Stat 316 - Stochastic Processes on Graphs

This course is oriented towards research in applied probability, and requires active participation from *all students attending it.* Students are asked to get involved in a (small) research project by:

1. Forming a small team (ideally 2-3 students).
2. Reviewing the relevant literature on a research problem chosen from a list proposed by us, and submitting a one-page proposal by October 26.
3. Thinking (!) independently to the problem. By 'think' we mean try a few approaches to solving, or make progress on the problem. Some of these will be suggested by us, and some (hopefully) will come from you.

   The conclusions will be presented at the end of the course: you are encouraged to report both negative ('we tried this and did not work') and positive ones ('we solved the problem').

Here is a list of possible possible topics. More details are available from us.

Bipartite Sherrington-Kirkpatrick model. For n an even integer, let $[n] = V_1 \cup V_2$ be a partition with $|V_1| = |V_2| = n/2$. Consider the Gibbs measure over $x \in \{+1, -1\}^n$ defined by

$$\mu(x) = \frac{1}{Z_n(\beta, J)} \exp\Big\{\frac{\beta}{\sqrt{n}} \sum_{1\le i<j\le n} J_{ij}x_ix_j\Big\}. \tag{1}$$

Here $(J_{ij})_{i<j}$ is a collection of independent centered Gaussian random variables with

$$\mathbb{E}\{J_{ij}^2\} = \begin{cases} a & \text{if } \{i,j\} \subseteq V_1 \text{ or } \{i,j\} \subseteq V_2, \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

The objective is to compute the limit free energy $\phi(\beta, a) = \lim_{n\to\infty} n^{-1}\mathbb{E}\log Z_n(\beta, J)$ (i.e. prove existence of the limit and obtain a Parisi-like formula for its value). (1) Can you solve the problem for a larger than a large constant? (2) What about all $a \ge 1$? (3) What about $a \in [0, 1)$?

Viana-Bray model with large degree. Let $G = (V, E)$ be an Erdös-Renyi random graph with vertex set $V = [n]$ and edge probability d/n. Assume $(J_{ij})_{(ij)\in E}$ is a collection of i.i.d. random variables $J_{ij} \sim \mathsf{Unif}(\{+1, -1\})$, and consider the free energy density

$$\phi(\beta, d) \equiv \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\log\Big\{\sum_{x\in\{+1,-1\}^n} \exp\Big[\frac{\beta}{\sqrt{d}}\sum_{(i,j)\in E} J_{ij}x_ix_j\Big]\Big\}, \tag{3}$$

where the limit is known to exist by the subadditivity argument [1,2] It was proven in [3] that, for large d,

$$\phi(\beta, d) = \phi_{\rm SK}(\beta) + o_d(1), \tag{4}$$

where $\phi_{\rm SK}(\beta)$ is the free energy density of the Sherrington-Kirkpatrick model.

(1) Can you derive a quantitative upper bound on the $o_d(1)$ term? (2) Can you get the correct order of this term as $d \to \infty$? (It should be d^{-c} for a certain exponent c that you have to determine? (3) Can you characterize the constant in front of this leading order?

[1] Franz, Silvio, and Michele Leone. "Replica bounds for optimization problems and diluted spin systems." Journal of Statistical Physics 111.3 (2003): 535-564.
[2] Panchenko, Dmitry, and Michel Talagrand. "Bounds for diluted mean-fields spin glass models." Probability Theory and Related Fields 130.3 (2004): 319-336.
[3] Guerra, Francesco, and Fabio Lucio Toninelli. "The high temperature region of the VianaBray diluted spin glass model." Journal of statistical physics 115.1 (2004): 531-555.

Low temperature behavior of the Parisi solution. Let α_β be the Parisi measure for the Sherrington-Kirkpatrick model at inverse temperature β. This is a probability measure over the interval $[0, 1]$ that optimizes the Parisi functional, and represents the asymptotic distribution of the two replicas overlap. In a recent paper [4], Auffinger, Chen and Zeng prove that for any k there exists β_k such that, for $\beta > \beta_k$, the support of α_β contains at least k points. (1) Can you extract the quantitative dependance of β_k on k? (2) What is the best behavior that you can obtain for β_k as a function of k?

[4] Auffinger, Antonio, Wei-Kuo Chen, and Qiang Zeng. "The SK model is Full-step Replica Symmetry Breaking at zero temperature." arXiv:1703.06872 (2017).

Ising models on random hypergraphs. Hypergraphs are generalization of ordinary graphs, whereby each edge (a 'hyperdge') may contains $k \geq 3$ vertices. For our purposes, it is actually convenient to use the equivalent formulation of 'factor graphs.' These are bipartite graphs $G = (V, F, E)$, whereby each vertex $i \in V = [n]$ corresponds to a variable, and each vertex $a \in F$ to a 'factor' in the energy function. We assume factor node to have uniform degree $k \geq 3$.

Given such a factor graph, and parameters $\beta, h \geq 0$, we can consider the energy function E_G : $\{+1, -1\}^n \to \mathbb{R}$ by

$$U_G(x) \equiv -\beta \sum_{a \in F} \prod_{i \in \partial a} x_i - h \sum_{i \in V} x_i \,, \tag{5}$$

and the corresponding Gibbs measure over $\{+1, -1\}^n$:

$$\mu_G(x) = \frac{1}{Z_G(\beta, h)}\, e^{-U_G(x)} \,. \tag{6}$$

For sequences of factor graphs $\{G_n = (V_n = [n], F_n, E_n)\}_{n \geq 1}$ of diverging size, we also consider the free energy density (whenever it exists)

$$\phi(\beta, h) = \lim_{n \to \infty} \frac{1}{n} \log Z_{G_n}(\beta, h) \,. \tag{7}$$

The project aims at investigating: (i) Existence of the limit free energy density; (ii) Characterization of the limit value; (iii) Local limit of the measures μ_{G_n}. We suggest to consider the following random models for the graphs G_n

- Random regular factor graphs. These are uniformly random factor graphs, with degree k at the factor nodes, and degree l at variable nodes. If $|F| = m$, we have, of course, $mk = nl$.

