Math 205A Real Analysis

Practice Problems for Midterm

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Solutions should be complete and concisely written. Please, mark clearly the beginning and end of each problem.

You have 80 minutes but you are not required to solve all the problems!

Just solve those that you can solve within the time limit. Points assigned to each problem are indicated in parenthesis. I recommend to look at all problems before starting.

For any clarification on the text, the TA's will be outside the room.

You can consult textbooks (Billinglsely, Cohn) and your notes. You cannot use computers, and in particular you cannot use the web. You can cite theorems (propositions, corollaries, lemmas, etc.) from textbooks by number, and exercises you have done as homework by number as well. Any other non-elementary statement must be proven!

Problem 1 (30 points)

Throughout this problem $(\mathbb{X}, \mathcal{F}, \mu)$ is a probability space, that is a measure space with $\mu(\mathbb{X}) = 1$. For a measurable function $f : \mathbb{X} \to \mathbb{R}$, and for $p > 0$, we define the p pseudo-norm of f via

$$\|f\|_p \equiv \left(\int |f(x)|^p \mathrm{d}\mu(x) \right)^{1/p} \tag{1}$$

(a) Assume that $\|f\|_q < \infty$ for some $q > 0$. Show that

$$\lim_{p \to 0} \|f\|_p = \exp\left\{ \int \log |f(x)| \mathrm{d}\mu(x) \right\} . \tag{2}$$

(b) Assume that $\|f\|_q < \infty$ for some $q > 0$. What is the value of $\lim_{p\to 0} \|f\|_p^p$?

(c) The 'weak ℓ_p norm' $\|f\|_{w\ell_p}$ of a measurable function f is defined as

$$\|f\|_{w\ell_p} \equiv \sup_{t \geq 0} \left\{ t \left[\mu(|f| \geq t) \right]^{1/p} \right\} . \tag{3}$$

Notice that this is not a norm. Prove that $\|f\|_{w\ell_p} \leq \|f\|_p$.

Problem 2 (20 points)

Let Ω be the space of functions $\omega : [0, 1] \to \mathbb{R}$, and, for each $t \in [0, 1]$, define $f_t(\omega) = \omega(t)$. Let $\mathcal{F} \equiv \sigma(\{f_t\}_{t\in[0,1]})$ be the smallest σ-algebra such that f_t is measurable for each $t \in [0, 1]$.

Also, for any $S \subseteq [0, 1]$, le $\mathcal{F}_S \equiv \sigma(\{f_t\}_{t\in S})$ be the smallest σ-algebra such that X_t is measurable for each $t \in S$.

(a) Prove that

$$\mathcal{F} = \bigcup_{S \text{ countable}} \mathcal{F}_S \, . \tag{4}$$

(b) Show that, for any measurable function g on $(\Omega, \mathcal{F})$ there exists S countable such that g is measurable on $(\Omega, \mathcal{F}_S)$.