Math 205A Real Analysis

Homework 7

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- Return your assignments via Gradescope
- Solutions should be complete and concisely written. You can reference results/statements in either of the textbooks. Any other non-elementary fact must be proven.
- You are welcome to discuss problems with your colleagues, but should write and submit your own solution.
- Solutions are due on Thu, by 11:59PM.

Problem 1

Consider a finite measure μ on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$ and let λ_n denote the Lebesgue measure on $\mathbb{R}^n$.

(*a*) Define for each $i \in \{1, \dots, n\}$, and $A \in \mathcal{B}_{\mathbb{R}}$

$$\mu_i(A) = \mu\big(\{x : \ x_i \in A\}\big) \,. \tag{1}$$

Prove that μ_i isa measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

(*b*) Assume that μ has density f with respect to λ_n. prove that μ_i defined above has density f_i with respect to λ_1, where

$$f_i(x) \equiv \int_{\mathbb{R}^{n-1}} f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n) \, \lambda_{n-1}(\mathrm{d}x_1, \dots, \mathrm{d}x_{i-1}, \mathrm{d}x_{i+1}, \dots, \mathrm{d}x_n) \,. \tag{2}$$

(*c*) Assume that $\mu = \nu_1 \times \nu_2 \times \cdots \times \nu_n$, for finite measures ν_i on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Prove that there are constants $c_i > 0$ such that $\mu_i = c_i \nu_i$ for all i.

[In solving this point, we do not assume that μ has a density.]

(*d*) Assume that $\mu = \nu_1 \times \nu_2 \times \cdots \times \nu_n$ as at the last point, and that ν_i has density f_i with respect to λ_1. Prove that μ has density f with respect to λ_n, where

$$f(x_1, \dots, x_n) = f_1(x_1) \cdot \cdots \cdot f_n(x_n) \,. \tag{3}$$

(*e*) Prove that the product form assumption at the last point is required. Namely, give an example of a measure μ on $\mathbb{R}^2$ such that both μ_1 and μ_2 have a density with respect to λ_1, but μ does not have a density with respect to λ_2. Prove your claim.

Problem 2

Suppose f and g are Lebesgue measurable functions on $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, we let

$$(f * g)(x) = \int_{y \in \mathbb{R}^n} f(x - y) g(y) \, \mathrm{d}y$$

provided the integral exists. Suppose $\phi : \mathbb{R}^n \to \mathbb{R}$ is a bounded measurable function such that $\phi(x) = 0$ for $\|x\| > 1$ and such that $\int \phi = 1$. For $r > 0$, define $\phi_r : \mathbb{R}^n \to \mathbb{R}$ by $\phi_r(x) = r^{-n}\phi(x/r)$. Prove that if $f \in \mathcal{L}^1(\mathbb{R}^n)$, then

$$\lim_{r\to 0}(\phi_r * f)(x) = f(x)$$

for almost every $x \in \mathbb{R}^n$.

[Hint: to understand what's going on, you might first consider the case when ϕ is a constant times the indicator function of a ball centered at the origin.]

Problem 3

(*a*) As above, $f * g(y) \int_{y\in\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y$. Prove that, if $f, g \in L^2(\mathbb{R}, \mathcal{B}, \lambda)$, then $f * g$ is continuous.

(*b*) Suppose that A and B are Lebesgue measurable subsets of $\mathbb{R}$ with $\lambda(A) > 0$ and $\lambda(B) > 0$. Prove that the set $A - B := \{x - y : x \in A, y \in B\}$ contains an interval.

Problem 4

Prove that if $F : \Omega \to \mathbb{R}_{\geq 0}$ is a non-negative measurable function on the measure space $(\Omega, \mathcal{F}, \mu)$. Prove that

$$\int F \mathrm{d}\mu = \int_{\mathbb{R}_{\geq 0}} \mu\big(\{x :\ F(x) \geq t\}\big)\,\lambda(\mathrm{d}t)\,. \tag{4}$$

Problem 5

For $f : \mathbb{R}^d \to \mathbb{R}$ a measurable function, and $h \in \mathbb{R}^d$, let $\mathsf{U}_h f$ be the function defined by $(\mathsf{U}_h f)(x) := f(x+h)$.

(*a*) Recall that $C(\mathbb{R}^d)$ (continuous functions) is dense in $\mathcal{L}^p(\mathbb{R}^d, \mathcal{B}, \lambda)$ for all $p \in [1, \infty)$. Prove that $C_c(\mathbb{R}^d)$ (continuous functions with compact support) is also dense in $\mathcal{L}^p(\mathbb{R}^d, \mathcal{B}, \lambda)$.

(*b*) Prove that, for any $p \in [1, \infty)$ and any $f \in \mathcal{L}^p(\mathbb{R}^d, \mathcal{B}, \lambda)$

$$\lim_{h\to 0} \|\mathsf{U}_h f - f\|_p = 0\,. \tag{5}$$