathcal{F}_2 \, . \tag{12}$$

On the other hand for any set O open in the uniform topology, and $\omega \in O$, let

$$\epsilon_O(\omega) \equiv \sup \big\{\epsilon : \; \mathsf{B}(\omega, \epsilon) \subseteq O\big\} > 0 \, . \tag{13}$$

(The last inequality follows from the assumption that O is open.) Letting $\mathbb{P}$ the the set of polynomial with rational coefficients, we notice that

$$O = \cup_{\omega \in O \cap \mathbb{P}} \mathsf{B}(\omega, \epsilon_O(\omega)/2) \, . \tag{14}$$

and therefore $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

(c) Consider the function $G : (\Omega, \mathcal{F}) \to (\Omega, \mathcal{F})$, $\omega \mapsto G(\omega)$ defined by letting, for $t \in [0,1]$,

$$G(\omega)(t) \equiv \int_0^t \omega(u) \, \mathrm{d}u \tag{15}$$

Prove that G is a measurable mapping (with respect to the σ-algebra $\mathcal{F}$).

Solution: This follows from the remark that G is continuous. Indeed, using linearity and monotonicity of the integral,

$$\big|G(\omega_1)(t) - G(\omega_2)(t)\big| = \bigg|\int_0^t \big(\omega_1(u) - \omega_2(u)\big) \, \mathrm{d}u\bigg| \tag{16}$$

$$\leq \int_0^t \big|\omega_1(u) - \omega_2(u)\big| \, \mathrm{d}u \leq \int_0^t d(\omega_1, \omega_2) \, \mathrm{d}u \leq d(\omega_1, \omega_2) \, . \tag{17}$$

In other words

$$d(G(\omega_1), G(\omega_2)) \leq d(\omega_1, \omega_2), . \tag{18}$$

Hence G is continuous in the uniform topology, and therefore measurable.