Stabilizing Wage Policy

by
Mordecai Kurz,
Department of Economics Stanford University, Stanford, CA.
(This version: May 27, 2015)

Summary: A rapid recovery from deflationary shocks that result in transition to the Zero Lower Bound (ZLB) requires that policy generate an inflationary counter-force. Monetary policy cannot achieve it and the lesson of the 2007-2015 Great Recession is that growing debt give rise to a political gridlock which prevents restoration to full employment with deficit financed public spending. Even optimal investments in needed public projects cannot be undertaken at a zero interest rate. Hence, failure of policy to arrest the massive damage of eight year's Great Recession shows the need for new policy tools. I propose such policy under the ZLB called "Stabilizing Wage Policy" which requires public intervention in markets instead of deficit financed expenditures. Section 1 develops a New Keynesian model with diverse beliefs and inflexible wages. Section 2 presents the policy and studies its efficacy.

The integrated New Keynesian (NK) model economy consists of a lower sub-economy under a ZLB and upper sub-economy with positive rate, linked by random transition between them. Household-firm-managers hold heterogenous beliefs and inflexible wage is based on a four quarter staggered wage structure so that mean wage is a relatively inflexible function of inflation, of unemployment and of a distributed lag of productivity. Equilibrium maps of the two sub-economies exhibit significant differences which emerge from the relative rates at which the nominal rate, prices and wage rate adjust to shocks. Two key results: first, decline to the ZLB lower sub-economy causes a powerful debt-deflation spiral. Second, output level, inflation and real wages rise in the lower sub-economy if all base wages are unexpectedly raised. Unemployment falls. This result is explored and explained since it is the key analytic result that motivates the policy.

A Stabilizing Wage Policy aims to repair households' balance sheets, expedite recovery and exit from the ZLB. It raises base wages for policy duration with quarterly cost of living adjustment and a prohibition to alter base wages in order to nullify the policy. I use demand shocks to cause recession under a ZLB and a deleveraging rule to measure recovery. The rule is calibrated to repair damaged balance sheets of US households in 2007-2015. Sufficient deleveraging and a positive rate *in the upper sub-economy without a wage policy* are required for exit hence at exit time inflation and output in the lower sub-economy are irrelevant for exit decision. Simulations show effective policy selects high policy intensity at the outset and given the 2007-2015 experience, a constant 10% increased base wages raises equilibrium mean wage by about 5.5%, generates a controlled inflation of 5%-6% at exit time and attains recovery in a fraction of the time it takes for recovery without policy. Under a successful policy *inflation exceeds the target at exit time* and when policy terminates, inflation abates rapidly if the inflation target is intact. I suggest that a stabilizing wage policy with a constant 10% increased base wages could have been initiated in September 2008. If controlled inflation of 5% for 2.25 years would have been politically tolerated, the US would have recovered and exited the ZLB in 9 quarters and full employment restored by 2012. Lower policy intensity would have resulted in smaller increased mean wage, lower inflation but increased recession's duration. The policy would not have required any federal budget expenditures, it would have reduced public deficits after 2010 and the US would have reached 2015 with a lower national debt.

The policy negates the effect of demand shocks which cause the recession and the binding ZLB. It attains it's goal with strong temporary intervention in the market instead of generating demand with public expenditures. It does not solve other long term structural problems that persist after exit from the ZLB and which require other solutions.

JEL classification: D21, E12, E24, E3, E4, E52, E6, H3, J3, J6.
Keywords: New Keynesian Model; wage scale; reference wage; inflexible wages; sticky prices; heterogenous beliefs; market belief; Rational Beliefs; Great Recession; Depression; monetary policy; Stabilizing Wage Policy.

For Correspondence:
Professor Mordecai Kurz
Department of Economics, Landau Building
Stanford University
Stanford, CA. 94305 - 6072
Email: mordecai@stanford.edu

Stabilizing Wage Policy[1]

by

Mordecai Kurz

This Version: May 27, 2015

The Great 2007-2015 Recession has lingered on for almost eight years with very high effective rates of unemployment, altering quality of life across the world, curtailing public services, retarding career launching of a generation of recent graduates with long term adverse effects and accelerating the rise of income inequality. Lost potential US output in 2014 prices exceeds $6 Trillion which will grow in 2015. Its most disturbing aspect is the failure of policy to contain the recession's effects. Leaving aside its causes or failure to forecast its depth, most economists agree the central problem is lack of demand. Hence, regardless of one's views about long run structural problems, immediate reduction in unemployment is attained only by demand oriented policies and the initial US policy in 2008 actually aimed to boost demand. Since then, despite a vast waste of social resources, policy has mostly ignored the unemployed, including Congress' terminating long term unemployment insurance in 2013, restricting it to 26 weeks. Three groups of reasons explain this attitude. First, politicians' inadequate understanding of Economics, particularly regarding deep recessions under a Zero Lower Bound (in short ZLB). It is reflected in inference of austerity policy from individuals' financial distress and opposition to "currency debasement," in contrast to the fact that inflation is optimal policy under the ZLB. Second, an ideological perspective that opposes all New Deal programs, insists unemployment insurance is an "entitlement"and views unemployment as voluntary on the wrong ground that there exist posted job openings. Third is the cost of deficit financed policy which is tied to the second factor. In a major recession all standard policies such as public spending, tax cuts or tax incentives are deficit financed. Even with full Ricardian offset such spending decreases unemployment since demand is increased during recession with output gap while savings to pay the debt occur in full employment, at future dates. And recession multipliers are larger than 1.

Textbook public investment theory holds that at zero interest rate we should upgrade every US public facility, from airports to roads, bridges or schools: massive infrastructure investments are optimal, disregarding unemployment. This was rejected by a refusal to borrow regardless of public benefits. Borrowing requires future taxes for repayments. With progressive taxation some well-to-do individuals perceive that much of the

[1] I thank Maurizio Motolese for extensive comments and for invaluable and dedicated help that was essential to bring this work to completion. Howei Wu's valuable assistance was vital at the early stages and Giulia Piccillo provided helpful comments. I thank Ken Judd for essential technical suggestions and Truman Bewley for a conversation about his 1999 book that was pivotal. I also thank Kenneth Arrow and Truman Bewley for constructive suggestions that helped shape the present version.

future tax burden of current debts will fall on them. Some make little use of public facilities and those who view all cost to boost employment as "entitlement" transfer payments rationally oppose borrowing to finance stabilization policies. The essential fact is that standard stabilization policy entails income redistribution and political resistence to redistribution becomes policy gridlock. Full employment policy must then be based on two basic principles. First, it must be publicly clear and politically popular so that with strong public support a President can promote it and overcome resistance of opposing interests. Second, it should not require public funding and, instead, *employ temporary but stronger public intervention in the functioning of markets*.

This paper proposes a new tool for stabilization policy in deep recessions under the ZLB which I call a "Stabilization Wage Policy." It is developed in two parts. Sections 1 develops an integrated New Keynesian (in short NK) Model with diverse beliefs and *inflexible* wages in which an economy with positive or zero nominal rates is formulated as consisting of two different sub-economies linked by a random transition between them. Section 2 introduces the Stabilizing Wage Policy and studies with simulations the efficacy of its different variants. Section 3 concludes.

Each part covers different issues hence references to prior work are in the text as it is developed. My main debt is to the basic work on the ZLB of Eggertsson (2006), (2008), (2010), (2011), Eggertsson and Krugman (2012) and Eggertsson and Woodford (2003) and to Bewley's (1999) work on wages. Other papers on the ZLB include Benhabib et. al. (2001), (2002), Christiano et. al. (2011), Cochrane (2014), Correia et. al. (2013), Del Negro et. al. (2012), Del Negro, Giannoni and Schorfheide (2013), Fernández-Villaverde et. al. (2014), Kiley (2013), Mertens and Ravn (2014), Werning (2012) and Wieland (2014).

1. An Integrated New Keynesian Model

1.1 General Formulation and Log Linearization

I study stabilizing wage policy with a model based on the widely used NK model with sticky prices (e.g.Woodford (2003), Galí (2008), Walsh (2010)) which I modify in three ways. First, flexible wage is replaced with the inflexible wage. Second, I model an *integrated* economy with random transition between the "upper" sub-economy with positive nominal rate and a "lower" sub-economy with nominal rate at the ZLB. Equilibrium maps of the two are different due to changed policy rule with discontinuities between the two sub-economies. Third, I replace the Rational Expectations (RE) assumption with the model of rational diverse beliefs of Kurz (1994),(2012) (in short RB). My development extends Kurz, Piccillo and Wu (2013) (in short

KPW (2013)) who use flexible wage and do not study the ZLB.

There is a continuum of consumer-producer households and of products. Firm i produces a differentiated intermediate good i sold at price p_{it}. Firms are monopolistic competitors who select optimal prices for the goods they produce, given demand and wage. The production function is

(1) $$Y_{it} = \zeta_t N_{it}^{1-\alpha} \ , \ \zeta_t > 0 \text{ a random variable with } E^m(\zeta_{t+1}) = 1 \ , \ 0 < \alpha < 1$$

where N is labor input, hence there is a fixed non-reproducible capital (e.g. a unit of land) in use. With diverse beliefs there are diverse model probabilities. In (1) m is a stationary empirical probability deduced from past data which is common knowledge. A belief is a model, explained later, specifying how a subjective probability *differs from m*. Household j buys all intermediate goods to produce its final consumption priced at P_t with a transformation

$$C_t^j = [\int_{[0,1]} (C_{it}^j)^{\frac{\theta-1}{\theta}} di]^{\frac{\theta}{\theta-1}} \quad , \quad P_t \equiv [\int_{[0,1]} p_{it}^{1-\theta} di]^{\frac{1}{1-\theta}} \quad , \quad \theta > 1 \ .$$

P_t is then also "The Price Level." The budget of household j that owns firm j is

(2) $$C_t^j + \frac{M_t^j}{P_t} + \frac{B_t^j}{P_t} + \frac{T_t^j}{P_t} = (\frac{W_t^N}{P_t})L_t^j + [\frac{B_{t-1}^j(1+r_{t-1}) + M_{t-1}^j}{P_{t-1}}](\frac{P_{t-1}}{P_t}) + \frac{1}{P_t}[p_{jt}Y_{jt} - W_t N_{jt}] \ , \ (M_0^j, B_0^j) \text{ is given, all j.}$$

C is consumption, L -labor supplied, M -money held, T -transfers, W^N- nominal wage, B - bonds held and r - a nominal interest rate. Households solve an optimum problem with penalty on excessive borrowing and lending:

(3) $$\operatorname{Max} E_t^j \sum_{\tau=0}^{\infty} \beta^{t+\tau} \Psi_{t+\tau} [\frac{1}{1-\sigma}(C_{t+\tau}^j)^{1-\sigma} - \frac{1}{1+\eta}(L_{t+\tau}^j)^{1+\eta} + \frac{1}{1-b}(\frac{M_{t+\tau}^j}{P_{t+\tau}})^{1-b} - \frac{\tilde{\tau}_b}{2}(\frac{B_{t+\tau}^j}{P_{t+\tau}})^2], \sigma>0, \eta>0, b>0$$

where demand shocks, via a stochastic discount rate, are

$$\Psi_{t+\tau} = \psi_{-1}\psi_0\psi_1, ..,.\psi_{t+\tau-1} \text{ hence } \psi_t \text{ discounts t+1 utility and is known at t; } E^m(\psi_{t+1}) = 1 \ .$$

Demand shocks are proxies *used only to generate a fall in the natural rate and cause a depression which, in turn, calls for a wage stabilization policy*. I set $\tilde{\tau}_b = 10^{-4}$ to implement transversality conditions hence a solution with explosive borrowing is not an equilibrium. The central bank sets a nominal interest rate and provides required liquidity to satisfy demand for money via transfers. I log-linearize the Euler equations and depart from my notation of $\hat{x}_t = (X_t - \bar{X})/\bar{X}$ with three exceptions: $\hat{w}_t$ is *always* a real wage instead of $(\hat{w}_t^N - \hat{p}_t)$ and the nominal rate $\hat{r}_t$ and borrowing $\hat{b}_t^j$ are defined by

$$\hat{r}_t = \frac{r_t - \bar{r}}{(1 + \bar{r})} \quad , \quad \bar{r} = \frac{1 - \beta}{\beta} \quad , \quad \hat{b}_t^j = \frac{B_t^j}{P_t \bar{Y}} .$$

For all variables $\hat{x}_{it}$ or $\hat{v}_t^j$, the corresponding aggregates are

$$\hat{x}_t = \int_{[0,1]} \hat{x}_{it} di \quad \text{or} \quad \hat{v}_t = \int_{[0,1]} \hat{v}_t^j dj \ .$$

Inflation is denoted $\hat{\pi}_t = \hat{p}_t - \hat{p}_{t-1}$, its target and steady state are assumed zero.

