**Stat 316 Stochastic Processes on Graphs**

# Introduction and the Curie-Weiss model

*Andrea Montanari* *Lecture 1-2 - 9/24/2007*

The Curie-Weiss model is deceivingly simple, but is a good pretext for explaining what are we talking about.

## 1 A story about opinion formation

At time 0, each of $N$ individuals takes one of two opinions $x_i \in \{+1, -1\}$ independently and uniformly at random for $i \in \{1, \dots, N\}$. At each subsequent time, one individual $i$ chosen uniformly at random considers whether changing its opinion. If the majority of the other individuals disagree with her, then she changes her opinion. This is just *conformism.*

In the other case, she takes an *anti-conformist* behavior with probability that becomes larger when the aggregate opinion is close to neutral. To be definite, she computes the opinion imbalance

$$M \equiv \sum_{j=1}^{N} x_j \, . \tag{1}$$

and $M^{(i)} \equiv M - x_i$. Then she changes her opinion with probability

$$p_{\rm flip}(x) = \exp\left\{-2\beta |M^{(i)}|/N\right\} \, . \tag{2}$$

Her choice depends on the history so far only through the current value of $M^{(i)}$.

Let $M(t)$ be the opinion imbalance after $t$ steps. Despite its simplicity, this model raises several interesting questions.

1. How long does is take for the process to become approximately stationary?
2. How often do individuals change opinion in stationary state?
3. Is the typical opinion pattern strongly polarized (*herding*)?
4. If this is the case, how often does the popular opinion change?

These lectures are not concerned with question 1, but with (some version of) questions 2 to 4.

To be more precise, notice that the above dynamics is an aperiodic irreducible Markov chain whose (unique) stationary state is

$$\mu_{N,\beta}(x) = \frac{1}{Z_N(\beta)} \exp\left\{\frac{\beta}{N}\sum_{(i,j)} x_i x_j\right\} \, . \tag{3}$$

To prove this it is sufficient to check that the above dynamics is *reversible* with respect to the measure $\mu_{N,\beta}$, i.e. that $\mu_{N,\beta}(x)\mathbb{P}(x \to x') = \mu_{N,\beta}(x')\mathbb{P}(x' \to x)$ for any two configurations $x$, $x'$ (where $\mathbb{P}(x \to x')$ is the one-step transition probability).

In writing $\mu_{N,\beta}$, we emphasized the dependence of this distribution on $N$ (population size) and $\beta$ (interaction strength). We will be particularly interested in the large-$N$ behavior, and its dependence on $\beta$.

Then we can ask the following 'static' version of the above questions

2'. What is the distribution of $p_{\rm flip}(x)$ when $x$ has distribution $\mu_{N,\beta}(\,\cdot\,)$.

3'. What is the distribution of the opinion imbalance $M$. Is it concentrated near 0 (evenly spread opinions), or far from 0 (herding)?

4'. In the herding case: how unlikely are balanced ($M \approx 0$) configurations?

**Exercise 1:** Which measures replaces $\mu_{N,\beta}(\,\cdot\,)$ of (3) if the flipping probability is a general function of $M^{(i)}$ and possibly $x_i$ but nothing else?

[You are not required to solve the exercises in these notes. They are pretty difficult and represent suggestions for those students who are interested in a deeper understanding of the material.]

# 2 Stochastic processes on graphs

Recall that a graph $G = (V, E)$ is defined by a set of vertices $V$ and of edges $E$ (an edge being an unordered pair of vertices.) We shall always assume $G$ to be finite with $|V| = N$ and often make the identification $V = [N]$.

In these lectures a **stochastic process on graph** $G$ is a process, i.e. a collection of random variables, indexed by the vertices of $G$, $X = \{X_i : i \in V\}$. We shall further assume that the joint distribution $\mu(x) = \mathbb{P}\{X = x\}$ of such variables factorizes according to $G$. By this we mean that there exist non-negative weights $\psi_{ij}$ such that

$$\mu(x) = \frac{1}{Z} \prod_{(i,j)\in E} \psi_{ij}(x_i, x_j)\,. \tag{4}$$

Finally we shall restrict our attention to the case of variables $X_i$ taking value in a finite alphabet $\mathcal{X}$. It is not hard to see that the distribution (3) takes this form (with $G$ being the complete graph over $N$ vertices.)

