Practice Final Solutions (sketches)

1)

(a) X_1, X_2 are compact since they are homeomorphic, respectively to S^1 (the unit circle) and $[0,1]$.

Hence

$C_c(X_1) = C(X_1)$ is the set of functions $f : \mathbb{R} \cup \{\infty\} \to \mathbb{R}$ st $f|_{\mathbb{R}}$ is cont. (as a function on $\mathbb{R}$)

and $f(\infty) = \lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x)$

$C_c(X_2) = C(X_2)$ is the set of functions $f : \mathbb{R} \cup \{+\infty, -\infty\} \to \mathbb{R}$ st $f|_{\mathbb{R}}$ is cont and

$f(+\infty) = \lim_{x \to +\infty} f(x)$, $f(-\infty) = \lim_{x \to -\infty} f(x)$.

(b) L is well defined because

$\lim_{x \to +\infty} f(x)$ exists finite both ~~on~~ for

$f \in C_c(X_1)$ and for $f \in C_c(X_2)$.

Linear because of

$$\lim_{x \to +\infty} (a f_1 + b f_2) = a \lim_{x \to +\infty} f_1 + b \lim_{x \to +\infty} f_2$$

for any $f_1, f_2 \in C_c(X_1)$ or $C_c(X_2)$.

$$\mu_1 = \delta_\infty, \quad \mu_2 = \delta_{+\infty}.$$

(c) on X_1, $T(f) = 0 \ \forall f \in C_c(X_1)$

hence can take $\nu_1 = 0$.

on X_2 $T(f)$ well defined by prev

point $\nu_2 = \delta_{+\infty} - \delta_{-\infty}$.

Problem 2

(a) If $x \in [0,1] \setminus C$ then there exists $n_0 < \infty$ such that ~~x~~ $x \in [0,1] \setminus C_{n_0}$

Since $[0,1] \setminus C_n$ is open, ~~$(x-\epsilon, x+\epsilon) \in$~~

$(x-\epsilon, x+\epsilon) \in [0,1] \setminus C_n \quad \forall n \geq n_0$

$\Rightarrow F_n(t) = F_n(x) \quad \forall t \in (x-\epsilon, x+\epsilon)$

Further $F_n(x) = F(x) \quad \forall n \geq n_0$ (explain)

hence $F_n(t) = F(x) \quad \forall t \in (x-\epsilon, x+\epsilon)$

Hence F differentiable at x with $F'(x) = 0$.

(b) Let $\underline{k} = (k_1, k_2, \ldots k_n) \in \{0,2\}^n$

$a(\underline{k}) := \sum_{i=1}^{n} \frac{k_i}{3^i}$, $\quad b(\underline{k}) := \sum_{i=1}^{n} \frac{k_i}{3^i} + \frac{1}{3}$

Then

$$C_n := \bigcup_{\underline{k}\in\{0,2\}^n} [a(\underline{k}), b(\underline{k})]$$

and $\forall\, x \in \mathcal{C}$ $\cancel{\exists}$ $\forall n$ $\exists\, \underline{k}^{(n)} \in \{0,2\}^n$

st $\quad a(\underline{k}^{(n)}) \leq x \leq b(\underline{k}^{(n)})$

(Indeed $\quad x = \sum_{i=1}^{\infty} \frac{c_i}{3^i}$ for $c \in \{0,2\}^{\mathbb{N}_{>0}}$.

Just ~~k~~ take $\cancel{c}$ $k_i^{(n)} = c_i \;\forall i \leq n$.)

Further $\quad \mu_n([a(\underline{k}^{(n)}), b(\underline{k}^{(n)})]) = \frac{1}{2^n}$,

$$\Rightarrow \mu([a(\underline{k}^{(n)}), b(\underline{k}^{(n)})]) = \frac{1}{2^n}$$

$$\Rightarrow F(b(\underline{k}^{(n)})) - F(a(k^{(n)})) = \frac{1}{2^n}$$

$\Rightarrow$ $\cancel{\limsup_{a,b\to x}}$ $\forall \epsilon > 0$

$$\cancel{\limsup_{\substack{a\to 0+\\ b\to 0+}}} \quad \sup_{\substack{x < b \leq x+\epsilon \\ x-\epsilon \leq a < x}} \frac{F(b)-F(a)}{b-a} \geq \frac{1/2^n}{1/3^n} \xrightarrow{n\to\infty} \infty$$

$\Rightarrow$ not differentiable.

(c) Lebesgue theorem implies that

$x \mapsto F(x)$ is differentiable at λ-almost every x.

Further $F'(x) = D\mu(x)$ is the RN derivative $\frac{d\mu_{ac}}{d\lambda}(x)$ a.e. where

μ_{ac} is the a.c. part of μ.

Since μ is singular wrt. λ.

This implies $F'(x) = 0$ a.e.