The log linearized Euler equations for bond holdings and labor supply are then

(4a) $$\hat{c}_t^j = E_t^j(\hat{c}_{t+1}^j) - (\frac{1}{\sigma})[\hat{r}_t - E_t^j(\hat{\pi}_{t+1}) + \hat{\psi}_t] + \tau_b \hat{b}_t^j \quad , \quad \tau_b = \frac{\tilde{\tau}_b}{\sigma}\bar{Y}^{1+\sigma} \quad , \quad \tau_{\tilde{b}} \leq 10^{-4}$$

(4b) $$\hat{\ell}_t^j = \frac{\sigma}{\eta}\hat{c}_t^j + \frac{1}{\eta}\hat{w}_t .$$

I distinguish between a true *endogenous* constant values of variables around which equilibrium time paths fluctuate and a near-by riskless steady state of an economy with a positive rate at which the linear approximation is made. In the latter steady state $\bar{\zeta} = 1$, $\bar{C}^j = \bar{Y} = \bar{Y}_i = \bar{N}_i^{1-\alpha}$. Market clearing conditions are

$$\hat{c}_t = \hat{y}_t \quad , \quad \hat{b}_t = 0 \quad , \quad \hat{n}_t = \frac{1}{1-\alpha}(\hat{y}_t - \hat{\zeta}_t).$$

In full employment $\hat{n}_t = \hat{\ell}_t$ which, together with aggregating (4b), imply that under flexible wages

$$\hat{w}_t = \frac{\eta + \sigma(1-\alpha)}{1-\alpha}\hat{y}_t - \frac{\eta}{1-\alpha}\hat{\zeta}_t .$$

I briefly explain Calvo (1983) procedure for optimal pricing but avoid computations which are available in Kurz (2012) and KPW (2013). To remove income effects of optimal pricing by heterogenous firms these papers make an assumption which I adopt here as well:

***Insurance Assumption* 1**: A household-firm chooses an optimal price subject to budget constraints (2) and views transfers as lump sums. Transfers made ensure all firms share profits equally. Hence, actual transfers to firm j equal $(T_t^j/P_t) = PR_t - PR_t^j$, $PR_t = \int_{[0,1]} PR_t^j dj$ where

(5) $$PR_t^j = \frac{1}{P_t}[p_{jt}Y_{jt} - W_t N_{jt}] .$$

Let $p_{jt}^{\star}$ be the optimal price of j and define $q_{jt} = \frac{p_{jt}^{\star}}{P_t}$. KPW (2013) show that if ω is the probability a firm cannot adjust its price at each date then

(6) $$\hat{q}_t \equiv \int_{[0,1]} \hat{q}_{jt} dj = \frac{\omega}{1-\omega}\hat{\pi}_t .$$

Using standard methods (see Walsh (2010) page 338 or Galí (2008) Chapter 3, KPW (2013)) the linearized solution of the optimal pricing problem of firm j is

(7a) $$\hat{q}_{jt} = (1-\beta\omega)\Lambda\hat{\varphi}_t + \beta\omega E_t^j[\hat{q}_{j,t+1} + \hat{\pi}_{t+1}] \quad , \quad \Lambda = \frac{1-\alpha}{1-\alpha+\alpha\theta}$$

(7b) $$\hat{\varphi}_t = \hat{w}_t - \frac{1}{1-\alpha}[\hat{\zeta}_t - \alpha\hat{y}_t] .$$

Labor demand is determined by optimal pricing of a firm who satisfies demand but the inflexible wage leaves $\hat{n}_t^j$, the employment of household j, undetermined. In addition, log-linearizing the budget constraint under the Insurance Assumption 1 implies a demand function for bond holdings

$$\hat{b}_t^j = \frac{1}{\beta}\hat{b}_{t-1}^j + (1-\alpha)\frac{\theta-1}{\theta}(\hat{n}_t^j - \hat{n}_t) + [\hat{y}_t - \hat{c}_t^j]$$

which also depends upon $\hat{n}_t^j$. I thus make the following assumption:

***Unemployment Assumption* 2:** Unemployment is uniform over households, consequently for all j the unemployment rate is the same in all households hence $u_t = \hat{\ell}_t^j - \hat{n}_t^j$.

By the unemployment assumption and (4b)

$$\hat{n}_t^j = -\frac{\sigma}{\eta}(\hat{c}_t^j) + \frac{1}{\eta}\hat{w}_t - u_t \quad \Rightarrow \quad \hat{n}_t = -\frac{\sigma}{\eta}(\hat{y}_t) + \frac{1}{\eta}\hat{w}_t - u_t.$$

Hence, bond demand is

(8) $$\hat{b}_t^j = \frac{1}{\beta}\hat{b}_{t-1}^j + [1 + (1-\alpha)\frac{\theta-1}{\theta}\frac{\sigma}{\eta}](\hat{y}_t - \hat{c}_t^j).$$

To compute total profits I log linearize the profit function (26) and aggregate it to deduce

(9) $$\hat{pr}_t = \frac{\theta}{\theta-(1-\alpha)(\theta-1)}\left\{\hat{y}_t - (1-\alpha)(\theta-1)\hat{w}_t + (\theta-1)\hat{\zeta}_t\right\}.$$

Throughout I use standard parameter values (e.g. Galí (2008), Woodford (2003)): $\beta = 0.99$, $\alpha = \frac{1}{3}$, $\sigma = 0.90$, $\eta = 1.0$, $\tau_{\tilde{b}} = 10^{-4}$, $\theta = 6$, $\lambda_\zeta = 0.90$ but I discuss my values of ω = 0.40 and $\sigma_\zeta = 0.0045$ later.

1.2 The Belief Structure

For beliefs to be diverse there must be something agents do not know and about which they disagree. Here it is the distribution of (ζ_t, ψ_t) which is non-stationary, with structural changes (see Kurz (2009)). Appendix A to this paper and KPW (2013) offer detailed explanations of the structure of belief diversity and my presentation here is very brief.

Under a Rational Belief approach (see Kurz (1994), (1997)), agents use data to deduce the stationary probability m used in (1) called the "empirical probability" described with the Markov transitions

(10a) $$\hat{\zeta}_{t+1} = \lambda_\zeta\hat{\zeta}_t + \rho_{t+1}^\zeta$$

(10b) $$\hat{\psi}_{t+1} = \lambda_\psi\hat{\psi}_t + \rho_{t+1}^\psi$$

$$\begin{pmatrix}\rho_{t+1}^\zeta \\ \rho_{t+1}^\psi\end{pmatrix} \sim N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{bmatrix}\sigma_\zeta^2, & 0, \\ 0, & \sigma_\psi^2\end{bmatrix}\right), \quad \text{i.i.d.}$$

Real business cycles theory assumes only technology shocks with $\sigma_\zeta = 0.0072$ based on Solow's residuals. It is widely recognized technology has a smaller effect (e.g. Eichenbaum (1991), Summers (1991), Basu (1996), King and Rebelo (1999)) and $\sigma_\zeta \approx 0.0035 - 0.0045$ hence I set $\sigma_\zeta = 0.0045$. Volatility of aggregates is then far too small and belief diversity provides an alternate mechanims that explains the observed volatility.

In comparison with (10a)-(10b) the truth is

$$\text{(11a)} \quad \zeta_{t+1} = \lambda_\zeta \zeta_t + s_t + \tilde{\rho}^\zeta_{t+1} \qquad \begin{pmatrix} \tilde{\rho}^\zeta_{t+1} \\ \tilde{\rho}^\psi_{t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \tilde{\sigma}^2_\zeta, & 0, \\ 0, & \tilde{\sigma}^2_\psi \end{bmatrix} \right)$$

$$\text{(11b)} \quad \psi_{t+1} = \lambda_\psi \psi_t + \lambda^s_\psi s_t + \tilde{\rho}^\psi_{t+1}$$

where s_t are unobserved. Agent j's belief is described with a state variable g^j_t which is the *subjective expected value of s_t* and which pins down all perceived transitions. Agent j observes the distribution of the g^k_t over k hence its mean Z_t but assumes his own belief has no effect on Z_t. Household j's date t perception of the two exogenous shocks is denoted by $(\zeta^j_{t+1}, \psi^j_{t+1})$ and take the following form

$$\hat{\zeta}^j_{t+1} = \lambda_\zeta \zeta^j_t + g^j_t + \rho^{j\zeta}_{t+1} \quad , \quad \rho^{j\zeta}_{t+1} \sim N(0, \sigma^2_{\zeta g}) \quad , \quad \hat{\psi}^j_{t+1} = \lambda_\psi \psi^j_t + \lambda^g_\psi g^j_t + \rho^{j\psi}_{t+1} \quad , \quad \rho^{j\psi}_{t+1} \sim N(0, \sigma^2_{\psi g}).$$

Appendix A and KPW (2013) explain that three Rationality Principles imply g^j_t has a transition

$$\text{(12)} \quad g^j_{t+1} = \lambda_Z g^j_t + \lambda^\zeta_Z [\zeta_{t+1} - \lambda_\zeta \zeta_t] + \rho^{jg}_{t+1} \quad , \quad \rho^{jg}_{t+1} \sim N(0, \sigma^2_g) \ , \ \rho^{jg}_{t+1} \text{ are correlated across j.}$$

$\lambda^\zeta_Z[\zeta_{t+1} - \lambda_\zeta \zeta_t]$ measures learning from current data. Since Z_t aggregates (12) and ρ^{jg}_t are correlated, their average does not vanish causing uncertainty about Z_{t+1} to emerge. It is a *belief externality* but since Z_t is common knowledge, there is no infinite regress. Its empirical transition is

$$\text{(13)} \quad Z_{t+1} = \lambda_Z Z_t + \lambda^\zeta_Z [\zeta_{t+1} - \lambda_\zeta \zeta_t] + \tilde{\rho}^Z_{t+1}.$$

