Math 205A Real Analysis

Homework 3

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- Return your assignments via Gradescope
- Solutions should be complete and concisely written. You can reference results/statements in either of the textbooks. Any other non-elementary fact must be proven.
- You are welcome to discuss problems with your colleagues, but should write and submit your own solution.
- Solutions are due on Thu, by 11:59PM.

Problem 1

Let Ω be the space of probability measures ω over $((0,1], \mathcal{B}_{(0,1]})$. As usual, given such a measure $\omega \in \Omega$ and $A \in \mathcal{B}_{(0,1]}$, we write $\omega(A)$ for the probability of set A. (Pay attention, Ω is not the interval $(0,1]$!).

For each $A \in \mathcal{B}_{(0,1]}$, define the mapping

$$\begin{aligned} X_A : &\Omega \to \mathbb{R} \\ &\omega \mapsto \omega(A)\,. \end{aligned} \tag{1}$$

We let $\mathcal{F}$ be the smallest σ-algebra on Ω such that all of the X_A are measurable, i.e.

$$\mathcal{F} \equiv \sigma(\{X_A : \; A \in \mathcal{B}_{(0,1]}\})\,. \tag{2}$$

For $f : (0,1] \to \mathbb{R}$ a bounded Borel function, let

$$\begin{aligned} X_f : &\Omega \to \mathbb{R} \\ &\omega \mapsto \int_{(0,1]} f(x)\,\omega(\mathrm{d}x)\,. \end{aligned} \tag{3}$$

Prove that X_f is measurable on $(\Omega, \mathcal{F})$.

Problem 2

Recall that, given functions $f_\alpha : \Omega \to \mathbb{R}$, $\alpha \in S$, we denote by $\sigma(\{f_\alpha\}_{\alpha \in S})$ the smallest σ-algebra $\mathcal{F}$ such that all the f_α are measurable with respect to $\mathcal{F}$. Verify the following relations and show that each generating collection of sets on the right hand side is a π-system.

(a) $\sigma(f) = \sigma(\{x : f(x) \le t\} : t \in \mathbb{R})$.

(b) $\sigma(\{f_i : i \le n\}) = \sigma(\{x : f_1(x) \le t_1, \dots f_n(x) \le t_n\} : t_1, \dots, t_n \in \mathbb{R})$.

(c) $\sigma(\{f_i : i \in \mathbb{N}\}) = \sigma(\{x : f_1(x) \le t_1, \dots f_n(x) \le t_n\} : t_1, \dots, t_n \in \mathbb{R}, n \in \mathbb{N})$.

Problem 3

Let $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a Borel function and define the set function λ_f on $(\mathbb{R}, \mathcal{B})$ via

$$\lambda_f(A) := \lambda(f\mathbf{1}_A)\,, \tag{4}$$

for every $A \in \mathcal{B}$.

(a) Prove that λ_f is a measure on $(\mathbb{R}, \mathcal{B})$

(b) Prove that $\lambda_f(h) = \lambda(fh)$ for every Borel function h such that $\lambda(f|h|) < \infty$.

(c) Prove that λ_f is σ-finite.

Problem 4

Let $(f_n)_{n\geq 1}$ be a sequence of functions on a measure space $(\Omega, \mathcal{F}, \mu)$.
(a) Show that $f_n \to 0$ almost everywhere if and only if for any $\varepsilon > 0$,

$$\mu\Big(\cap_{n=1}^{\infty} \cup_{k\geq n} \big\{x : |f_k(x)| \geq \varepsilon\big\}\Big) = 0\,. \tag{5}$$

(b) Show that, if $\mu(\Omega) < \infty$, then $f_n \to 0$ almost everywhere if and only if for any $\varepsilon > 0$, there is n so that, for any measurable function $M : \Omega \to \mathbb{N}$, with $M(x) \geq n$ for all $x \in \Omega$ we have that $\mu(\{x : |f_{M(x)}(x)| > \varepsilon\}) < \varepsilon$.