Math 205A Real Analysis

Homework 7

Andrea Montanari *Due on*

Problem 1

(a) Omitted. This amounts just to checking the axioms.

(b) For any $A \in \mathcal{B}$, we have

$$\begin{aligned}\mu_i(A) &= \mu(\mathbb{R} \times .. \times A ... \times \mathbb{R}) \\ &= \int_{\mathbb{R}\times..\times A...\times\mathbb{R}} f(x_1, ..., x_i, ..., x_n)\lambda_n(dx_1, ..., dx_n) \\ &= \int_{\mathbb{R}\times...\times\mathbb{R}\times A} f(x_1, ..., x_i, ..., x_n)\lambda_n(dx_1, .., dx_{i-1}, dx_{i+1}., dx_n, dx_i) \\ &= \int_A \left(\int_{\mathbb{R}\times...\times\mathbb{R}} f(x_1, ..., x, ..., x_n)\lambda_{n-1}(dx_1, .., dx_{i-1}, dx_{i+1}., dx_n) \right) \lambda_1(dx)\end{aligned}$$

where in the last step we used Fubini and the fact that $\lambda_n = \lambda \otimes \cdots \otimes \lambda$. By the given definition of density, we conclude that
$f_i(x) = \int_{\mathbb{R}^{n-1}} f(x_1, ..., x, ..., x_n)\lambda_{n-1}(dx_1, .., dx_{i-1}, dx_{i+1}., dx_n)$.

(c) We only need to show the equality on any generating set in $\mathbb{R}^n$. We proved that $\mathcal{B}_{R^n} = \sigma(A_1 \times ... \times A_n)$ for $A_1, ..., A_n \in \mathcal{B}$. Hence, for any measurable set $A_1, A_2, ..., A_n$, μ is of product form, we have

$$\begin{aligned}\mu(A_1 \times ... \times A_n) &== \prod_{i=1}^{n} \nu_i(A_i) = \prod_{i=1}^{n} \int_{A_i} f_i(x_i)\lambda_1(dx_i) \\ &= \int_{A_1} ... \int_{A_n} f_1(x_1)...f_n(x_n)\lambda_1(dx_1)...\lambda_1(dx_n) \\ &= \int_A f_1(x_1)...f_n(x_n)\lambda_n(dx_1, ..., dx_n) \text{ by property of Lesbegue measure.}\end{aligned}$$

Since this is true for any $A_1 \times ... \times A_n$, and by the definition of density we can conclude that the density of $(X_1, ..., X_n)$ exists and $f(X_1, ..., X_n) = f_1(X_1)...f_n(X_n)$.

(d) Counter example: let $F : \mathbb{R} \to \mathbb{R}^2$ be defined by $F(x) = (x, x)$, and $\mu = F_{\#}\lambda_{[0,1]}$, where $\lambda_{[0,1]}$ is the Lebesgue measure restricted to $[0, 1]$/ Clearly μ_1, μ_2 both have density $f(x) = \mathbf{1}(x \in [0, 1])$. However, the support of μ is $A = \{\omega = (\omega_1, \omega_2) : \omega_1 = \omega_2, 0 \le \omega_1, \omega_2 \le 1\}$. Thus, A is a line, and has Lesbegue measure 0 in dimension 2. If μ had a density in dimension 2, then $\mu(\mathbb{R}^2) = \int_A f(X_1, X_2)\lambda_2(dx_1, dx_2) = 0$ for some density f. However, $\mu(\mathbb{R}^2) = 1$ by constriction. Thus, we get a contradiction.

Problem 2

Let x be a Lebesgue point of f.

$$\begin{aligned}(f * \phi_r)(x) - f(x) &= r^{-n} \int f(x-y)\phi(y/r)\,dy - f(x) \\ &= r^{-n} \int f(z)\phi((x-z)/r)\,dz - f(x) \\ &= r^{-n} \int_{B(x,r)} f(z)\phi((x-z)/r)\,dz - r^{-n} \int_{B(x,r)} f(x)\phi((x-z)/r)\,dz \\ &= r^{-n} \int_{B(x,r)} (f(z) - f(x))\phi((x-z)/r)\,dz\end{aligned}$$

(Note we did a change of variable $z = x - y$.) Thus

$$|(f * \phi_r)(x) - f(x)| \leq \left(r^{-n} \int_{B(x,r)} |f(z) - f(x)|\,dz \right) \|\phi\|_\infty.$$

The integral tends to 0 as $r \to 0$ as x is a Lebesgue point of f. Since almost every point is a Lebesgue point of f, we are done.

Problem 3

We can assume that A and B are bounded. (Otherwise replace A and B by $A \cap [-n, n]$ and $B \cap [-n, n]$ where n is large enough that these sets have positive measure.) Thus 1_A and 1_B are in L^p for every p. In particular, they are both in L^2. Thus the function $1_A * 1_B$ is continuous. Note that

$$\begin{aligned}\int (1_A * 1_B)(x)dx &= \int_x \int_y 1_A(x-y)1_B(y)\,dy\,dx \\ &= \int_y \int_x 1_A(x-y)1_B(y)\,dx\,dy \\ &= \int_y 1_B(y) \int_x 1_A(x-y)\,dx\,dy \\ &= \int_y 1_B(y)\lambda(A)\,dy \\ &= \lambda(A)\lambda(B) \\ &> 0.\end{aligned}$$

Now $1_A * 1_B$ is a continuous function (by hw 6, problem 3) with positive integral, so there is an interval I on which is it is positive. Now for $x \in I$,

$$\begin{aligned}0 &< (1_A * 1_B)(x) \\ &= \int_y 1_A(x-y)1_B(y)\,dy\,dx \\ &= \lambda\{y : x - y \in A \text{ and } y \in B\}\,dx \\ &= \lambda\{y : y \in x - A \text{ and } y \in B\}\,dx \\ &= \lambda((x - A) \cap B)\,dx.\end{aligned}$$

In particular, $(x - A) \cap B$ is non-empty, or, equivalently, $x \in A + B$. We have shown: there is an interval I such that $I \subset A + B$. Of course the same is true for the sets $-A := \{-a : a \in A\}$ and B.