Since the empirical distribution of all three variables (ζ_t, ψ_t, Z_t) is known, the final models are about the state variables $(\zeta^j_{t+1}, \psi^j_{t+1}, Z^j_{t+1}, g^j_{t+1})$[2]. Agent j's belief is then a perception model

$$\begin{aligned} &\text{(14a)} \quad \zeta^j_{t+1} = \lambda_\zeta \zeta_t + \lambda^g_\zeta g^j_t + \rho^{j\zeta}_{t+1} \\ &\text{(14b)} \quad \psi^j_{t+1} = \lambda_\psi \psi_t + \lambda^g_\psi g^j_t + \rho^{j\psi}_{t+1} \\ &\text{(14c)} \quad Z^j_{t+1} = \lambda_Z Z_t + \lambda^\zeta_Z [\zeta_{t+1} - \lambda_\zeta \zeta_t] + \lambda^g_Z g^j_t + \rho^{jZ}_{t+1} \\ &\text{(14d)} \quad g^j_{t+1} = \lambda_Z g^j_t + \lambda^\zeta_Z [\zeta_{t+1} - \lambda_\zeta \zeta_t] + \rho^{jg}_{t+1} \end{aligned} \qquad \begin{pmatrix} \rho^{j\zeta}_{t+1} \\ \rho^{j\psi}_{t+1} \\ \rho^{jZ}_{t+1} \\ \rho^{jg}_{t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \hat{\sigma}^2_\zeta, & 0, & 0, & 0 \\ 0, & \hat{\sigma}^2_\psi, & 0, & 0 \\ 0, & 0, & \hat{\sigma}^2_Z, & \hat{\sigma}_{Zg} \\ 0, & 0, & \hat{\sigma}_{Zg}, & \sigma^2_g \end{bmatrix} \right).$$

g^j_t then measures how agent j's expects t+1 states to be different from normal, where "normal" is m.

KPW (2013) show that agents' belief are restricted by the RB conditions

$$\text{(15)} \quad \sigma_g \le \sigma_\zeta \sqrt{(1 - \lambda^2_Z)} \quad , \quad |\lambda^g_\psi| \le \frac{\sigma_\psi}{\sigma_\zeta} \quad , \quad |\lambda^g_Z| \le \frac{\sigma_Z}{\sigma_\zeta} \le 0.9\sqrt{(1 - \lambda^2_Z)}.$$

A Bayesian model in KPW (2013, Appendix) and evidence in Kurz and Motolese (2011) lead to estimated parameters of λ_Z in the range of [0.6 , 0.8] hence with $\lambda_Z \simeq 0.7$, $\sigma_g = 0.003$, $\sigma_Z = 0.9\sigma_g$, $\lambda^\zeta_Z \simeq 0.70$, $\lambda^g_Z = 0.05$, $\lambda^g_\psi = \frac{\sigma_\psi}{\sigma_\zeta}$, $\sigma_\zeta = 0.0045$ and normalization $\lambda^g_\zeta = 1$, conditions (15) is satisfied.

---

[2] The notation $(\zeta^j_{t+1}, \psi^j_{t+1}, Z^j_{t+1})$ indicates agent i's *perception* of $(\zeta_{t+1}, \psi_{t+1}, Z_{t+1})$. Since there is no difference between $E^j_t \zeta^j_{t+1}$ and $E^j_t \zeta_{t+1}$, I write $E^j_t \zeta_{t+1}$ to express expectations of ζ_{t+1} by j, *in accordance with his perception*.

1.3 *Inflexible Wages*

Empirical evidence for inflexible wages is extensive (e.g.Tobin (1972), Abraham and Haltwinger (1995), Brandolini (1995), Malcomson (1999), and references there). Indirect evidence can also be deduced from a conflict between actual volatility and predictions of flexible wage models. If $(\sigma_w, \sigma_y, \sigma_n)$ are standard deviations of the real wage, output per capita and labor employed then data for the U.S. since 1947 show $(\sigma_w/\sigma_y) = 0.38$, $(\sigma_n/\sigma_y) = 0.99$ while models with flexible wages imply $(\sigma_w/\sigma_y) > 1.0$, $(\sigma_n/\sigma_y) < 0.75$ (e.g. see Kurz, Piccillo and Wu (2013), Table 3). That is, competitive flexible wages are too volatile and *output volatility is mostly associated with fluctuations in unemployment,* not in wages. A growing volume of research incorporates inflexible wages and some recent papers (e.g.Christiano, Eichenbaum Evans (2005), Blanchard and Galí (2010), Gertler and Trigari (2009), Hall (2005a), (2005b), Shimer (2004)) show that wage stickiness resolves much of the issues raised above. The specific form of wage inflexibility I adopt in this work has two components, based on known results. First, the Implicit Contract literature explored the need for smoothing consumption. However, my treatment adopts only the wage smoothing component. It rejects the Pareto Optimal contracting environment and considers *involuntary unemployment as socially inefficient.* Second, the literature on staggered wage contracts (e.g. Fischer (1977), Taylor (1979), (1980), (1999), Gertler and Trigari (2009) and references) together with the direct empirical evidence in support of this form of wage inflexibility (e.g.Taylor's (1999) survey and references there as well as papers by Taylor (1983), Cecchetti (1984), McLaughlin (1994), Lebow, Stockton and Wascher (1995) and Card and Hyslop (1997)). My formulation is motivated by the empirical record hence it is useful to sum it up:

- wages and benefits of *regular employees* are adjusted at discrete intervals, mostly once a year;
- adjustment of the mean wage rate to inflation is only partial;
- the effect of unemployment on the mean wage rate is very small;
- in general, wages do not respond to contemporary shocks except for productivity.

In the discussion below I will then distinguish between the wage of regular employees with long term relation to the firm and irregular employees without such a relation.

In accord with the staggered wage model the duration of all wage offers to regular employees is four quarters and these offers are distributed equally among the four quarters. The average deviation of the date t offered real wage from steady state wage for employment at date t+1 is then the expected average deviation of the competitive real wages from steady state at dates t+1, t+2, t+3 and t+4:

(16) $$\frac{W_t^e - \bar{W}}{\bar{W}} = \sum_{k=1}^{4} \frac{\beta^{k-1}}{1+\beta+\beta^2+\beta^3} E_t[\frac{W_{t+k}^F - \bar{W}}{\bar{W}}] \quad \Rightarrow \quad \hat{w}_t^e = \sum_{k=1}^{4} E_t \frac{\beta^{k-1}\hat{w}_{t+k}^F}{1+\beta+\beta^2+\beta^3}.$$

where W_t^e is the offered four period wage and W_{t+k}^F are future competitive wages. In a growing economy steady state is the deterministic growth trajectory. In the simple case of an equilibrium under Rational Expectations with flexible wages of the model above the wage has the form

$$\hat{w}_{t+k}^F = A_{wF}^{\zeta}\hat{\zeta}_{t+k} + A_{wF}^{\psi}\hat{\psi}_{t+k}.$$

where A_{wF}^{ζ} is the elasticity of the competitive real flexible wage with respect to productivity.

FIGURE 1

Figure 1 shows mean wage traces productivity per man hour closely *when the real wage is computed with the PPI,* not the CPI. It is well known CPI based real wages have been stagnant and deterioration in the labor market was also caused by internal polarization (see, Autor and Dorn (2013), Autor (2014)). However, the issue is the relation between productivity and real wage offers based on *optimal firm's inputs*. A rational firm considers relative cost of labor to other inputs hence a PPI is the appropriate index and Figure 1 shows deviations in the tight relation of mean wage with productivity occurred at the height of the Viet Nam war and during the elimination of corporate pensions that started around 2004-2005. Such factors are transitory. In sum, Figure 1 and the evidence cited earlier suggest that average wage setting of regular employees is responsive only to productivity shocks hence I assume average wage offers are made according to the scale

(17) $$\hat{w}_t^e = \sum_{k=1}^{4} \frac{\beta^{k-1}E_t[A_{wF}^{\zeta}\hat{\zeta}_{t+k}]}{1+\beta+\beta^2+\beta^3}.$$

Now use (10a) to conclude that

$$\hat{w}_t^e = \sum_{k=1}^{4} \frac{\beta^{k-1}\lambda_{\zeta}^{k}}{1+\beta+\beta^2+\beta^3}[A_{wF}^{\zeta}\hat{\zeta}_t].$$

For $\beta = 0.99$ this last expression is well approximated by

(18) $$\hat{w}_t^e = (\frac{1}{4})(\lambda_{\zeta}+\lambda_{\zeta}^2+\lambda_{\zeta}^3+\lambda_{\zeta}^4)A_{wF}^{\zeta}\hat{\zeta}_t.$$

I call this expression the *"wage scale."* It is a function that specifies the constant real wage offer made at date t for employment starting at t+1 and paid to regular employees if employed at t+1, t+2, t+3 and t+4. A new offer is made at dater t+4 but the firm reserves the right to lay off any worker at any time in the future. Since (18) is for a constant wage during four *quarters*, at any date t there are four regular wages in the market: $(W^e_{t-1}, W^e_{t-2}, W^e_{t-3}, W^e_{t-4})$. Since I study aggregate behavior with log-linear approximation, wage cost are additive hence it is the mean wage that has an effect on aggregate variables. I then define the reference wage, which is the mean real wage of *regular* employees

(19a) $$W^\star_t = [W^e_{t-1}W^e_{t-2}W^e_{t-3}W^e_{t-4}]^{(\frac{1}{4})}.$$

Taking percentage deviations from steady state

(19b) $$\hat{w}^\star_t = \frac{1}{4}[\hat{w}^e_{t-1} + \hat{w}^e_{t-2} + \hat{w}^e_{t-3} + \hat{w}^e_{t-4}] = B^\zeta_w(\hat{\zeta}_{t-1} + \hat{\zeta}_{t-2} + \hat{\zeta}_{t-3} + \hat{\zeta}_{t-4}) \quad , \quad B^\zeta_w = (\frac{1}{16})(\lambda_\zeta + \lambda^2_\zeta + \lambda^3_\zeta + \lambda^4_\zeta)A^\zeta_{wF}.$$

To see the empirical implication of inflexibility note that in a typical flexible wage model with $\lambda_\zeta = 0.90, \eta = 1, \sigma = 0.9, \alpha = 1/3$ equilibrium elasticity is $A^\zeta_{wF} = 0.98$ hence $(\lambda_\zeta + \lambda^2_\zeta + \lambda^3_\zeta + \lambda^4_\zeta) = 3.0952$ and $B^\zeta_w = (1/16)(0.98)(3.0952) = 0.19$. It means a 1% change in $\hat{\zeta}_t$ changes the flexible wage by 0.98% but the t+1 reference wage changes initially only by 0.19% but it continues to change slowly as future scales adjust.

The staggered wage literature starts from evidence on union wage contracting which contributes to wage inflexibility. But the obvious fact observed in daily life is that the compensation for most regular jobs is constant for a period, typically one year, with different job starting dates and offer renewal dates. That is, the implied staggered mean regular wages are almost universal and have little to do with union contracts. One may then naturally ask why they exist at all, a question that I ignore here but explore somewhere else[3].

