Math 205A Real Analysis

Homework 4

Andrea Montanari *Due on October 31, 2024*

- Return your assignments via Gradescope
- Solutions should be complete and concisely written. You can reference results/statements in either of the textbooks. Any other non-elementary fact must be proven.
- You are welcome to discuss problems with your colleagues, but should write and submit your own solution.
- Solutions are due on Thu, by 11:59PM.
- Credit: These problems were originally written by Brian White.

Problem 1

Let $(X, \mathcal{A}, \mu)$ be a probability space, i.e., a measure space with $\mu(X) = 1$.

(a) Let $f : X \to \mathbb{R}^k$ be $\mathcal{A}$-measurable, integrable (which means that all of its components are integrable), ag $\Phi : \mathbb{R}^k \to \mathbb{R}$ convex proper. Prove the following inequality (Jensen's inequality):

$$\int \Phi(f)\mathrm{d}\mu \geq \Phi(\int f\mathrm{d}\mu)\,. \tag{1}$$

(b) Prove that if $1 \leq p \leq r$, then $\|f\|_p \leq \|f\|_r$.

Problem 2

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $1 \leq p < q < r < \infty$. (Here p and q need not be conjugate exponents.) Show that if $f \in L^p$ and if $f \in L^r$, then $f \in L^q$.

Problem 3

Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function with $\int |f|\, d\lambda < \infty$. Prove that

$$\lim_{a\to\infty} \int f(t)\sin(at)\,\mathrm{d}t = 0.$$

Problem 4

Let $(X, \mathcal{A}, \mu)$ be a measure space and suppose f_n $(n = 1, 2, \dots)$ and f are functions in $\mathcal{L}^p(X, \mathcal{A}, \mu)$ such that

$$\int f_n g\,\mathrm{d}\mu \to \int f g\,\mathrm{d}\mu$$

for every $g \in \mathcal{L}^q(X, \mathcal{A}, \mu)$, where $p, q \in (1, \infty)$ are conjugate exponents. Prove that $\|f\|_p \leq \liminf_{n\to\infty} \|f_n\|_p$.