Rather than studying stochastic processes on graphs in this generality, we shall mostly focus on a few concepts/tools that have been object of recent research effort.

**Phase transitions.** Roughly speaking, we will say that a phase transition occurs when the measure $\mu$ decomposes into the convex combination of well separated lumps. In order to formalize this notion, we consider sequences of measures $\mu_N$ on graphs $G_N = ([N], E_N)$, and say that a phase transition occurs if, for each $N$, there exists a partition of the configuration space into subsets $\Omega_{1,N}, \dots, \Omega_{r,N}$ such that the following happens

1. The measure of any of the subsets in the partition is bounded away from 1:

$$\max\{\mu_N(\Omega_{1,N}), \dots, \mu_N(\Omega_{r,N})\} \le 1 - \delta\,. \tag{5}$$

2. The subsets are separated by 'bottlenecks.' More precisely, for $\Omega \subseteq \mathcal{X}^N$, define its $\epsilon$-boundary as

$$\partial_\epsilon \Omega \equiv \{x \in \mathcal{X}^N : 1 \le d(x, \Omega) \le N\epsilon\}\,. \tag{6}$$

where $d$ is the Hamming distance[1]. Then we require

$$\frac{\mu_N(\partial_\epsilon \Omega_{s,N})}{\mu_N(\Omega_{s,N})(1 - \mu_N(\Omega_{s,N}))} \to 0\,, \tag{7}$$

for some $\epsilon > 0$ and all $s \in \{1, \dots, r\}$. The normalization by $\mu_N(\Omega_{s,N})$ is introduced to avoid false bottlenecks due to small $\Omega_{s,N}$. The term $1 - \mu_N(\Omega_{s,N})$ is there just for the sake of symmetry.

Occasionally, we will require the above limit to be approached at some specific rate.

[1] The Hamming distance $d(x, x')$ between configurations $x$ and $x'$ is the number of positions in which the two configurations differ. Given $\Omega \subseteq \mathcal{X}^N$, $d(x, \Omega) \equiv \min\{d(x, x') : x' \in \Omega\}$.

**Mean field models.** Again roughly speaking, mean field models are models that lack any (finite-dimensional) geometrical structure. For instance, typically models on the complete graphs or on standard random graphs are mean field. On the other hand, models on (finite portions) of finite dimensional grids are not.

A particular class of mean field models is defined by the requirement that $\mu(x_1, \dots, x_N)$ is exchangeable.

A wider class is obtained by considering *random* distributions[2] $\mu$. Given $\mu$, consider $k$ iid configurations $X^{(1)}, \dots, X^{(k)}$ each having distribution $\mu$. These are called 'replicas' in statistical physics. The unconditional, joint distribution of these $k$-copies is

$$\mu^{(k)}(x^{(1)}, \dots, x^{(k)}) = \mathbb{E}\left\{\mu(x^{(1)}) \cdots \mu(x^{(k)})\right\}, \tag{8}$$

which we view as a distribution over $(\mathcal{X}^k)^N$. For the model to be mean field we require $\mu^{(k)}$ to be exchangeable with respect to permutations of the vertices indexes in $[N]$.

While such requirement is sufficient in 'natural' examples, there are examples of models that intuitively are not mean-field and yet meet the requirement. For instance, given a non-random measure $\mu$, and a uniformly random permutation $\pi$, define $\mu_\pi(x_1, \dots, x_N) \equiv \mu_\pi(x_{\pi(1)}, \dots, x_{\pi(N)})$. Then $\mu_\pi$ meets the requirement. A satisfactory formalization of the intuitive notion of 'mean field model,' is an open problem.

Mean field models are a pretty restrictive class, but a rich array of phenomena can be studied in detail.