[3]The Implicit Contracts theory shows such wage arrangements are desired by workers who wish to smooth their consumption but cannot accomplish such smoothing due to incomplete markets. One important novelty of the present research reported in Kurz (2015) is that, in contrast to the common view (also found in the implicit contract literature), the firm with profit function (5) has a strong interest in the institution of inflexible wages that result in staggered wages. Indeed, Kurz (2015) shows that if the firm offers its workers at date t to work for a fixed wage like (16) or (19b), to be paid in four future dates t+1, t+2, t+3 and t+4, the workers accept such an offer *and the firm's expected profits rise*. Such offer has two provisions. First, the firm may lay-off any worker at any future date hence the offer is for a wage not for guaranteed employment. Second, the firm is committed to make only such offers for regular jobs and this becomes the socio-economic norm. To clarify this result recall the standard argument which shows a firm benefits from wage volatility due to convexity of the profit function with respect to the wage. This theorem ignores equilibrium relations between shocks and the wage and prices. In assessing the risk of wage volatility a rational firm takes into account the general equilibrium effects of shocks on output, inputs and the wage. Convexity of profits with respect to the wage is then an incorrect basis for judging if it is optimal for the firm to hedge against risky shocks. Kurz's (2015) technical result shows that if General Equilibrium effects of shocks are taken into account then, for shocks with direct effects on output such as technology and demand shocks, *the profit function is convex in the wage but concave in the shocks!* The firm gains from hedging wage cost at the expected value of wages. The wage in (16) or (19b) can then be derived as a perfect equilibrium of a game in which the firms compete in making constant wage offers for a fixed duration and workers accept or reject such offers by searching across firms. The interest of the firm in inflexible wage is even greater when prices are sticky since such prices imply the firm may not be able to respond to changes in wage hence constant wage is a superior institution.

Although these results provide a compelling explanation why the constant-wage-for-a-period of regular jobs have become standard, these explanations are not needed for the development of a wage stabilization policy. To formulate the policy it is sufficient to specify wage offers like (16) made to regular employees and the implied staggered wage structure of the mean wage, as in (19b). These can then be deduced from the empirical evidence available, hence I will show below that the final wage equation is in fact familiar from the empirical literature of the 1970's.

The development leading to (19a)-(19b) has not considered the role of irregular employees whose wages impact the mean wage in the market. Moreover, (16) and (19b) are formulated as real wages without considering the effect of inflation. I thus turn to these questions now.

1.3.1 Wage of Irregular Jobs

A typical firm has many regular jobs, many wage scales and many reference wages since different jobs surely pay different wages but (19b) is a wage scale of a single job. It is due to the fact that this work is a study of wage inflexibility hence I examine differences among jobs *only in terms of their wage response to the current state and a single task is sufficient for that.* The distinction between"regular" and "irregular" employment focuses on the difference between a bus driver that holds a regular job with expectation of long term work vs. a temporary bus driver. They do the same work but have different relation to the firm. This distinction is complicated by two facts. First, irregular jobs are concentrated in categories such as personal services, hospitality jobs, seasonal agricultural jobs, fixed duration projects like oil drilling or construction etc. They tend to be temporary or short duration, with high turnover and without long term relation with the employing firm. Second, irregular jobs are generally lower paid and with much higher rates of involuntary unemployment. However, irregular jobs are also filled by highly paid consultants or professionals hired for specific projects and all entry level wages are also irregular. They become regular wages when work status is changed to regular employment. Notwithstanding these difficulties simplicity requires consideration of only one category of "irregular" jobs. The key feature of (19b) is that, as a wage of regular employees, it exhibits small response to current conditions while irregular wages exhibit much more flexibility. Well established empirical results show that irregular wages and wages of newly hired workers respond more to unemployment and to current state than wages of regular employees (e.g. Bils (1985), Keane, Moffitt and Runkle (1988), Solon, Barsky and Parker (1994), Pissarides (2009) and references there).

What is the wage of irregular jobs relative to a regular counterpart? Although a regular engineer does the same work as a short term engineer, there is a difference. Regular employees are perceived as loyal and trusted to perform to the best of their ability. Irregular employees have no such incentives, their productivity is more risky, they have less responsibility and with short term employment status competition causes their wage to be responsive to current conditions. The two wages are then assumed the same except for the greater response of the irregular wage to current conditions. The way this wage responds to unemployment is discussed here and it's response to inflation is evaluated in the next sub-section.

To introduce involuntary unemployment I define it first. I do not adopt Search and Matching theory since my interest is *involuntary unemployment* while search theory assumes unemployment is a productive use of time that results from friction in matching of workers with jobs. Equilibrium search explains well how a voluntary natural unemployment rate of about 5.0% arises but the theory's premises and analytic machinery are unsuitable for understanding of involuntary unemployment. In the equilibrium used in this paper the natural rate is assumed constant ϱ = 5% and this rate is anticipated by job offers. If desired employment at date t by firms is N_t then, expecting a fraction ϱ of accepted jobs to end due to quits and bad matches, offers are made for $(1+\varrho)N_t$. Labor supplied is L_t hence involuntary unemployment rate is

(20) $$u_t = 1-(1+\varrho)\frac{N_t}{L_t} \approx \hat{\ell}_t - \hat{n}_t.$$

It is technically simpler to make the unrealistic but inconsequential assumption that search takes place in steady state so that in steady state ϱ is voluntary unemployment rate, $\bar{u} = 0$ and $(1+\varrho)\bar{N} = \bar{L}$. I now return to wage setting of irregular workers.

Let (W_t^R, W_t^{IR}) be mean wages of regular and irregular workers, then their response to involuntary unemployment is expressed by

(21) $$W_t^R = W_t^\star \qquad\qquad W_t^{IR} = ((1+\varrho)\frac{N_t}{L_t})^\mu W_t^\star .$$

Since $(1+\varrho)\bar{N} = \bar{L}$ and $\bar{W}^\star = \bar{W}$, in steady state all wages are equal. Mean irregular wage responds to unemployment with elasticity μ which I set equal 1 based on evidence in Abraham and Haltiwanger (1995) and Pissarides (2009) although estimates of μ vary widely. Data from earlier in the 20th century suggest that wage response to unemployment declined. Not distinguishing between regular and irregular new matches, Pissarides (2009) selects $\mu = 3.0$ which implies, in his framework, that an unemployment rate of 9% lowers entry level wages by 27%! The 2008-2014 data do not exhibit such a fall in wages of irregular jobs. Since in my framework 5% is a natural unemployment rate, $\mu = 1$ means that a 9% unemployment rate entails *involuntary* unemployment of 4% and this drives wages of irregular workers down by 4%. I later show that changes in μ have small effect on the policy results.

If the proportion of regular employees is a_w (estimated at about 0.85) then the mean wage is

$$W_t = (W_t^\star)^{a_w}[((1+\varrho)\frac{N_t}{L_t})^\mu W_t^\star)]^{(1-a_w)}$$
$$\hat{w}_t = \hat{w}_t^\star - \lambda_u u_t + \rho_t^w \qquad u_t = \hat{\ell}_t - \hat{n}_t \quad , \quad \lambda_u = \mu(1-a_w).$$

Flexibility of irregular wages imply mean wage also responds to unemployment. With $(a_w=0.85, \mu=1)$, I compute $\lambda_u = \mu(1 - a_w) = 1.0 \times 0.15 = 0.15$. It implies that if unemployment rises to 9% (i.e. $u_t = 4\%$), mean wage will decline by 0.60%, which is consistent with the empirical evidence.

1.3.2 The Effect of Inflation

Developments up to now specify wages in real terms but inflation puts it into question. Full indexation of regular wages creates *asymmetry in risk bearing* between the firm and its workers. In an economy with sticky prices full indexation means workers face no inflation risk while the firm faces the risk of being unable to adjust its price in response to unexpected inflation. This suggests we cannot expect full wage indexation but the evidence shows that significant indexation does take place but it is complex. Suppose that no indexation takes place and wage offers are adjusted to inflation only four quarters later. Actual real regular wage $W_{t+j}^{(t)}$ offered at t but paid at t+j would satisfy $P_{t+j} W_{t+j}^{(t)} = P_t W_t^e$ for $j = 1, 2, 3, 4$. It follows that with no indexation mean real regular wage is a function of a distributed lag of at least 3 past inflation rates, a result not found in the data (e.g. Tobin (1972) who reviews Hirsch (1972), de Menil and Enzler (1972) and Hymans (1972), and Ashenfelter and Card (1982)). On a positive side, several inflation adjustments do take place which imply partial indexation:

(i) the fringe benefits components of the wage are stated in nominal terms hence adjust with inflation;

(ii) some wage offers (e.g. some union contracts) contain either explicit indexation or premia for *expected inflation,* providing implicit indexation (for empirical evidence see Tobin (1972));

(iii) a quarterly turnover of about 12% generates an automatic adjustment to inflation for all new hires;

(iv) the frequency of work review and wage adjustments is changed with inflation (e.g. Cecchetti (1984)).

What is then the empirical evidence? Two empirical results about the effect of short term inflation on the real wage stand out. The first shows the inflation rate that matters is the *current* rate. Second, the elasticity of the real wage with respect to inflation was estimated between 0.3 and 0.6 (see Tobin (1972) and Blanchard and Fischer (1989) Chapter 10). This elasticity expresses a short run Phillips Curve that also arises from an inflation risk sharing between the firm and its workers. Elasticity of 0.6 means inflation of 1% reduces the real wage in that period by 0.6% hence for one period the heavier weight is born by workers. But this is only for one period and full indexation occurs if inflation lasts only this period. For a persistent effect on the real wage inflation must exhibit persistence. This is then the assumption I adopt for regular employees. For irregular employees I assume inflation adjustment is immediate since they are offered a current nominal wage which fully adjusts.

Translating these assumptions, the real wages of past offers for regular jobs at the current date t are

$$W_t^{(t-j)} = (\frac{P_{t-1}}{P_t})^{\upsilon} W_{t-j}^e \quad , j = 1,2,3,4 \quad , \quad 0 < \upsilon < 1 .$$

Mean real wage of regular employees is then

$$W_t^R = (\frac{P_{t-1}}{P_t})^{\upsilon} [W_{t-1}^e W_{(t-2)}^e W_{(t-3)}^e W_{(t-4)}^e]^{\frac{1}{4}} = (\frac{P_{t-1}}{P_t})^{\upsilon} W_t^{\star} .$$

Real wages of regular and irregular employees are then

$$W_t^R = (\frac{P_{t-1}}{P_t})^{\upsilon} W_t^{\star} \quad , \quad W_t^{IR} = [((1+\varrho)\frac{N_t}{L_t})^{\mu} W_t^{\star}]$$

hence the mean real wage is

$$W_t = [(\frac{P_{t-1}}{P_t})^{\upsilon} W_t^{\star}]^{a_w} [(1+\varrho)\frac{N_t}{L_t})^{\mu} W_t^{\star}]^{(1-a_w)} .$$

The final wage incorporated in all models used later is then specified by the system

(22a) $$\hat{w}_t = \hat{w}_t^{\star} - \lambda_{\pi}\hat{\pi}_t - \lambda_u u_t \; , \quad u_t = \hat{\ell}_t - \hat{n}_t \quad , \quad \lambda_{\pi} = \upsilon a_w \quad , \quad \lambda_u = \mu(1-a_w) .$$

(22b) $$\hat{w}_t^{\star} = B_w^{\zeta}(\hat{\zeta}_{t-1} + \hat{\zeta}_{t-2} + \hat{\zeta}_{t-3} + \hat{\zeta}_{t-4}) \quad , \quad B_w^{\zeta} = \frac{A_w^{\zeta}}{16}(\lambda_{\zeta} + \lambda_{\zeta}^2 + \lambda_{\zeta}^3 + \lambda_{\zeta}^4) .$$

Parameter values used are those estimated empirically for the post war era: $\lambda_{\pi} = 0.6$, $\lambda_u = 0.15$. With the approximate value of $a_w = 0.85$, they imply $\upsilon = 0.71$, $\mu = 1$.