  If convenient, you should consider the corresponding configuration model.

- Uniformly random factor graphs, with $m = \lceil n\gamma \rceil$ factors of degree k, and n variables, with $\gamma > 0$ a constant.

  If convenient, you should consider the corresponding 'independent hyper-edge' model, whereby for each of the $\binom{n}{k}$ subset of k variable nodes, the corresponding degree-k factor node is present, independently, with probability $n\gamma/\binom{n}{k}$.

Hint 1: There is an obvious connection between the $\beta \to \infty$ limit of this model and XORSAT. Hint 2: The appendix of Ref. [8] contains some simple moment calculations for this model.

[5] S. Franz, M. Mézard, F. Ricci-Tersenghi, M. Weigt, and R. Zecchina, (2001). "A ferromagnet with a glass transition." EPL (Europhysics Letters), 55(4), 465
[6] A. Montanari, G. Semerjian, "Rigorous inequalities between length and time scales in glassy systems", Journal of statistical physics, 125(1), 23-54.

Glauber dynamics for Ising models on random hypergraphs. Consider the model in the previous problem, and assume the case of a random regular hypergraph. Further assume that the degree ℓ is smaller than the hyper-edge size k (take for instance $k = 5$ and $l = 3$). Further assume $h = 0$ (no magnetic field). It can then be proved that the model has no theromodynamic phase transition. Namely, for each $\beta \geq 0$

$$\lim_{n\to\infty} \frac{1}{n} \log Z_G(\beta, 0) = \log 2 + \frac{l}{k} \log\cosh\beta \,. \tag{8}$$

In this project we consider Glauber dynamics for the measure $\mu(x)$. This is the reversible Markov chain that, at each time step, chooses a uniformly random vertex i and resamples x_i from its conditional distribution given all the other spins.

It is conjectured that there exists $\beta_{\rm d} \in \mathbb{R}$, such that the mixing time of this Markov chain is $\exp\{\Theta(n)\}$ if $\beta > \beta_{\rm d}$ and $O(n \log n)$ if $\beta < \beta_{\rm d}$.

Can you bring evidence in favor, or against this conjecture?

Random Unique Games. We consider an integer k, a graph $G = (V = [n], E)$, and for each directed edge $i \to j$ with $(i, j) \in E$, a permutation $\pi_{i\to j} : [q] \to [q]$ with $\pi_{i\to j} = \pi^{-1}_{j\to i}$. We then consider the energy function $E_{G,\pi} : [q]^n \to \mathbb{R}$ defined by

$$U_{G,\pi}(x) \equiv \sum_{(i,j)\in E, i<j} \mathbb{I}\big(x_j = \pi_{i\to j}(x_i)\big) \,. \tag{9}$$

We suggest to consider the following random model for G and π:

1. The permutations $(\pi_{i\to j})_{(i,j)\in E}$ (assume here that a direction is selected arbitrarily for each edge) are i.i.d. and uniformly random.

2. The graph is a uniformly random k-regular graph over n, vertices.

3. Alternatively, you can consider the Erdös-Renyi random graph with edge probability γ/n.

For such graphs, we consider the maximum energy density

$$u_*(G_n,\pi) \equiv \frac{1}{|E|} \max_{x\in[q]^n} U_{G_n,\pi}(x)\,, \tag{10}$$

and its limit $\lim_{n\to\infty} u_*(G_n,\pi)$ (whenever it exists). The objective of the project is (*i*) Establish upper/lower bounds on the limit; (*ii*) Prove that the limit exists.

Spin glass models in large dimension. Let $G_{L,d}$ a d-dimensional discrete torus of size L (this is the graph with vertex set $\{1,\dots,L\}^d$ and edges $(x,x+e_i)$ where e_i is the canonical basis, and sums are modulo L). Consider the corresponding Ising partition function

$$Z_L(\beta;d) = \sum_{\sigma\in\{+1,-1\}^d} \exp\Big\{\frac{\beta}{\sqrt{d}} \sum_{(x,y)\in E(G_{L,d})} J_{xy}\sigma_x\sigma_y\Big\}\,, \tag{11}$$

where $\{J_{xy}\}$ is a collection of standard normal random variables. Define the free energy density

$$\phi(\beta,d) \equiv \lim_{L\to\infty} \frac{1}{L^d}\mathbb{E}\log Z_L(\beta;d)\,. \tag{12}$$

(Why does this limit exist?) A natural conjecture is that, as the dimension diverges, this quantity converges to the free energy of the Sherrington-Kirkpatrick model:

$$\lim_{d\to\infty} \phi(\beta;d) \stackrel{?}{=} \phi_{\rm SK}(\beta)\,. \tag{13}$$

Can you prove or disprove this conjecture? (A related question was studied in [9].)

[7] S. Franz, and F.L. Toninelli. "Kac limit for finite-range spin glasses," Physical Review Letters 92.3 (2004): 030602.