**Mean field equations.** The problem with the model (4) is that distinct variables can be correlated in very subtle ways. Nevertheless, mean field models are often tractable because an effective 'reduction' to local marginals[3] takes place asymptotically for large sizes (i.e. as $N \to \infty$).

Thanks to this reduction it is often possible to write a close system of equations for the local marginals that hold in the large size limit. Such equations allow to determine the local marginals up to (eventually) a finite multiplicity. Finding a good formalization of this notion is an open problem. We shall instead provide specific examples throughout the course.

## 3 The Curie-Weiss model: Phase transition

The model (3) appeared for the first time in the physics literature as a model for ferromagnets[4]. In this context, the variables $x_i$ are called *spins* and their value represents the direction in which a localized magnetic moment (think of a tiny compass needle) is pointing. In some materials different magnetic moments like to point in the same direction (as people like to have similar opinions). Physicists want to understand whether this interaction might lead to a macroscopic magnetization (imbalance), or not.

In order to study the model, it is convenient to generalize it slightly by introducing a linear term in the exponent ('magnetic field')

$$\mu_{N,\beta}(x) = \frac{1}{Z_N(\beta)} \exp\left\{\frac{\beta}{N} \sum_{(i,j)} x_i x_j + h \sum_{i=1}^{N} x_i\right\}. \tag{9}$$

In this context $1/\beta$ is referred to as the 'temperature.' We shall always assume $\beta \geq 0$ (positive interaction) and (without loss of generality) $h \geq 0$.

The good question to ask for understanding the Curie-Weiss model (9) is: what is the distribution of the magnetization?

**Lemma 1.** *For $m \in [-1, +1]$, define*

$$\psi_\beta(m) = hm + \frac{1}{2}\beta m^2 + H\left(\frac{1+m}{2}\right), \tag{10}$$

[2] A random distribution over $\mathcal{X}^N$ is just a random variable taking values on the $(|\mathcal{X}|^N - 1)$-dimensional standard simplex.

[3] In particular, single variable marginals, or joint distributions of two variables connected by an edge.

[4] A ferromagnet is a material that acquires a macroscopic spontaneous magnetization at low temperature.

*where $H(x) = -x\log x - (1-x)\log(1-x)$ is the binary entropy function.*

*Let $M \in \{-N, -N+2, \dots, N-2, N\}$, and $X = (X_1, \dots, X_N)$ be a random configuration from the Curie-Weiss model. Then,*

$$\frac{e^{-\beta/2}}{N+1}\,\frac{1}{Z_N(\beta)}e^{N\psi_\beta(M/N)} \le \mathbb{P}\left\{\sum_{i=1}^N X_i = M\right\} \le \frac{1}{Z_N(\beta)}\,e^{N\psi_\beta(M/N)}\,. \tag{11}$$

**Proof** It is immediate to see that

$$\mathbb{P}\left\{\sum_{i=1}^N X_i = M\right\} = \frac{1}{Z_N(\beta)}\binom{N}{(N+M)/2}\exp\left\{hM + \frac{\beta M^2}{2N} - \frac{1}{2}\beta\right\}\,. \tag{12}$$

Our thesis then follows by Stirling approximation of thre binomial coefficient, cf. [CT91], Theorem 12.1.3. □

A major role in this course is played by the free-entropy density (the term 'density' refers here to the fact that we are dividing by the number of variables),

$$\phi_N(\beta) = \frac{1}{N}\log Z_N(\beta)\,. \tag{13}$$

**Lemma 2.** *Let*

$$\phi_*(\beta) \equiv \sup\left\{\psi_\beta(m) : m \in [-1,1]\right\}\,, \tag{14}$$

*and $\phi_N(\beta)$ be the free entropy density of the Curie-Weiss model. Then, for all $N$ large enough*

$$\phi_*(\beta) - \frac{\beta}{2N} - \frac{1}{N}\log\{N(N+1)\} \le \phi_N(\beta) \le \phi_*(\beta) + \frac{1}{N}\log(N+1)\,. \tag{15}$$

