**Math 205A Real Analysis**

# Homework 5

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- Return your assignments via Gradescope
- Solutions should be complete and concisely written. You can reference results/statements in either of the textbooks. Any other non-elementary fact must be proven.
- You are welcome to discuss problems with your colleagues, but should write and submit your own solution.
- Solutions are due on Thu, by 11:59PM.
- Credit: These problems were originally written by Brian White.

## Problem 1

Suppose that $(\mathbb{X}, \mathcal{A}, \mu)$ is a finite measure space and that $f$ is a $\mu$-integrable function. Let $\mathcal{F}$ be a $\sigma$-algebra contained in $\mathcal{A}$. Prove that there is an $\mathcal{F}$-measurable function $g$ such that

$$\int f \mathbf{1}_S \, \mathrm{d}\mu = \int g \mathbf{1}_S \, \mathrm{d}\mu$$

for every $S \in \mathcal{F}$.

## Problem 2

Recall that $M(\mathbb{X}, \mathcal{F})$ decotes the Banach space of signed measures on the measurable space $(\mathbb{X}, \mathcal{F})$, endowed with the total variation norm. Given a positive measure $\mu$ on $(\mathbb{X}, \mathcal{F})$, define

$$L(\mu) := \big\{ \nu \in M(\mathbb{X}, \mathcal{F}) \ : \ \nu \perp \mu \big\} . \tag{1}$$

(Here $\nu \perp \mu$ means that $\mu, \nu$ are mutually singular.)

Prove that $L(\mu)$ is a closed vector space.

## Problem 3

Let $\nu_1$, $\nu_2$, $\nu_3$ be $\sigma$-finite measures on a common measurable space $(\mathbb{X}, \mathcal{F})$, such that $\nu_1 \ll \nu_2$ and $\nu_2 \ll \nu_3$.

(*a*) Prove that $\nu_1 \ll \nu_3$.

(*b*) Prove that, if $f$ is a Radon-Nikodym derivative of $\nu_1$ with respect to $\nu_2$ and $g$ is a Radon-Nikodym derivative of $\nu_2$ with respect to $\nu_3$, then $fg$ is Radon-Nikodym derivative of $\nu_1$ with respect to $\nu_3$. In other words, the following holds $\nu_3$-almost everywhere

$$\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_3} = \frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2} \cdot \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_3} . \tag{2}$$

## Problem 4

Let $\mathbb{X} := \{0,1\}^{\mathbb{Z}}$ be endowed with the $\sigma$-algebra generated by sets $C(\boldsymbol{z};n)$, $\boldsymbol{z} \in \{0,1\}^{[-n,n]\cap\mathbb{Z}}$ defined by

$$C(\boldsymbol{z};n) := \Big\{\boldsymbol{x} \in \mathbb{X} :\ x_i = z_i \forall |i| \le n\Big\}. \tag{3}$$

For $p \in [0,1]$ define the probability measure $\mu_p$ by

$$\mu_p(C(\boldsymbol{z};n)) = (1-p)^{2n+1-|\boldsymbol{z}|} p^{|\boldsymbol{z}|}, \tag{4}$$

where $|\boldsymbol{z}|$ is the number of nonzeros in $\boldsymbol{z}$. (Informally, under $\mu_p$, $\boldsymbol{x}$ is a sequence of coin tosses with bias $p$.)

($a$) Prove that the above indeed defines a unique probability measure $\mu_p$.

($b$) Letting $S_n(\boldsymbol{x}) = \sum_{i=-n}^{n} x_i/p$, prove that for any $\varepsilon > 0$ there exists $C(\varepsilon)$ finite such that, for all $n \ge 1$

$$\mu_p\Big(\big\{|S_n(\boldsymbol{x}) - p| \ge \varepsilon\big\}\Big) \le \frac{C(\varepsilon)}{n^2}. \tag{5}$$

($c$) Deduce the 'strong law of large numbers.' Namely, for any $\varepsilon > 0$,

$$\mu_p\Big(\big\{ \limsup_{n\to\infty} |S_n(\boldsymbol{x}) - p| \ge \varepsilon\big\}\Big) = 0. \tag{6}$$

($d$) Conclude that, for $p \neq q$, $\mu_p$ and $\mu_q$ are mutually singular.