Math 205A Real Analysis

Homework on Functional Inequalities

Andrea Montanari *April 15, 2026*

- This homework is to help you test your undestanding of the topics, and your ability to fill in certain type of details detailsthat we miss in class.
- You are not expected to submit solutions, but you are welcome to discuss them during office hours.

Problems

1. Let $f \in H^1(\mathbb{R}^n)$. Prove that $|f| \in H^1(\mathbb{R}^n)$ and $\|\nabla|f|\|_{L^2(\mathbb{R}^n)} \le \|\nabla f\|_{L^2(\mathbb{R}^n)}$.

2. Let $f, g : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ be tro non-negative Borel functions and denote by f^* and g^* their spherically symmetric decreasing rearrangements. Prove that $\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x \le \int_{\mathbb{R}^n} f^*(x)g^*(x)\,\mathrm{d}x$.

3. Complete the proof of the following extension theorem whose proof was sketched in class.

   Theorem 1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, $\Gamma := \partial\Omega$ be its boundary and assume Γ is C^1. Then, for $p \in [0, \infty)$ there exists a linear extendion operator $T : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ such that:*

   (a) $Tf|_\Omega = f$.

   (b) $\|f\|_{L^p(\Omega)} \le \|Tf\|_{L^p(\mathbb{R}^n)} \le C_{\Omega,p}\|f\|_{L^p(\Omega)}$.

   (c) $\|\nabla f\|_{L^p(\Omega)} \le \|\nabla Tf\|_{L^p(\mathbb{R}^n)} \le C_{\Omega,p}\|f\|_{W^{1,p}(\Omega)}$.

4. Construct an example of an open set $\Omega \subseteq \mathbb{R}^n$ with finite Lebesgue measure and C^1 boundary, such that the (normalized) uniform measure over Ω, denoted by μ does not admit a Poincaré inequality. Namely

$$\inf_{f \in C^\infty(\Omega)} \frac{\|\nabla f\|_{L^2(\mu)}}{\|f - \mu(f)\|_{L^2(\mu)}} = 0\,. \tag{1}$$

5. Prove the following properties of the entropy of discrete variables that we used without proof in proving the Loomis-Whitney inequality:

   (a) $0 \le H(X) \le \log|A|$ for any radom variable X taking values in the finite set A.

   (b) Chain rule: $H(X, Y) = H(X) + H(Y|X)$.

6. In class we proved the Sobolev embedding inequality $\|f\|_{L^q(\mathbb{R}^n)} \le C_{p,n}\|\nabla f\|_{L^p(\mathbb{R}^n)}$ for $1/q = 1/p - 1/n$ and any $f \in C_c^\infty(\mathbb{R}^n)$. Write the approximation argument to extend the inequality to any $f \in W^{1,p}(\mathbb{R}^n)$.