Homework 2

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- Return your assignments via Gradescope

- Solutions should be complete and concisely written. You can reference results/statements in either of the textbooks. Any other non-elementary fact must be proven.

- You are welcome to discuss problems with your colleagues, but should write and submit your own solution.

- Solutions are due on Thu, by 11:59PM.

Problem 1

Let S^{d-1} be the unit sphere in $\mathbb{R}^d$:

$$S^{d-1} = \left\{ x \in \mathbb{R}^d : ||x|| = 1 \right\}. \tag{1}$$

The sphere S^{d-1} can be given the topology induced by $\mathbb{R}^d$. More precisely $A \subseteq S^{d-1}$ is open if for any $x \in A$, there exists $\varepsilon > 0$ such that $\{y \in S^{d-1} : ||x - y|| \leq \varepsilon\} \subseteq A$.

Let $\mathcal{B}(S^{d-1})$ be the corresponding Borel σ-algebra. For any $A \in \mathcal{B}(S^{d-1})$, define

$$\Gamma(A) = \left\{ rx : r \in [0,1], \ x \in A \right\}, \tag{2}$$

(a) Show that, for any $A \in \mathcal{B}(S^{d-1})$, $\Gamma(A) \in \mathcal{B}(\mathbb{R}^d)$.

(b) Let λ_d be the Lebesgue measure on $\mathbb{R}^d$, and define, for $A \in \mathcal{B}(S^{d-1})$,

$$\mu(A) = d \, \lambda_d(\Gamma(A)). \tag{3}$$

Prove that μ is a finite measure on $(S^{d-1}, \mathcal{B}(S^{d-1}))$.

(c) For $A \in \mathcal{B}(S^{d-1})$ and $0 \leq a \leq b$, define the set $C_{a,b}(A) \in \mathcal{B}(\mathbb{R}^d)$ as $C_{a,b}(A) = \{rx : a < r \leq b, \ x \in A\}$. Prove that

$$\lambda_d(C_{a,b}(A)) = \frac{b^d - a^d}{d} \, \mu(A). \tag{4}$$

[Hint: You could use the fact that, for $\gamma > 0$ and $B \in \mathcal{B}(\mathbb{R}^d)$, $\lambda_d(\gamma B) = \gamma^d \lambda_d(B)$ (with γB the set obtained by 'dilating' B by a factor γ).]

Problem 2

Let λ_2 be the Lebesgue measure on $(\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2})$. We know already that it is invariant under translation i.e. that $\lambda_2(B + x) = \lambda_2(B)$ for any Borel set B and $x \in \mathbb{R}^2$ (whereby $B + x = \{y \in \mathbb{R}^2 : y - x \in B\}$).

(a) Show that it is invariant under rotations as well, i.e. that for any $\alpha \in [0, 2\pi]$, and any Borel set $B \subseteq \mathbb{R}^2$, $\lambda_2(\mathsf{R}(\alpha)B) = \lambda_2(B)$ (whereby $\mathsf{R}(\alpha)$ denotes a rotation by an angle α and $\mathsf{R}(\alpha)B = \{x \in \mathbb{R}^2 : \mathsf{R}(-\alpha)x \in B\}$).

(b) For $s \in \mathbb{R}_+$, and $B \subseteq \mathbb{R}^2$ Borel, let $sB \equiv \{x \in \mathbb{R}^2 : s^{-1}x \in B\}$. Prove that $\lambda_2(sB) = s^2 \lambda_2(B)$.

(c) For $r > 0$, $0 \leq \alpha < \beta \leq 2\pi$, let

$$C_{r,\alpha,\beta} \equiv \left\{ x = (u\cos\theta, u\sin\theta) : u \in [0,r], \theta \in [\alpha, \beta] \right\}. \tag{5}$$

Prove that $\lambda_2(C_{r,\alpha,\beta}) = (\beta - \alpha)r^2/2$.

Problem 3

Let $A \subseteq \mathbb{R}$ be a Lebesgue measurable set with positive Lebesgue measure. Prove that for every $\epsilon > 0$, there is an nonempty open interval I such that $\lambda(A \cap I) > (1 - \epsilon)\lambda(I)$.

Problem 3

Consider the space $\Omega = C([0,1])$ of continuous functions on the interval $[0,1]$. For any two such functions, define the distance

$$d(\omega_1, \omega_2) = \sup_{t \in [0,1]} \left| \omega_1(t) - \omega_2(t) \right|.$$

We endow Ω with the uniform topology, i.e. the topology induced by this distance. We define the following σ-algebras

$$\mathcal{F}_1 \equiv \sigma(\{O \subseteq \Omega : O \text{ is open in the uniform topology }\}), \tag{6}$$

$$\mathcal{F}_2 \equiv \sigma(\{A(t_1, S_1; t_2, S_2; \ldots; t_k; S_k) : t_i \in [0,1], S_i \in \mathcal{B}_{\mathbb{R}}, k \in \mathbb{N}\}), \tag{7}$$

$$\mathcal{F}_3 \equiv \sigma(\{A(t, (a,b]) : t \in [0,1], a, b \in \mathbb{R}, a < b\}), \tag{8}$$

where for $t_1, \ldots, t_k \in [0,1]$, $S_1, \ldots, S_k \subseteq \mathbb{R}$ we define

$$A(t_1, S_1; t_2, S_2; \ldots; t_k; S_k) \equiv \left\{ \omega \in C([0,1]) : \omega(t_1) \in S_1, \omega(t_2) \in S_2, \ldots, \omega(t_k) \in S_k \right\}. \tag{9}$$

(a) Prove that $\mathcal{F}_2 = \mathcal{F}_3 \subseteq \mathcal{F}_1$.

(b) Prove that $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Call this σ algebra $\mathcal{F} = \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3$.

[This is mostly a topology exercise, and you might want to keep it last, and solve point (c) before. It might be useful to remember Weierstrass approximation theorem. This says that the (countable) set of polynomials with rational coefficients is dense in $C([0,1])$ with respect to the uniform topology.]

(c) Consider the function $G : (\Omega, \mathcal{F}) \to (\Omega, \mathcal{F})$, $\omega \mapsto G(\omega)$ defined by letting, for $t \in [0,1]$,

$$G(\omega)(t) \equiv \int_0^t \omega(u) \, du \tag{10}$$

Prove that G is a measurable mapping (with respect to the σ-algebra $\mathcal{F}$).