Problem 3

(a) $$(\mu_1 \otimes \mu_2)(\mathbb{R}^2 \setminus L_\epsilon) = \int 1(|x-y|>\epsilon)\ \mu_1 \otimes \mu_2(dx,dy)$$

$$\overset{(1)}{=} \int \left[\int 1(|x-y|>\epsilon)\ \mu_2(dy) \right] \mu_1(dx)$$

$$= \int \mu_2([x-\epsilon, x+\epsilon]^c)\ \mu_1(dx)$$

where (1) holds by Fubini

Since $\int f\, d\mu = 0$ for $f \geq 0$ implies $f = 0$ μ-a.e. the claim follows.

(b) The question as stated is trivial.

~~(see below for a more interesting one)~~

However the claim holds with

$x_1 = x_2$, (see below)

(c) The bound on $\max(|a_1-b_1|, |a_2-b_2|)$ cannot be improved. P7

Take for instance $\mu_1 = \delta_x$ ~~μ_2 = Unif~~

$\mu_2 = 1_S \lambda$ where $S = [x-\epsilon, x+\epsilon]$.

However the following more general bounds also hold

$$b_2 - a_1 \leq \epsilon, \quad b_1 - a_2 \leq \epsilon.$$

To see for instance the first one, let $\delta > 0$ be smaller than ϵ and

$$f(x,y) = 1(|x-y| > \epsilon)\, 1_{x < a_1 + \delta}\, 1_{y > b_2 - \delta}$$

$$0 = \int f(x,y)\, \mu_1 \otimes \mu_2(dx, dy)$$

$$= \cancel{\int \left[\int 1(|x-y| > \epsilon) \, 1_{y > b_2 - \delta} \right]}$$

$$= \int \left[\int 1(|x-y|>\epsilon)\, 1_{y>b_2-\delta}\, \mu_2(dy) \right] 1_{x<a_1+\delta}\, \mu_1(dx)$$

Assume $b_2 - a_1 > \epsilon$ by contradiction

then taking $\delta < \dfrac{b_2 - a_1 - \epsilon}{3}$ we have

$$y > b_2 - \delta,\quad x < a_1 + \delta \implies |x-y| > \epsilon$$

Therefore the above implies

$$0 = \int \mu_2([b_2-\delta, b_2])\, 1_{x<a_1+\delta}\, \mu_1(dx)$$

$$= \mu_2([b_2-\delta, b_2]) \cdot \mu_1([a_1, a_1+\delta))$$

but both of the above are nonzero by def. hence we have a contradiction

~~We~~ Choosing wlog $a_1 \leq a_2$ we therefore have

$$a_1 \leq a_2 \leq b_2 \leq a_1 + \epsilon$$

and $b_1 \in (a_1, a_2 + \epsilon]$.

Therefore taking $x = b_2$ we have that μ_1, μ_2 are supported in $[x-\epsilon, x+\epsilon]$

Problem 4

(a) Let $\nu := f\mu$ and define $\nu|_{\mathcal{G}}$, $\mu|_{\mathcal{G}}$ to be the restrictions of ν, μ to the σ-algebra $\mathcal{G}$.

Note that $\nu \ll \mu$ and therefore $\nu|_{\mathcal{G}} \ll \mu|_{\mathcal{G}}$ (indeed if $A \in \mathcal{G}$ is s.t. $\mu(A) = 0$, then $\nu(A) = 0$ because $\nu \ll \mu$).

By the RN thm $\nu|_{\mathcal{G}} = g \cdot \mu|_{\mathcal{G}}$ for some $\mathcal{G}$-meas function g.

Hence $\forall A \in \mathcal{G}$

$$\nu(A) = \int g \, 1_A \, d\mu.$$

By def

$$\nu(A) = \int f \, 1_A \, d\mu$$

Further
$$\nu(A) \leq \int f_+ \, d\mu < \infty \quad \forall A$$
$$\nu(A) \geq -\int f_- \, d\mu > -\infty$$

Hence
$$\int g \, 1_{g>0} \, d\mu \leq \int f_+ \, d\mu < \infty$$
$$\int g \, 1_{g\leq 0} \, d\mu \geq -\int f_- \, d\mu > -\infty$$

$$\Rightarrow \quad g \in L^1(\mathbb{X}, \mathcal{G}, \mu).$$

(b) Assume two such fcts g_1 g_2 exist

Then

$$\int g_1 1_A \, d\mu = \int g_2 1_A \, d\mu. \quad \forall A \in \mathcal{G}$$

Take $A = \{g_1 \geq g_2\}$

$$\Rightarrow \int (g_1 - g_2)\, 1_{g_1 \geq g_2}\, d\mu = 0$$

$$\Rightarrow (g_1 - g_2)_+ = 0 \quad \text{a.e.}$$

Inverting the role of g_1, g_2, we get the claim.