Math 205A Real Analysis

Homework 8

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- Return your assignments via Gradescope
- Solutions should be complete and concisely written. You can reference results/statements in either of the textbooks. Any other non-elementary fact must be proven.
- You are welcome to discuss problems with your colleagues, but should write and submit your own solution.
- Solutions are due on Thu, by 11:59PM.

Problem 1

Let $\boldsymbol{x} : [0,1] \to \mathbb{R}^d$ be injective and continuously differentiable. For $f \in C_c(\mathbb{R}^d$, define the positive linear functional

$$L(f) := \int_{[0,1]} f(\boldsymbol{x}(t))\, \mathrm{d}\lambda(t)\,. \tag{1}$$

Find the regular Borel measure associated to L by the Riesz representation theorem.

Problem 2

Construct a topological space $\mathbb{X}$ by letting the underlying set be $\mathbb{R}^2$ and declaring that the open subsets U of $\mathbb{X}$ are those for which each section of the form $U_x := \{(x, y) : y \in \mathbb{R}\}$ is an open subset of $\mathbb{R}$. Given $f : \mathbb{X} \to \mathbb{R}$, we define $f_x : \mathbb{R} \to reals$ via $f_x(y) = f(x, y)$.

(*a*) Show that $\mathbb{X}$ is locally compact and Hausdorff.

(*b*) Show that $f : \mathbb{X} \to \mathbb{R}$ is in $C_c(\mathbb{X})$ if and only if f_x is continuous for every x and $f_x \neq 0$ for at most finitely many x.

(*c*) Show that the formula

$$L(f) := \sum_x \int f_x \mathrm{d}\lambda \tag{2}$$

(where λ is Lebesgue measure on $\mathbb{R}$) defines a positive linear functional on $C_c(\mathbb{X})$ and find the regular Borel measure associated to L by the Riesz representation theorem.

(*d*) Show that if μ is the regular Borel measure on $\mathbb{X}$ that corresponds to L, then there exist a Borel set A such that the following strict inequality holds

$$\mu(A) > \sup\{\mu(K) : K \subseteq A,\ K\ \text{compact}\}\,. \tag{3}$$

Problem 3

For $U, V \subseteq \mathbb{R}^d$ bounded open sets, let $F : \overline{U} \to \overline{V}$ be a bijection such that $F \in C^1(\overline{U})$, $F^{-1} \in C^1(\overline{V})$. Recall that we define the measure $F_\# \lambda$ on V by letting, for any $A \in \mathcal{B}_V$,

$$F_\# \lambda(A) = \lambda(F^{-1}(A)) \,. \tag{4}$$

In this exercise, we denote by $\mathrm{d}\varphi$ the differential of a map φ.

(*a*) Let $M := \sup_{y \in V} \|\mathrm{d}F^{-1}(y)\|_{\mathrm{op}}$. Prove that, for any $A \in \mathcal{B}_V$,

$$F_\# \lambda(A) \leq 2M^d \lambda(A) \,. \tag{5}$$

Deduce that $F_\# \lambda$ is absolutely continuous with respect to λ.

(*b*) Prove that there exist $\delta(\varepsilon)$ with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that the following happens. For any axis parellel cube Q, $y \in Q$, with side length $\ell(Q) \leq \varepsilon$, we have

$$\left| \frac{F_\# \lambda(Q)}{\lambda(Q)} - |\det(\mathrm{d}F(F^{-1}(y)))| \right| \leq \delta(\varepsilon) \,. \tag{6}$$

(*c*) Deduce that, for $\mu = F_\# \lambda$,

$$D\mu(y) = \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(y) = |\det(\mathrm{d}F(F^{-1}(y)))| \,. \tag{7}$$