**Proof** The upper bound follows upon summing the upper bound in Eq. (11) over $M$. From the lower bound in the same equation, we get

$$\phi_N(\beta) \ge \max\left\{\psi_\beta(m) : m \in S_N\right\} - \frac{\beta}{2N} - \frac{1}{N}\log(N+1)\,. \tag{16}$$

where $S_N \equiv \{-1, -1+2/N, \dots, 1-2/N, 1\}$. A little calculus shows that maximum of $\psi_\beta(m)$ over the finite set $S_N$ is not smaller that the maximum over the interval $[-1,+1]$ minus $(\log N)/N$, for all $N$ large enough. □

Consider the optimization problem in Eq. (14). Since $\psi_\beta(m)$ is continuous in the interval $[-1,1]$ and differentiable in its interior, with $\psi'_\beta(m) \to \pm\infty$ as $m \to \mp 1$, the maximum is achieved at points $m \in (-1,1)$ such that $\psi'_\beta(m) = 0$. A direct calculation shows that this condition is equivalent to

$$m = \tanh(\beta m + h)\,. \tag{17}$$

It is not hard to study the solutions of this equation. Here we limit ourself to presenting the results.

For $\beta \le 1$, the equation admits a unique solution $m_*(\beta,h)$ increasing in $h$ with $m_*(\beta,h) \downarrow 0$ as $h \downarrow 0$. Obviously $m_*(\beta,h)$ maximizes $\psi_\beta(m)$.

For $\beta \ge 1$ there exists $h_*(\beta) > 0$ continuously increasing in $\beta$ with $\lim_{\beta\to 1} h_*(\beta) = 0$ such that the following happens. For $0 \le h < h_*(\beta)$, Eq. (17) admits three distinct solutions $m_-(\beta,h) < m_0(\beta,h) \le 0 \le m_+(\beta,h)$. For $h = h_*(\beta)$ two of these solutions coincide $m_-(\beta,h) = m_0(\beta,h)$ and for $h > h_*(\beta)$ only the positive one $m_+(\beta,h)$ survives.

Further, for any $h > 0$, $m_\pm(\beta,h)$ correspond to local maxima of $\psi_\beta(m)$, while $m_0(\beta,h)$ is a local minimum. The global maximum coincides with $m_+(\beta,h)$, that we shall henceforth denote as $m_*(\beta,h)$.

At $h = 0$ (and always $\beta > 1$), $\psi_\beta(m)$ is an even function of $m$. As a consequence $m_0(\beta,0) = 0$ and $m_\pm(\beta,0) = \pm m_*(\beta,0)$.

The theorem below answers question 3' in Section 1 of these notes.

**Theorem 3.** *Let $m_*(\beta, h)$ be defined as above, $X$ be a random configuration of the Curie-Weiss model and $\overline{X} \equiv N^{-1}\sum_{i=1}^N X_i$. For $h > 0$ or $h = 0$ and $\beta \le 1$, and for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that, for all $N$ large enough*

$$\mathbb{P}\left\{\left|\overline{X} - m_*(\beta, h)\right| \le \varepsilon\right\} \ge 1 - e^{-NC(\varepsilon)}\,. \tag{18}$$

*For $h = 0$ and $\beta > 1$, and for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that, for all $N$ large enough*

$$\mathbb{P}\left\{\left|\overline{X} - m_*(\beta, 0)\right| \le \varepsilon\right\} = \mathbb{P}\left\{\left|\overline{X} + m_*(\beta, 0)\right| \le \varepsilon\right\} \ge \frac{1}{2} - e^{-NC(\varepsilon)}\,. \tag{19}$$

**Proof** Consider first the case $\beta \le 1$ or $h > 0$. Under this assumption, $\psi_\beta(m)$ has a non-degenerate maximum at $m = m_*(\beta, h)$. Then, by Lemma 1

$$\mathbb{P}\left\{|\overline{X} - m_*(\beta, h)| \ge \varepsilon\right\} \le \frac{1}{Z_N(\beta)}\,(N+1)\,\exp\Big\{N \max[\psi_\beta(m) : |m - m_*(\beta, h)| \ge \varepsilon]\Big\}\,. \tag{20}$$

Using Lemma 2 we get

$$\mathbb{P}\left\{|\overline{X} - m_*(\beta, h)| \ge \varepsilon\right\} \le (N+1)^3 e^{-\beta/2}\,\exp\Big\{N \max[\psi_\beta(m) - \phi_*(\beta) : |m - m_*(\beta, h)| \ge \varepsilon]\Big\}\,. \tag{21}$$

whence Eq. (18) follows.