Is it necessary to test (23a)-(23b) empirically? Fortunately, this family of wage models was extensively tested in the 1970-1980 literature on wage behavior within aggregate econometric models and Tobin (1972) offers a summary that reviews such equations in Hirsch (1972), de Menil and Enzler (1972) and Hymans (1972). The models perform very well, providing evidence (22a)-(22b) are compatible with the data. The practice at the time was to estimate the *rate of change* of variables hence tests of (22a)-(22b) use time differences of wage, growth, inflation and unemployment (e.g. Tobin (1972) Eq. (2), Abraham and Haltwinger (1995), Brandolini (1995) and references). I recognize the debate about the 1970-1980 macro models. It does not apply here; (22a)-(22b), are formulated as diviations from steady state growth and the empirical test they passed in 1970-1980 remain valid as standard formal statistical tests. As I noted earlier, Kurz (2015) offers a formal argument why wage inflexibility is expected to take the form in (22a)-(22b) but the real virtue of this specification for policy evaluation is its empirical foundation.

1.4 An Integrated Economy

Recall that $\hat{y}_t^F = (1+\eta)/[\alpha+\eta+\sigma(1-\alpha)]\hat{\zeta}_t$ is potential output and $\hat{r}_t = \frac{r_t}{(1+\bar{r})} - (1-\beta)$. A Taylor type monetary rule is $\hat{r}_t = \xi_y(\hat{y}_t - \hat{y}_t^F) + \xi_\pi \hat{\pi}_t$ if $r_t \geq 0$ and $\hat{r}_t = -(1-\beta)$ if $r_t = 0$ but a transition to a ZLB is caused by random "crash" events defined by a function $\mathbb{C}_t$ on states. The true policy rule is then

$$\hat{r}_t = \begin{cases} \xi_y(\hat{y}_t - \hat{y}_t^F) + \xi_\pi \hat{\pi}_t & \text{if} \quad \mathbb{C}_t = 1 \\ -(1-\beta) & \text{if} \quad \mathbb{C}_t = 0. \end{cases} \tag{23}$$

In single agent problems this discontinuity reflects a Kuhn-Tucker multiplier. More generally, equilibrium functions have two branches one with $r_t \geq 0$ and one with $\hat{r}_t = -(1-\beta)$. There are then two sub-economies with different policy rules, *different expectations* and different equilibrium maps, all linked by a random transition due to $\mathbb{C}_t$. I refer to the economy with $r_t \geq 0$ as the "upper sub-economy" and to the ZLB economy with $r_t = 0$ as the "lower sub-economy" with natural notation. For example, $\hat{c}_t^{jU}$ is consumption of j in the upper sub-economy while $\hat{c}_t^{jL}$ is its value in the lower sub-economy. I avoid notational complexity with a convention that takes each model equation like (4a)-(4b), (7a),(8), (9) *as a vector of two equations*. Three model components cannot be treated this way: transition of $\mathbb{C}_t$, the policy rule and the expectation operator.

Starting with transition, I assume agents believe transition between upper and lower sub-economies is Markov with a transition matrix

	U	L
U	Ω_U	$1 - \Omega_U$
L	$1 - \Omega_L$	Ω_L

As to the policy rule, *it is crucial to understand that the rule is*

$$\hat{r}_t = \begin{cases} \xi_y(\hat{y}_t^U - \hat{y}_t^F) + \xi_\pi \hat{\pi}_t^U & \text{if} \quad \xi_y(\hat{y}_t^U - \hat{y}_t^F) + \xi_\pi \hat{\pi}_t^U \geq -(1-\beta) \text{ when } \mathbb{C}_t = 1 \\ -(1-\beta) & \text{if} \quad \xi_y(\hat{y}_t^U - \hat{y}_t^F) + \xi_\pi \hat{\pi}_t^U < -(1-\beta) \text{ when } \mathbb{C}_t = 0 \end{cases} \tag{24}$$

hence, *the implied nominal rate in the lower sub-economy is irrelevant for exit decision from the ZLB* and execution of a policy rule in such circumstances is a complex task. The computed nominal rate based on inflation and gap in the lower sub-economy can be misleading and errors may result in too early exit. I will later show a successful stabilization policy generates temporary inflation in the lower sub-economy *with positive implied rate in that sub-economy*. Those who would insist on raising rates would be wrong since the nominal rate in the upper sub-economy would be negative due to its own equilibrium map and the continued deflation under $\mathbb{C}_t = 0$ but without the policy's help. Exit requires the natural rate to be positive, enabling the nominal rate to be positive in the upper sub-economy when deflation's pressures are dissipated and without the help of a stabilization policy.

Expectations of any endogenous variable $\hat{v}_{t+1}$ (individual or aggregate) is naturally defined by

$$(25) \qquad E_t^j(\hat{v}_{t+1}) \equiv \begin{cases} \Omega_U E_t^j(\hat{v}_{t+1}^U) + (1-\Omega_U)E_t^j(\hat{v}_{t+1}^L) & \text{if current state is U} \\ \Omega_L E_t^j(\hat{v}_{t+1}^L) + (1-\Omega_L)E_t^j(\hat{v}_{t+1}^U) & \text{if current state is L .} \end{cases}$$

Belief in constant transition probabilities is a useful simplification. Great Recessions under the ZLB like 2007-2015 are hard to predict. My estimate $\Omega_U = 0.9976$ is based on data of a depression once in 70 years. Exit from a ZLB has virtually no statistics and with ambiguity about adopted policies and uncertainty about their effects. Given durations observed and some survey data from Japan, my estimate is $\Omega_L = 0.96$ with expected duration of 25 quarters. (Ω_L, Ω_U) is *fixed throughout and represents the real economy*.

Definition : A competitive equilibrium under policy rule (23) is a set of stochastic processes $\{(\hat{c}_t^j, \hat{\ell}_t^j, \hat{q}_{jt}, \hat{b}_t^j, \hat{y}_t, \hat{\pi}_t, \hat{w}_t, u_t, \hat{r}_t),\ t = 1, 2, \ldots\}$ for all j in which agents hold beliefs (14a)-(14d) and which satisfy the following *pairs* of equations for the upper and lower sub-economies

(26a) consumption demand $\quad \hat{c}_t^j = E_t^j(\hat{c}_{jt+1}) - (\frac{1}{\sigma})[\hat{r}_t - E_t^j(\hat{\pi}_{t+1}) + \hat{\psi}_t] + \tau_b \hat{b}_t^j \quad , \tau_b = \frac{\tilde{\tau}_b}{\sigma} \bar{Y}^{1+\sigma}$

(26b) labor supply $\quad \hat{\ell}_t^j = \frac{\sigma}{\eta} \hat{c}_t^j + \frac{1}{\eta} \hat{w}_t$

(26c) wages rate $\quad \hat{w}_t = \hat{w}_t^\star - \lambda_\pi \hat{\pi}_t - \lambda_u u_t \quad , \quad \hat{w}_t^\star = [\frac{A_w^\zeta}{16}(\lambda_\zeta + \lambda_\zeta^2 + \lambda_\zeta^3 + \lambda_\zeta^4)](\hat{\zeta}_{t-1} + \hat{\zeta}_{t-2} + \hat{\zeta}_{t-3} + \hat{\zeta}_{t-4})$

(26d) optimal pricing $\quad \hat{q}_{jt} = (1-\beta\omega)\Lambda[\hat{w}_t - \frac{1}{1-\alpha}(\hat{\zeta}_t - \alpha\hat{y}_t)] + \beta\omega E_t^j[\hat{q}_{j(t+1)} + \hat{\pi}_{t+1}] \ , \ \Lambda = \frac{1-\alpha}{1-\alpha+\alpha\theta}$

(26e) bond holding $\quad \hat{b}_t^j = \frac{1}{\beta} \hat{b}_{t-1}^j + [1 + (1-\alpha)\frac{\theta - 1}{\theta} \frac{\sigma}{\eta}](\hat{y}_t - \hat{c}_t^j)$

(26f) profits earned $\quad \hat{pr}_t = \frac{\theta}{\theta - (1-\alpha)(\theta-1)} \left\{ \hat{y}_t - (1-\alpha)(\theta-1)\hat{w}_t + (\theta-1)\hat{\zeta}_t \right\}$

(26g) market clearing $\quad \hat{c}_t = \hat{y}_t \quad , \quad \hat{n}_t = \frac{1}{1-\alpha}(\hat{y}_t - \hat{\zeta}_t) \quad , \quad \hat{\pi}_t = \frac{1-\omega}{\omega} \hat{q}_t \quad , \quad \hat{b}_t = 0 \quad , \quad u_t = \hat{\ell}_t - \hat{n}_t .$

1.5 The Equilibrium Map of the Integrated Economy

The microeconomic equilibrium concept I use is the Standard equilibrium as in Kurz (2012), KPW (2013) and is explored in detail in Appendix B where I discuss the general problem of multiple expectation equilibria. Individual decisions are then linear in $(1, Z_t, \zeta_t, \zeta_{t-1}, \zeta_{t-2}, \zeta_{t-3}, \zeta_{t-4}, \psi_t, \hat{b}_{t-1}^j, g_t^j)$ which are *individual* *states* and aggregates are linear in *aggregate states* $(1, Z_t, \zeta_t, \zeta_{t-1}, \zeta_{t-2}, \zeta_{t-3}, \psi_t, 0, Z_t)$. For example consumption, optimal pricing of j, income and inflation in the upper and lower sub-economies are

(27a) $\hat{c}_t^{jU} = A_y^{U} \bullet (1, Z_t, \zeta_t, \zeta_{t-1}, \zeta_{t-2}, \zeta_{t-3}, \zeta_{t-4}, \psi_t, \hat{b}_{t-1}^{j}, g_t^{j})$, $\hat{c}_t^{jL} = A_y^{L} \bullet (1, Z_t, \zeta_t, \zeta_{t-1}, \zeta_{t-2}, \zeta_{t-3}, \zeta_{t-4}, \psi_t, \hat{b}_{t-1}^{j}, g_t^{j})$

(27b) $\hat{y}_t^{U} = A_y^{U} \bullet (1, Z_t, \zeta_t, \zeta_{t-1}, \zeta_{t-2}, \zeta_{t-3}, \zeta_{t-4}, \psi_t, 0, Z_t)$, $\hat{y}_t^{L} = A_y^{L} \bullet (1, Z_t, \zeta_t, \zeta_{t-1}, \zeta_{t-2}, \zeta_{t-3}, \zeta_{t-4}, \psi_t, 0, Z_t)$.

