Math 205A Real Analysis

Homework 6

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- Return your assignments via Gradescope
- Solutions should be complete and concisely written. You can reference results/statements in either of the textbooks. Any other non-elementary fact must be proven.
- You are welcome to discuss problems with your colleagues, but should write and submit your own solution.
- Solutions are due on Thu, by 11:59PM.
- Credit: These problems were originally written by Brian White.

Problem 1

Show that if $F : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, then F is of finitevariation on any bounded interval.

Problem 2

Let $(\mathbb{X}, \mathcal{F}, mu)$ be a measure space, and define $T_g(f) := \int fg\mathrm{d}\mu$ for $g \in L^1(\mathbb{X}, \mathcal{F}, mu)$, $f \in L^\infty(\mathbb{X}, \mathcal{F}, mu)$ Show that $g \mapsto T_g$ is an isometry of $L^1(\mathbb{X}, \mathcal{F}, mu)$ into $(L^\infty(\mathbb{X}, \mathcal{F}, mu))^*$.

Problem 3

Let μ be the counting measure on $(\mathbb{R}, \mathcal{B})$, namely $\mu(A)$ is the cardinality of the set A.

(*a*) Show that the Lebesgue measure λ is absolutely continuous with respect to μ.

(*b*) Show that there is no measurable function f such that $\lambda = f\mu$.

(*c*) Why the Radon-Nikodym theorem does not apply to this case?

Problem 4

Suppose $f \in L^p(\mathbb{R}, \mathcal{B}, \lambda)$, where λ is the Lebesgue measure and $1 \le p < \infty$. Let $T_h f$ be the function given by $T_h f(x) = f(x + h)$.

(*a*) Prove that

$$\lim_{h \to 0} \|f_h - f\|_p = 0.$$

(*b*) Suppose $g \in L^q(\mathbb{R})$ (where $\frac{1}{p} + \frac{1}{q} = 1$). Define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = \int_{t \in \mathbb{R}} f(x - t)g(t)\, dt.$$

Prove that ϕ is bounded and uniformly continuous.

Problem 5

Let $f \in L^p(\mathbb{R}, \mathcal{B}, \lambda)$ and $g \in L^1(\mathbb{R}, \mathcal{B}, \lambda)$ $p \in [1, \infty)$. Prove that

$$f * g(x) := \int f(x-t)g(t)\,\mathrm{d}t \tag{1}$$

is well defined (and finite) almost everywhere.

[Hint: Use the inequality $|a| \leq 1 + |a|^p$ to prove that $\int |f(x-t)g(t)|\,\mathrm{d}t$ is a.e. finite.]