Equation (19) is proved analogously, using the symmetry of the model for $h = 0$. □

We just encountered our first example of phase transition.

**Theorem 4.** *The Curie-Weiss model undergoes a phase transition if and only if $h = 0$ and $\beta > 1$.*

**Proof** We will limit ourselves to the 'if' part of this statement: for $h = 0$, $\beta > 1$, the Curie-Weiss model has a phase transition. Consider the partition of the configuration space given by $\{+1, -1\}^V = \Omega_+ \cup \Omega_-$, whereby $\Omega_+ \equiv \{x : \sum_i x_i \ge 0\}$ and $\Omega_- \equiv \{x : \sum_i x_i < 0\}$. We have to check that such partition satisfies the conditions in Section 2.

It follows immediately from Eq. (19) that, choosing $\epsilon < m_*(\beta, 0)/2$, we have

$$\mu_{\beta,N}(\Omega_\pm) \ge \frac{1}{2} - e^{-CN}, \qquad \mu_{\beta,N}(\partial_\epsilon \Omega_\pm) \le e^{-CN}\,, \tag{22}$$

for some $C > 0$ and all $N$ large enough, which is the thesis. □

# 4 The Curie-Weiss model: Mean field equations

We have just encountered the first example of a mean field model, the first example of phase transition, and also the first example of *mean field equation*, namely Eq. (17). In the present Section, we will rederive this equation, using a somewhat more general type of argument.

Before doing this, it is worth trying to 'interpret' Eq. (17) and verify that indeed it matches the general definition of a mean field equation in Section 2. Throughout this section we will assume not to be on the phase transition line $h = 0$, $\beta > 1$. It then follows from Theorem 3 that $\mathbb{E}\,X_i = \mathbb{E}\,\overline{X} \approx m_*(\beta, h)$ (we will use $\approx$ whenever we do not want to get into a precise mathematical definition.) Therefore, Eq. (17) can be rewritten as

$$\mathbb{E}\,X_i \approx \tanh\left\{h + \frac{\beta}{N}\sum_{j \in V}\mathbb{E}\,X_j\right\}\,. \tag{23}$$

In agreement with our general description of mean field equations, this is a closed form relation between the local marginals under the measure $\mu$.

In fact, (23) follows just from the fact that $\overline{X}$ concentrated in probability, and does not require such a fine control as in Theorem 3. We start by proving an auxiliary result.

**Lemma 5.** *Denote by $\mathbb{E}_{N,\beta}$ expectation with respect to the Curie-Weiss model with $N$ variables at inverse temperature $\beta$ (and magnetic field $h$). Let $\overline{X} = N^{-1}\sum_{i=1}^{N} X_i$ and $\beta' = \beta(1+1/N)$. Then, for any $i \in [N]$:*

$$|\mathbb{E}_{N+1,\beta'}X_i - \mathbb{E}_{N,\beta}X_i| \le \beta \sinh(\beta+h)\mathrm{Var}_{N,\beta}(\overline{X}) \tag{24}$$

**Proof** By direct computation, for any function $F : \{+1,-1\}^N \to \mathbb{R}$,

$$\mathbb{E}_{N+1,\beta'}\{F(X)\} = \frac{\mathbb{E}_{N,\beta}\{F(X)\cosh(h+\beta\overline{X})\}}{\mathbb{E}_{N,\beta}\{\cosh(h+\beta\overline{X})\}}\,. \tag{25}$$

Therefore

$$|\mathbb{E}_{N+1,\beta'}\{F(X)\} - \mathbb{E}_{N,\beta}\{F(X)\}| \le ||F||_\infty\sqrt{\mathrm{Var}(\cosh(h+\beta\overline{X}))} \le ||F||_\infty \beta\sinh(h+\beta)\sqrt{\mathrm{Var}(\overline{X})}\,. \tag{26}$$

Here the first inequality follows from $\cosh a \ge 1$ and Cauchy-Schwarz, and the second from the Lipschitz behavior of $x \mapsto \cosh(h+\beta x)$ together with $|\overline{X}| \le 1$. □

The following Theorem implies (a formal version of) Eq. (23) for $\beta \le 1$ or $h > 0$.