(27c) $\hat{q}_{jt}^{U} = \dfrac{\omega}{(1-\omega)} A_\pi^{U} \bullet (1, Z_t, \zeta_t, \zeta_{t-1}, \zeta_{t-2}, \zeta_{t-3}, \zeta_{t-4}, \psi_t, \hat{b}_{t-1}^{j}, g_t^{j})$, $\hat{q}_{jt}^{L} = \dfrac{\omega}{(1-\omega)} A_\pi^{L} \bullet (1, Z_t, \zeta_t, \zeta_{t-1}, \zeta_{t-2}, \zeta_{t-3}, \zeta_{t-4}, \psi_t, \hat{b}_{t-1}^{j}, g_t^{j})$

(27d) $\hat{\pi}_t^{U} = A_\pi^{U} \bullet (1, Z_t, \zeta_t, \zeta_{t-1}, \zeta_{t-2}, \zeta_{t-3}, \zeta_{t-4}, \psi_t, 0, Z_t)$, $\hat{\pi}_t^{L} = A_\pi^{L} \bullet (1, Z_t, \zeta_t, \zeta_{t-1}, \zeta_{t-2}, \zeta_{t-3}, \psi_t, 0, Z_t)$.

The constants imply (26a)-(26g) fluctuate around values slightly different from steady state with $\bar{r} = (1-\beta)/\beta$. To see why, insert the linear maps and (24) into (26a)-(26g) to show variables are not zero even if all state variables are zero due to the ZLB. I next exclude a Ponzi equilibrium with details provided in Appendix B.

Excluding the Ponzi Micro Equilibrium. By Taylor's theorem equilibrium has a decomposition property that implies equilibrium elasticities of endogenous variables relative to an exogenous variable is solved for each such variable separately. With this principle I solve parameters of b_{t-1}^{j} and examine (A_b^{bU}, A_b^{bL}) to determine boundedness properties of bond holdings. Let $\Xi = 1 + (1-\alpha)\dfrac{\theta - 1}{\theta}\dfrac{\sigma}{\eta} > 0$, then I have

Proposition: In any equilibrium $A_y^{b} = A_y^{bL} = A_y^{bU}$ and $A_b^{bU} = A_b^{bL} = A_b^{b} = (1/\beta) - A_y^{b}\Xi$. However, the equation system in the Definition has two solutions, one with $A_y^{b} < 0$ and one with $A_y^{b} > 0$. The case of $A_y^{b} < 0$ implies a dynamically unstable "Ponzi" solution since $A_b^{b} > 1$. This solution is not an equilibrium and is thus excluded.

I finally specify the last set of parameters: $\lambda_\psi = 0.20$, $\sigma_\psi = 0.001$, $\xi_\pi = 1.5$, $\xi_y = 0.5$ and explain them as we go. Policy parameters (ξ_π, ξ_y) are annual, equal to about what most consider the policy to have been.

Equilibrium maps for RE are in Table 1A and for RB in Table 1B. By decomposition all wage parameters which are relevant to policy analysis are the same in the two tables hence the effects of a wage stabilization policy are the same under RE and RB. The key analytic result which is the foundation of the wage stabilization policy is that the upper sub-economy's map is drastically different from the map of the lower sub-economy, a fact not adequately recognized in recent policy discussions. Virtually all equilibrium parameters are different and some have opposite signs. The inflation constant of -0.00017 in the upper sub-economy is a small deflation effect due to a transition probability to the lower sub-economy with a strong deflation constant of -0.01078, equivalent to -4.4% annually and a deflation constant of -1% on output.

Table 1A: Equilibrium Map Under Rational Expectations

upper sub-economy							
	$\hat{c}_t^j$	$\hat{y}_t$	b_t^j	$\hat{\pi}_t$	$\hat{w}_t$	u_t	$\hat{pr}_t$
constant	0.00009	0.00009	0	-0.00017	0.00012	-0.00010	-0.00011
ζ_t	1.33488	1.33488	0	-0.34768	0.40362	-1.30009	1.87105
ζ_{t-1}	-0.11759	-0.11759	0	0.05928	0.08713	0.36936	-0.15301
ζ_{t-2}	-0.09598	-0.09598	0	0.05322	0.09706	0.32742	-0.15732
ζ_{t-3}	-0.06899	-0.06899	0	0.04245	0.11113	0.27671	-0.16478
ζ_{t-4}	-0.03673	-0.03673	0	0.02532	0.13016	0.21832	-0.17648
ψ_t	-0.90863	-0.90863	0	-0.18193	-0.18952	1.99120	-0.10384
b_{t-1}^j	0.01377	0	0.99335	0	0	0	0
lower sub-economy							
	$\hat{c}_t^j$	$\hat{y}_t$	b_t^j	$\hat{\pi}_t$	$\hat{w}_t$	u_t	$\hat{pr}_t$
constant	-0.00979	-0.00979	0	-0.01078	0.00256	0.02605	-0.00687
ζ_t	0.94143	0.94143	0	-0.27783	0.24401	-0.51542	1.92302
ζ_{t-1}	0.20508	0.20508	0	0.15543	0.13798	-0.35422	-0.09557
ζ_{t-2}	0.09960	0.09960	0	0.10401	0.13179	-0.10725	-0.12739
ζ_{t-3}	0.03312	0.03312	0	0.06439	0.13165	0.05216	-0.15214
ζ_{t-4}	0	0	0	0.03137	0.13851	0.13851	-0.17313
ψ_t	-1.46320	-1.46320	0	-0.29188	-0.30576	3.20592	-0.16650
b_{t-1}^j	0.01377	0	0.99335	0	0	0	0

A policy that alters wage scales have different effects in the upper and lower sub-economies and these are crucial for a stabilizing policy. Higher wage scales (attained by adding a constant to each $(\zeta_{t-1},\zeta_{t-2},\zeta_{t-3},\zeta_{t-4})$) causes an equilibrium reduction in consumption and output, and increased inflation and unemployment in the upper sub-economy. These are standard effects one would expect. *But in the lower sub-economy an increase of all wage scales by 0.05 added to each ζ_{t-j} increases output and consumption by 1.70%, it increases quarterly inflation by 1.78% and lowers unemployment by 1.35%!* These results are new. I show later they are caused by the fact that an economy with an inflexible price system finds alternate ways to respond to shocks. Here inflexible interest rate and inflexible wages combine to induce the economy to adjust to shocks in an unfamiliar ways that have profound implications to policy actions.

Table 1B presents maps under RB where varying (b_{t-1}^j, g_t^j, Z_t) contribute to volatility and explain the added volatility that needs explaining without large technology shocks. It is then appropriate to test the model's implied aggregate volatility but, more important, the two relative volatilities (σ_w/σ_y) and (σ_n/σ_y) where $(\sigma_w, \sigma_y, \sigma_n)$ are standard deviations of the real wage, output per capita and labor employed. Event $\mathbb{C}$ causing transition to the lower sub-economy in the simulations is a random configuration of aggregate states that imply a negative nominal rate. It is a very modest event since it excludes major unexpected crash event that causes such transitions in real data. Also, I select $(\lambda_\psi=0.20, \sigma_\psi=0.001)$ to have a small effect on volatility but it is essential in the next Section. Table 2 presents the results.

Table 1B: Equilibrium Map Under Rational Beliefs

upper sub-economy							
	$\hat{c}_t^j$	$\hat{y}_t$	b_t^j	$\hat{\pi}_t$	$\hat{w}_t$	u_t	$\hat{pr}_t$
constant	0.00009	0.00009	0	-0.00017	0.00012	-0.00010	-0.00011
z_t	-5.34729	-1.33811	4.87730	2.13031	-1.53035	1.68112	1.41115
ζ_t	1.33488	1.33488	0	-0.34768	0.40362	-1.30009	1.87105
ζ_{t-1}	-0.11759	-0.11759	0	0.05928	0.08713	0.36936	-0.15301
ζ_{t-2}	-0.09598	-0.09598	0	0.05322	0.09706	0.32742	-0.15732
ζ_{t-3}	-0.06899	-0.06899	0	0.04245	0.11113	0.27671	-0.16478
ζ_{t-4}	-0.03673	-0.03673	0	0.02532	0.13016	0.21832	-0.17648
ψ_t	-0.90863	-0.90863	0	-0.18193	-0.18952	1.99120	-0.10384
b_{t-1}^j	0.01377	0	0.99335	0	0	0	0
g_t^j	4.00917	0	-4.87730	0	0	0	0
lower sub-economy							
	$\hat{c}_t^j$	$\hat{y}_t$	b_t^j	$\hat{\pi}_t$	$\hat{w}_t$	u_t	$\hat{pr}_t$
constant	-0.00979	-0.00979	0	-0.01078	0.00256	0.02605	-0.00687
z_t	-1.40755	1.92310	4.05186	-0.58176	0.90554	-3.70991	-0.41077
ζ_t	0.94143	0.94143	0	-0.27783	0.24401	-0.51542	1.92302
ζ_{t-1}	0.20508	0.20508	0	0.15543	0.13798	-0.35422	-0.09557
ζ_{t-2}	0.09960	0.09960	0	0.10401	0.13179	-0.10725	-0.12739
ζ_{t-3}	0.03312	0.03312	0	0.06439	0.13165	0.05216	-0.15214
ζ_{t-4}	0	0	0	0.03137	0.13851	0.13851	-0.17313
ψ_t	-1.46320	-1.46320	0	-0.29188	-0.30576	3.20592	-0.16650
b_{t-1}^j	0.01377	0	0.99335	0	0	0	0
g_t^j	3.33066	0	-4.05186	0	0	0	0

Labor market models with flexible wages typically show $(\sigma_w/\sigma_y) > 1.25$, $(\sigma_n/\sigma_y) < 0.75$ (e.g. see KPW (2013), Table 3) and so does Table 2. It also shows that under wage inflexibility the economy makes much stronger quantity adjustments instead of wage adjustments. Policy changes the absolute volatilities.

Table 2: Simulated Model Volatilities

	σ_y	σ_π	σ_w/σ_y	σ_n/σ_y
US post war data	1.81	1.71	0.38	0.99
Model with flexible wage	1.19	3.99	1.38	0.70
Model with inflexible wage	1.59	2.56	0.46	0.90

1.6 Price Inflexibility vs. Wage Inflexibility

I assumed $\omega = 0.4$ which is lower than 2/3 commonly assumed in many NK papers and this is an important issues with implications to be examined. Note that wage inflexibility implies price inflexibility, particularly in models where labor is the only input, and this questions how much "own" price inflexibility is left. It is sharpened by inflexible interest rate under the ZLB which further limits market adjustment. Most NK models assume output is a linear function of labor under which a distinction between nominal wage and price level would have disappeared entirely had it not been for the opposite assumption that *wages are completely flexible* but prices are sticky, a combination not supported by empirical evidence.

The initial reason given for price inflexibility was a firms' choice to avoid small price changes in response to *small changes of cost* and that such choice has large aggregate implications (e.g. see Akerlof and Yellen (1985), Mankiw (1985), Blanchard and Fisher (1989) Chapter 8 and references). Later empirical work (e.g. Bils and Klenow (2004) , Hosken and Reiffen (2004), Nakamura and Steinsson (2008), Kehoe and Midrigan (2010), Eichenbaum, Jaimovich, and Rebelo (2011)) questions the degree of price inflexibility. Studies of retail prices show frequent "sales" reflect price flexibility but subsequent prices reverting to pre-sale prices reflect price inflexibility. But such inflexibility is with *respect to competitors' prices* when they have sales. No doubt, monopolistic competitors have a strong motive to promote customer loyalty with predictable price policy that persuades them not to search for alternate products. Therefore, such firms prefer to avoid unexpected price changes and take time to inform customers of any change.

