**Math 205A Real Analysis**

# Homework 3 - Solution

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## Problem 1

Let $\mathcal{H}$ be the set of bounded Borel functions $f$, such that $X_f$ is measurable on $(\Omega, \mathcal{F})$. By definition $X_{\mathbf{1}_A} = X_A$ and therefore $\mathbf{1}_A \in \mathcal{H}$. Further consider $f_1$, $f_2 \in \mathcal{H}$ and $c_1, c_2 \in \mathbb{R}$. Then

$$X_{c_1 f_1 + c_2 f_2}(\omega) = \int_{(0,1]} \big(c_1 f_1(x) + c_2 f_2(x)\big)\, \omega(\mathrm{d}x) \tag{1}$$

$$= c_1 \int_{(0,1]} f_1(x)\, \omega(\mathrm{d}x) + c_2 \int_{(0,1]} f_2(x)\, \omega(\mathrm{d}x) = c_1 X_{f_1}(\omega) + c_2 X_{f_2}(\omega)\,. \tag{2}$$

Since the sum of two random variables is a random variable, it follows that $c_1 f_1 + c_2 f_2 \in \mathcal{H}$.

Finally consider $f : [0,1] \to \mathbb{R}_+$ a bounded Borel function and assume that $f_n \uparrow f$ with $f_n \in \mathcal{H}$ non-negative functions. Then $X_{f_n}(\omega) = \int_{(0,1]} f_n(x)\omega(\mathrm{d}x)$ is non-negative and monotone increasing for each $\omega$. Further, by the monotone convergence theorem,

$$\lim_{n\to\infty} X_{f_n}(\omega) = \lim_{n\to\infty} \int_{(0,1]} f_n(x)\, \omega(\mathrm{d}x) = \int_{(0,1]} f(x)\, \omega(\mathrm{d}x) = X_f(\omega)\,. \tag{3}$$

Hence $X_f$ is a random variable (since is pointwise limit of random varables) and therefore $f \in \mathcal{H}$.

The proof is finished by applying the monotone class theorem, that implies that $\mathcal{H}$ contains all the bounded Borel functions.

## Problem 2 (sketch)

The same method works for all four parts.

1. Since $\mathcal{B} = \sigma(\{(-\infty, t] : t \in \mathbb{R}\})$, it follows that $f$ is measurable with respect to the right hand side (RHS), which hence also contains the left hand side (LHS). But the RHS is generated by elements of the $\sigma$-algebra on the LHS, so the LHS contains the RHS as well.

2. For $1 \le i \le n$, each $f_i$ is by Theorem 1.2.11 measurable with respect to the RHS. Therefore, the RHS contains the LHS. Again, the RHS is generated by sets from the LHS, so the latter contains the former.

3. Exactly the same method applies.

4. Since each $f_k$ is measurable with respect to the RHS, the latter contains the LHS. By definition $\sigma(f_k, k \le n)$ is contained in the LHS for each $n$, hence so is the union of these collections, implying that the LHS contains the RHS as well.

## Problem 3 (sketch)

**(a)** This can be checked just by checking the conditions that define measures. It is immediate that these holds by linearity and monotone convergence.

**(b)** If $h = I_B$ is an indicator function, this follows from the definition. Linearity of the integration extends the result to simple functions, and then the monotone convergence theorem gives the result for nonnegative functions. Finally, by taking positive and negative parts we get the result for all integrable functions.

**(c)** Take $A_n = [-n, n] \cap \{x : f(x) \le n\}$, and show that $\lambda_f(A_n) \le 2n^2$.

## Problem 4

**(a)** This point follows just by writing down the definition, and we omit details.

**(b)** Let $g_n(\omega) := \sup_{m \geq n} |f_m(\omega)|$ noting that $g_n \downarrow g$ for every $\omega$, with $g \geq 0$. Since $g_n(\omega) \to 0$ if and only if $g(\omega) = 0$, the a.e. convergence of $f_n$ to 0 is equivalent to $\mathbb{P}(g > \epsilon) = 0$ for each $\epsilon > 0$. Note that $\{g_n > \epsilon\} \downarrow \{g > \epsilon\}$, hence $g_n \stackrel{a.s.}{\to} 0$ if and only if for each $\epsilon > 0$ there is $n$ such that $\mathbb{P}(g_n > \epsilon) < \epsilon$. To complete the proof observe next that $\{|f_M| > \epsilon\} \subseteq \{g_n > \epsilon\}$ for any random integer $M(\omega) \geq n$, with set equality for

$$M(\omega) = \inf\{m \geq n \colon |f_m(\omega)| > \epsilon\}$$

in case $g_n(\omega) > \epsilon$ and $M(\omega) = n$ otherwise.