**Theorem 6.** *There exists a constant $C(\beta,h)$ such that*

$$\left|\mathbb{E}X_i - \tanh\left\{h + \frac{\beta}{N}\sum_{j\in V}\mathbb{E}\,X_j\right\}\right| \le C(\beta,h)\sqrt{\mathrm{Var}(\overline{X})}\,. \tag{27}$$

**Proof** Let $\mathbb{E}_{N,\beta}$, and $\mathbb{E}_{N+1,\beta'}$ be defined as in Lemma 5 and $\overline{X} = N^{-1}\sum_{i=1}^{N} X_i$. By direct computation

$$\mathbb{E}_{N+1,\beta'}\{X_{N+1}\} = \frac{\mathbb{E}_{N,\beta}\sinh(h+\beta\overline{X})}{\mathbb{E}_{N,\beta}\cosh(h+\beta\overline{X})}\,. \tag{28}$$

Notice that (by Lipschitz property of $\cosh(h+\beta x)$ and $\sinh(h+\beta x)$ together with $|\overline{X}| \le 1$)

$$|\mathbb{E}_{N,\beta}\cosh(h+\beta\overline{X}) - \cosh(h+\beta\mathbb{E}_{N,\beta}\overline{X})| \le \sinh(\beta+h)\sqrt{\mathrm{Var}(\overline{X})}\,, \tag{29}$$

$$|\mathbb{E}_{N,\beta}\sinh(h+\beta\overline{X}) - \sinh(h+\beta\mathbb{E}_{N,\beta}\overline{X})| \le \cosh(\beta+h)\sqrt{\mathrm{Var}(\overline{X})}\,. \tag{30}$$

Using the inequality $|a_1/b_1 - a_2/b_2| \le |a_1-a_2|/b_1 + a_2|b_1-b_2|/b_1b_2$ (for $a_i, b_i \ge 0$), the bound $|\overline{X}| \le 1$, and the fact that $\mathbb{E}_{N+1,\beta'}X_i$ is independent of $i$, this implies

$$\left|\mathbb{E}_{N+1,\beta'}\{X_i\} - \tanh\left\{h + \frac{\beta}{N}\sum_{j\in V}\mathbb{E}_{N,\beta}\,X_j\right\}\right| \le C(\beta,h)\sqrt{\mathrm{Var}(\overline{X})}\,. \tag{31}$$

The thesis then follows by applying Lemma 5. □

**Exercise 2:** Repeat the derivations in this lecture for the model

$$\mu(x) = \frac{1}{Z} \exp\left\{ \frac{\beta}{N^{p-1}} \sum_{(i_1 \dots i_p)} x_{i(1)} \cdots x_{i(p)} + h \sum_{i=1}^{N} x_i \right\} , \tag{32}$$

where $p \geq 2$ is a fixed integer and the first sum runs over all the $p$-uples of distinct indices (we solved the case $p = 2$).

**Exercise 3:** Consider the model

$$\mu_z(x) = \frac{1}{Z} \exp\left\{ \frac{\beta}{N} \sum_{(i,j)} x_i x_j + h \sum_{i=1}^{N} z_i x_i \right\} , \tag{33}$$

where $z_i$'s are iid standard gaussian random variables. What is the value of $Z$ for a 'typical' realization of the $z_i$'s? What about the the expectations $\mathbb{E}\{x_i\}$ for a typical realization of the $z_i$'s?

# References

[CT91] Thomas M. Cover and Joy A. Thomas. *Elements of Information Theory.* Wiley Interscience, New York, 1991.