Math 205A Real Analysis

Practice Midterm Solutions

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Problem 1

Throughout this problem $(\mathbb{X}, \mathcal{F}, \mu)$ is a probability space, that is a measure space with $\mu(\mathbb{X}) = 1$. For a measurable function $f : \mathbb{X} \to \mathbb{R}$, and for $p > 0$, we define the p pseudo-norm of f via

$$\|f\|_p \equiv \left(\int |f(x)|^p \mathrm{d}\mu(x) \right)^{1/p} \tag{1}$$

(a) Assume that $\|f\|_q < \infty$ for some $q > 0$. Show that

$$\lim_{p \to 0} \|f\|_p = \exp \left\{ \int \log |f(x)| \mathrm{d}\mu(x) \right\} . \tag{2}$$

Write $h(x) = \log |f(x)| = h_+(x) - h_-(x)$:

$$\lim_{p \to 0} \frac{1}{p} \big(\|f\|_p^p - 1 \big) = \int \frac{1}{p} \big(e^{ph(x)} - 1 \big) \mu(\mathrm{d}x) = A_+(p) - A_-(p) \,,$$
$$A_\pm(p) = \int \frac{\pm 1}{p} \big(e^{\pm p h_\pm(x)} - 1 \big) \mu(\mathrm{d}x) \,.$$

By monotone convergence

$$\lim_{p \to 0} A_-(p) = \int h_-(x) \mu(\mathrm{d}x) \,.$$

By dominated convergence, using $(e^{ph_+(x)} - 1)/p \le (e^{qh_+(x)} - 1)/q$ for $0 < p \le q$, we get

$$\lim_{p \to 0} A_+(p) = \int h_+(x) \mu(\mathrm{d}x) \,.$$

We thus have

$$\lim_{p \to 0} \frac{1}{p} \big(\|f\|_p^p - 1 \big) = \int \log |f(x)| \mathrm{d}\mu(x) \,,$$

and the claim follows by calculus.

(b) Assume that $\|f\|_q < \infty$ for some $q > 0$. What is the value of $\lim_{p \to 0} \|f\|_p^p$?

We have $|f(x)|^p \to \mathbf{1}_{f(x) \neq 0}$ as $p \to 0$. Hence applying dominated convergence (and the assumption $\|f\|_q < \infty$)

$$\lim_{p \to 0} \|f\|_p^p = \mu(\{f(x) \neq 0\}) \,. \tag{3}$$

(c) The 'weak ℓ_p norm' $\|f\|_{w\ell_p}$ of a measurable function f is defined as

$$\|f\|_{w\ell_p} \equiv \sup_{t\geq 0}\Big\{ t\,\big[\mu(|f|\geq t)\big]^{1/p}\Big\}\,. \tag{4}$$

Notice that this is not a norm. Prove that $\|f\|_{w\ell_p} \leq \|f\|_p$.

By Markov inequality $t^p, \big[\mu(|f|\geq t)\big] \leq \int |f|\,\mathrm{d}\mu$.

Problem 2

Let Ω be the space of functions $\omega : [0,1] \to \mathbb{R}$, and, for each $t \in [0,1]$, define $f_t(\omega) = \omega(t)$. Let $\mathcal{F} \equiv \sigma(\{f_t\}_{t\in[0,1]})$ be the smallest σ-algebra such that f_t is measurable for each $t \in [0,1]$.

Also, for any $S \subseteq [0,1]$, le $\mathcal{F}_S \equiv \sigma(\{f_t\}_{t\in S})$ be the smallest σ-algebra such that f_t is measurable for each $t \in S$.

(a) Prove that

$$\mathcal{F} = \bigcup_{S \text{ countable}} \mathcal{F}_S\,. \tag{5}$$

Solution: Let $\mathcal{A} \equiv \bigcup_{S \text{ countable}} \mathcal{F}_S$. It is clear that f_t is measurable on $\mathcal{A}$ for each $t \in [0,1]$. Indeed, $\mathcal{A}$ contains in particular $\mathcal{F}_{\{t\}} = \sigma(f_t)$.

Further $\mathcal{A} \subseteq \mathcal{F}$, since $\mathcal{F}_S \subseteq \mathcal{F}$ for each $S \subseteq [0,1]$ (indeed $\mathcal{F}_S$ is the *minimal* σ algebra such that f_t is measurable for each $t \in S$).

The claim follows if we show that $\mathcal{A}$ is a σ-algebra. Let $B \in \mathcal{A}$. Then $B \in \mathcal{F}_S$ for some S countable, whence $B^{\mathrm{c}} \in \mathcal{F}_S$ (because $\mathcal{F}_S$ is a σ-algebra) and thus $B^{\mathrm{c}} \in \mathcal{A}$. Therefore $\mathcal{A}$ is closed under complements.

Let $\{B_i\}_{i\in\mathbb{N}}$ be a countable collection in $\mathcal{A}$. Then there exist countable sets $S_i \subseteq [0,1]$ such that $B_i \in \mathcal{F}_{S_i}$ for each i. In particular $B_i \in \mathcal{F}_S$ with $S = \cup_{i=1}^{\infty} S_i$. Let $B \equiv \cup_{i=1}^{\infty} B_i$. By the σ-algebra property, $B \in \mathcal{F}_S$ as well. But S is countable (countable union of countable sets), whence $B \in \mathcal{A}$.

(b) Show that, for any measurable function g on $(\Omega, \mathcal{F})$ there exists S countable such that g is measurable on $(\Omega, \mathcal{F}_S)$.

Solution: Let $\mathbb{Q} = \{q_1, q_2, q_3, \dots\}$ be an ordering of the rationals. By point (a) above, for each i, there exist S_i countable, such that the set $B_i = \{\omega : g(\omega) \leq q_i\}$ is in in $\mathcal{F}_{S_i}$. As a consequence for each i, $B_i \in \mathcal{F}_S$ with $S \equiv \cup_{i=1}^{\infty} S_i$. This imply that $\{Z^{-1}((-\infty, q]) : q \in \mathbb{Q}\} \subseteq \mathcal{F}_S$. Since $\mathcal{P} = \{(-\infty, q] : q \in \mathbb{Q}\}$ is a π system which generates the Borel σ-algebra, the thesis follows.