This paper studies circumstances when firms face large common cost increase. Being *large* rather than small and *common* rather than firm specific, most firms may take time to inform customers but then adjust prices rapidly. The issue is sharper under a ZLB. With inflexible nominal rate and inflexible wage a firm's survival depends upon its ability to respond to rising cost with price adjustment, particularly if cost increases are large and common. This argument implies that price inflexibility with respect to large common changes in wage cost is lower compared to price inflexibility in the normal conduct of business. What is a plausible degree of own price inflexibility with respect to large wage changes? Full price flexibility with probability $\omega = 0$ implies a mean duration of 0 quarters after the present quarter. If a policy raises all real wages by 10%, how much time would a firm take to protect itself and raise its price? Expected durations beyond the present quarter are: 6 months if $\omega = 2/3$, 3 months if $\omega = 0.5$, 2 months if $\omega = 0.4$, 1.29 months if $\omega = 0.3$ and 0.75 month if $\omega = 0.2$. Given the cost, 0.75 month is too short and 6 months is much too long. 3 months is not implausible but is rather slow: it takes refineries less than 3 months to raise gas prices in response to a significant increase in crude prices.

Table 3 reports equilibrium elasticities of output, inflation, real wage, unemployment and profits relative to wage scales. They are deduced from Table 1B where ω is changed and each elasticity is the sum of the coefficients of $(\zeta_{t-1}, \zeta_{t-2}, \zeta_{t-3}, \zeta_{t-4})$. Focusing on *quarterly inflation rate*, if $\omega = 0.5$ the model predicts elasticity of 0.19 which is on the small side while for $\omega = 0.2$ the prediction is 1.31, which is far too high.

Table 3: Elasticities of Key Variables With Respect to the Wage Scale Under a ZLB (percent)

	ω = 0.2	ω = 0.3	ω = 0.4	ω = 0.5	ω = 2/3
η^{y}_{ws}	1.08	0.60	0.34	0.18	0.05
η^{π}_{ws} (quarterly)	1.31	0.67	0.35	0.19	0.05
η^{w}_{ws}	0.30	0.48	0.54	0.55	0.47
η^{u}_{ws}	-2.29	-0.96	-0.27	+0.11	+0.35
η^{pr}_{ws}	+0.03	-0.37	-0.55	-0.62	-0.57

I conclude that given my primary assumption of inflexible wages, *the added own price inflexibility should be moderate*. Plausible values are $0.3 \le \omega \le 0.5$ and most simulations are for $\omega = 0.40$. However, the results regarding the efficacy of policy discussed later would not be materially different if the true value is $\omega = 0.5$ or $\omega = 0.3$ since policy can be adapted to such circumstances, as will be explained later.

2. Stabilizing Wage Policy

My discussion is divided into three parts. First, I explain what a Stabilizing Wage Policy is. Second, to use my main tool which is the integrated economy model, I formulate an event ℂ that generates an artificial depression and, correspondingly, specify deleveraging conditions to end it. Third, I simulate the economy to assess the efficacy of the policy.

2.1 Stabilizing Wage Policy

A wage stabilization policy increases firms' cost, induces them to raise prices and generate inflationary pressure that breaks the deflationary forces under the ZLB. This lowers the real rate, boosts demand, increases output and in most cases lowers unemployment. Inflationary forces reduce debt and together with the increased output, the policy produces a deleveraging effect which enables the economy to move towards recovery. The policy does not rely on a central bank's presumed ability to raise inflation expectations by "committing to be irresponsible." It is achieved by aligning incentives of firms with the policy's objective *of creating a controlled inflationary spiral*. It has no residual debt burden effects compared with a Keynesian debt burden left by deficit financed public expenditures and if executed with sufficient intensity it reduces the real national debt. This intervention is temporary.

Recall that date t wage scale for offers made at t-j is $\hat{w}^{e}_{t-j} = (1/4)(\lambda_{\zeta}+\lambda^{2}_{\zeta}+\lambda^{3}_{\zeta}+\lambda^{4}_{\zeta}) A^{\zeta}_{w}\hat{\zeta}_{t-j}$ and date t reference wage is $\hat{w}^{\star}_{t} = B^{\zeta}_{w}(\hat{\zeta}_{t-1}+\hat{\zeta}_{t-2}+\hat{\zeta}_{t-3}+\hat{\zeta}_{t-4})$ where $B^{\zeta}_{w} = (A^{\zeta}_{w}/16)(\lambda_{\zeta}+\lambda^{2}_{\zeta}+\lambda^{3}_{\zeta}+\lambda^{4}_{\zeta})$. A "wage scale" is essential to my discussion and is equal to"base wage" or "base pay" commonly used.

A stabilizing wage policy is a sequence of legally imposed changes in the wage scales of all firms. Given offer duration of four quarters, the policy specifies the following:

(i) a starting date t^0 and a *rule for policy termination*, calling for termination *at the start of* date T;

(ii) a sequence $\hat{\vartheta}_t$, $t \geq t^0$ of percentage changes of all four effective real wage scales in the market at date t, making the effective scales for wage payments at date t be

$$
(28) \qquad \begin{pmatrix} \hat{w}^e_{t-1} \\ \hat{w}^e_{t-2} \\ \hat{w}^e_{t-3} \\ \hat{w}^e_{t-4} \end{pmatrix} = \begin{pmatrix} 4B^{\zeta}_{w}(\hat{\zeta}_{t-1} + \hat{\vartheta}_t) \\ 4B^{\zeta}_{w}(\hat{\zeta}_{t-2} + \hat{\vartheta}_t) \\ 4B^{\zeta}_{w}(\hat{\zeta}_{t-3} + \hat{\vartheta}_t) \\ 4B^{\zeta}_{w}(\hat{\zeta}_{t-4} + \hat{\vartheta}_t) \end{pmatrix} \qquad \hat{\vartheta}_t = [\hat{\vartheta}_t + \hat{\vartheta}_{t-1} + \ldots + \hat{\vartheta}_{t^0}]
$$

(iii) real wages to be changed at $t \geq t^0$ under the following rules adopted by each firm:

- date $t \geq t^0$ wages and salaries to be paid in accord with a firm's adjusted scale in (28) ;
- preserve the real value of the changes with their full cost of living adjustment at the end of the quarter hence the percentage changed scale $\hat{\vartheta}_t$ is the *cumulative real change* from t^0 to t;
- job offers, new or those made following work evaluation, are made with the adjusted scale;
- when the policy is in force it is unlawful for firms to lower their scales for any group of jobs together with the aim of eliminating the adjustments made by the policy;

(iv) Policy termination. After a policy's terminal date T, $\hat{\vartheta}_t = 0$, $t \geq T$ but since firms made earlier wage commitments, a residual policy effect is eliminated in four quarters by new wage offers and inflation. Cost of living adjustment are eliminated after T hence inflation reduces the adjustment of wage scales as follows:

$$\hat{\vartheta}_{T+j} = \text{Max}[0, \hat{\vartheta}_{T+j-1} - \hat{\pi}_{T+j-1}] \quad j = 1, 2, 3, 4.$$

- At T : 1/4 of firms make new offers $\hat{w}^e_T = 4B^{\zeta}_{w}\hat{\zeta}_T$, eliminate $\hat{\vartheta}_{T+1}$ and resort to the old scale;
- At T+1: 1/4 of firms make new offers $\hat{w}^e_{T+1} = 4B^{\zeta}_{w}\hat{\zeta}_{T+1}$ and eliminate $\hat{\vartheta}_{T+2}$;
- At T+2: 1/4 of firms make new offers $\hat{w}^e_{T+2} = 4B^{\zeta}_{w}\hat{\zeta}_{T+2}$ and eliminate $\hat{\vartheta}_{T+3}$;
- At T+3: 1/4 of firms make new offers $\hat{w}^e_{T+3} = 4B^{\zeta}_{w}\hat{\zeta}_{T+3}$ and eliminate $\hat{\vartheta}_{T+4}$;
- $\hat{w}_t^e = 4B^{\zeta}_{w}\hat{\zeta}_t$, all $t \geq T+4$ since by then all firms eliminate policy effects via new wage offers;

Discussion of the policy

(1) Changes of wage scale vs. wage control. Wage scales are changed by altering the state variables that define them rather than by direct control of market wages, which are endogenous variables. This is done to

integrate the policy into the wage setting process and thus interfere as little as possible with the market. The policy requires workers to be paid in accord with the adjusted scale for offers' durations and for the duration of all revised or new offers' and this fact introduces important persistence into the policy's effects. This is in contrast with a period-by-period nominal wage control.

(2) The objective: a controlled wage-inflation spiral. The required cost of living adjustments each quarter is crucial since the policy aims to create a controlled wage-inflation spiral: higher wages lead to higher prices but cost of living adjustments ensure the active spiral for the policy's duration. Cost of living adjustment also ensures the effect of a sequence of real wage changes on the inflationary spiral is *cumulative*. This raises several questions: is it more effective to change wages drastically at the start date t^0 or is a gradualist policy more effective if it is introduced at a slow cumulative rate?

(3) Future wage scale adjustments. Firms make future wage offers that include the adjusted scales. But wage scales are dynamic tools that change with productivity, raising enforcement problems that require a distinction between common changes in scale that reflect productivity and those that nullify the policy. Supervision can detects violations since quarterly productivity changes are measures in a *fraction of percentage point* while policy changed scales are measures in 5-10 percentage points.

(4) Residual policy termination effects. Due to firms' wage commitments a successful policy has a residual effect of real wages that may be too high at exit time from the ZLB. Termination policy sets $\tilde{\vartheta}_t = 0$ for all $t \geq T$ and a residual $\tilde{\vartheta}_T \geq 0$ is eliminated by firms that exclude it from new wage offers over four successive quarters. The simulations identify these four quarterly wage adjustments after a policy is terminated.

(5) Wages of irregular employees. (21) shows a changed regular scale applies to irregular wages. If a firm has a separate scale for irregular jobs the policy applies to scales used to set irregular wages.

(6) Inflation target and central bank credibility. The policy changes the inflation rate for the policy's duration and its termination phase. However, a consistent policy requires the monetary rule to remain intact with the same inflation target embodied in the rule. This implies that after exit from the ZLB the economy fluctuates *around the same steady state inflation built into the policy*. Ambiguity about this long run target can cause the central bank to lose control over the recovery, repeating the experience of the 1970's. Indeed, a stabilization policy is likely to fail if it does not distinguish between the short term objective attained under short term inflexibilities, and long term target which incorporates a vertical Phillips curve. If the inflation target is intact, then under a ZLB changes in base money have no effect on the inflation rate.

(7) Why not a combined wage tax and subsidy? An emergency to raise wages in a depression is a policy

which is well understood by the public hence a policy-maker can mobilize support for it. It is implemented with laws that grant executive power for emergency price control and can be instituted instantly with future changes in policy intensity as needed. A wage income tax *on employers* with a prohibited reduction in paid wages together with wage income subsidy can attain about the same goal but is rejected for three reasons: (i) a bi-weekly tax collection and subsidy payments are costly, (ii) it is politically vulnerable since once taxes are collected political pressure can divert funds to other programs, (iii) tax rates and changed tax rates require congressional approval which is slow, inefficient and subject to pressure from interest groups.

(8) Stabilization policy without public cost or borrowing. Deficit financed direct public expenditures and standard measures such as tax cuts, tax incentives for investments, all raise the public debt. Stabilizing wage policy does not use public funding and it actually benefits public financing. Higher wages increase public labor cost which are financed by larger tax revenues due to the policy. But the policy's induced inflation also lowers the public debt and, most important, shortens the duration of deflation and eliminates the automatic deficits due to higher social cost in the depression.

To construct the policy I focus on demand shocks to household balanced sheets, as in the 2007-2015 Great Recession and the proxy I use mimics forces that generate such deflation. The two exit conditions require first a positive nominal rate in the upper sub-economy and second, satisfaction of a deleveraging condition. I refer to the policy $\tilde{\mathfrak{V}}_{t^0} = 0.10$, $\tilde{\mathfrak{V}}_t = 0$ for $t^0 < t \leq T$ as the *Standard Active* policy as it raises all wage scales initially by 10% and maintains $\hat{\mathfrak{V}}_t = 0.10$ *real* level throughout.

2.2 Methodology for Implementing Wage Stabilization Policy

2.2.1 The Artificial Depression and the Event $\mathbb{C}$

I denoted by $\mathbb{C}=1$ an event that triggers a depression. It is now generally agreed that the event which triggered the great recession of 2007-2015 was the collapse of real estate prices that lead to a sharp decline in Asset Backed Securities which, in turn, lead to the financial crisis of 2009 which nearly bankrupted the banking system. Aggregate demand fell initially due to the loss of household wealth with a negative wealth effect and desire to lower debt. The resulting rise of unemployment made deleveraging harder and together with rising corporate and governmental austerity caused the great recession to linger on for long eight years. Despite signs of recovery early in 2015, no transition from the lower sub-economy has yet taken place. To describe such a complex configuration of decision makers, institutions and assets requires an expanded formal model which cannot be built with high precision. Indeed, the integrated model I use here does not contain real estate, mortgages or a stock market and short duration bonds do not lend themselves to a long process of deleveraging. In short, the model does not aim to explain what caused the great recession of 2007-2015 but then, it does not need to. One may recall that Keynes' General Theory is about policy to combat depressions but it does not explain what causes depressions. Since I study policies to counter the effects of such events, it is not clear a detailed expanded model is necessary. I agree with others (e.g. Koo (2008), Mian and Suffi (2011), Eggertson and Krugman (2013)) that the cause of persistent ZLB since 2009 has been *low demand due to lost household wealth and financial need (or pressure) to deleverage* and hence a recovery requires adequate repair of households' balance sheets.

My approach is then to create a proxy for the event ℂ=1 with *consequences to aggregate demand* that cause a transition to a binding ZLB. Next, using the model's computed equilibrium data of output and inflation I assess the implied approximate degree of deleveraging and implied restoration of household wealth that would occur at each date along the equilibrium path. These imply a date when cumulative deleveraging is sufficient to remove the deflationary pressure and exit the ZLB. My proxy to lower the natural rate is commonly used by others (eg. Eggertson and Woodford (2003), Eggertson (2011), Christiano, Eichenbaum, and Rebelo (2011), Correia, Farhi, Nicolini, and Teles (2011), Rendahl (2012), Schmitt-Grohe and Uribe (2012)). The event ℂ is a sequence of unanticipated shocks which are added to ψ_t. Recall that the empirical distribution of ψ_t implies a Markov transition

$$\psi_{t+1} = \lambda_\psi \psi_t + \rho^{\psi}_{t+1}$$

incorporated in agents' beliefs. ℂ=1 is then a sequence $\{\Delta\psi_t, t^0 \le t \le T\}$ of *unexpected* added shocks consisting of an initial shock $\Delta\psi_{t^0}>0$ followed by quarterly shocks $\Delta\psi_t, t>t^0$ that maintain pressure of deflation. To ensure it causes a transition to the ZLB, I use (24) and the equilibrium map of the *upper sub-economy* (Table 1B). If all other shocks are zero and after $\Delta\psi_{t^0}>0$ dissipates, it is necessary that

$$(\hat{r}_t)/(1+\bar{r}) + (0.38764)\Delta\psi_t > 0 \quad , \quad t>t^0.$$

For a ZLB to be maintained, $\Delta\psi_t > (0.01)/(0.38764) = 0.0259$ which is a lower bound on $\Delta\psi_t$. The actual transition function then becomes

(29) $$\psi_{t+1} = \lambda_\psi(\psi_t + \Delta\psi_t) + \rho^{\psi}_{t+1} \quad , \quad t^0 \le t \le T$$

To illustrate the event assume first that except for ρ^{ψ}_t all shocks are set at 0 hence simulated values are actually mean values. I set $\Delta\psi_{t^0} = 0.06$ which is $60\times\sigma_\psi$ and $\Delta\psi_t = 0.026$, $t^0<t\le T$ which is $26\times\sigma_\psi$. I later state a deleveraging condition and when satisfied at a random date T, it implies ℂ=0 and hence $\Delta\psi_t=0$ for $t\ge T$. The deflationary pressure is lifted and an exit from the ZLB occurs. Before discussing deleveraging I

report in Figure 2a simulated response to ℂ=1: output is initially 10% below potential, annualized deflation is -11%, involuntary unemployment +22% and profits fall by -1.7%. The depression settles down with output around -5%, annual deflation at -8%, unemployment at +14% and profits at -1.3%. The small decline in profits results from the ability of firms to immediately reduce employment which is their only cost. Profits would be much lower if invested capital, fixed cost and inventories are introduced. The deflationary pressure is lifted after 44 quarters, at which time the required deleveraging is attained, the nominal interest rate in the upper sub-economy turns positive and an exit from the ZLB takes place.

Figure 2a
The Canonical Artificial Depression

Figure 2a shows the key features of depressions are sustained unemployment and deflation, facts that are consistent with past depressions. As unemployment soared in the 1930's, deflation rates were -4% in 1930, -10% in 1931, -11% in 1932 and -3% in 1933, implying a *31% cumulative deflation in four years!* This motivated Fisher's (1933) Debt-Deflation spiral which causes the very slow recovery that takes 11 years in Figure 2a. To understand this slow rate I need to explain the deleveraging rule used for recovery.

2.2.2 Deleveraging Conditions

Deleveraging conditions are based on the 2006-2015 US record and quantify the repair of household balance sheets necessary to permit a resumption of normal economic growth. Table 4 shows that due to collapsed home prices household real estate wealth declined in 2007-2011 by $7.2 Trillion which is 51% of the 2007 GDP level which I use as a reference. Households have other assets which are ignored here for the moment. One effect of the Fed's Quantitative Easing policy was the rise in asset prices. This restored in 2012-2014 part of household real estate wealth amounting to $3.9 Trillion, leaving a deficit of $3.3 Trillion. Shiller-Case indexes reveal that most gains occurred in high income areas hence most of the $3.3 Trillion deficit, which is 24% of the reference 2007 GDP, is concentrated in middle and low income areas, a result supported by the empirical work of Mian and Sufi (2014). Accepting the approximate nature of the requirement, my objective in the model computations is to recover household wealth of $(0.24)\bar{Y}$[4]. How do

[4] This is a modest deleveraging requirement since the gains in home values is the result of the two active components of US policy during this period which are the Fed's QE and cumulative federal deficits in 2007 -2014 which amounted to $7.81 Trillion. These policies are not in the model and without them the required deleveraging would be much larger. It turns out that the efficacy of Stabilizing Wage Policy is little changed by tightening this requirement but without policy it increases the frequency of cases where spontaneous recovery in finite time is impossible.

we attain such a goal?

Table 4: US Household Rel Estate Assets and Debt 2006 - 2013
(Billions of current dollars)

year	Changed Value of Real Estate Assets	Household Debts	Home Mortgage	Consumer Credit
2006	320.7	12946.5	9910.1	2461.9
2007	-1815.6	13830.4	10611.7	2615.7
2008	-3657.8	13850.0	10579.4	2650.6
2009	-1138.9	13558.9	10418.1	2552.8
2010	- 395.9	13229.8	9914.2	2647.4
2011	- 155.9	13060.9	9698.5	2755.9
2012	1514.6	13063.5	9497.9	2923.6
2013	2386.3	13179.2	9415.9	3097.9

Source: *Federal Reserve Statistical Release Z.1 Financial Accounts of the United States*, September 18, 2014, Tables D.3 and S.3a

Deleveraging takes two forms: formal deleveraging and inflation. Formal deleveraging after interest payments is done by direct savings and debt reductions. Households save a fraction of income assumed the same as in 2007-2014 US. Some suggest it responds to liquidity constrains (e.g. Eggertsson and Krugman (2013)), an issue addressed later. But a major part of formal deleveraging entails reduction of debt by renegotiations, voluntary mortgage default, foreclosure, refinancing at lower rates and personal default. Saving of high income households is irrelevant since they do not need to deleverage. The fraction of GDP applicable to deleveraging is assumed 70% and if formal deleveraging is set at 10% of this available income (2.5% quarterly) then, using 2007 GDP, quarterly deleveraging is (0.025)(0.70)(14,000) = \$245 Billion which is \$980 Billion annually. Denote by $\mathbf{d}_\ell$ the effective deleveraging rate as a proportion of GDP hence $\mathbf{d}_\ell \mathbf{Y}_t$ is the total date t formal deleveraging by savings and adjustments to debt. The standard quarterly rate used in the simulations is $\mathbf{d}_\ell$=1.75%.

The second channel to repair or impair household balance sheets is inflation. To quantify it I use data for US in 2007 with GDP of \$14 Trillion and 61% share of wages and salaries that amount to \$8.54 Trillion and total household debt of \$14 Trillion. These debts consisted mostly of home mortgages of \$10.6 Trillion and consumer credit of \$2.62 Trillion (fixed rate student loans were only \$507 Billion in 2007). The question is how does the real value of these debts change with inflation or deflation since this is a central factor with an impact on the duration of a depression under a ZLB. It depends on the proportion of debt that has some effective protection and this is a complex issues. A large fraction of mortgages carry variable rates that respond with delay to inflation. In addition, a zero nominal rate with low penalty for refinancing offer borrowers refinance opportunities to lower debt burden. Consumer credit is mostly short term with rates that adjust for inflation. All these constitute some partial protection: inflation or expected inflation adjusts the nominal rate on these debts so that changed values are compensated by changed rate. Due to the complexity of this issue, only an approximation can be used to assess the magnitude of the effect of inflation. With that in mind I take all student loans and 60% of mortgage debt, for a total of $6.9 Trillion fixed rate household loans outstanding in 2007 which it is convenient to take as 0.49 of 2007 GDP.

I denote by $[Y_t^P - Y_t]$ the income difference with and without the wage policy. Since wage income was 61% of 2007 GDP, the wage income difference is $(0.61)[Y_t^P - Y_t]$, due to higher output and lower unemployment. Hence, it offers added deleveraging potential above the rate d_ℓ. The evidence also shows that regular income provides incentive to financial institutions to restructure debt. Savings out of this source depend on how workers treat the added income when they know the policy is temporary. The more permanent they believe the new jobs are, the lower are the savings. Considering all factors involved it is plausible the contribution to deleveraging from this source would be 20% of this added income source but I later report how the results change if this proportion takes different values.

To define a quarterly debt dynamics let (Y_t, D_t, π_t) be values of income, debt and inflation rate and (Y_t^P, D_t^P, π_t^P) are these quantities under a wage policy that starts at t^0 and ends at T. The quarterly formal deleveraging without policy is $d_\ell Y_t$ while $d_\ell Y_t^P + (0.20)(0.61)[Y_t^P - Y_t]$ is formal deleveraging under the policy. It is hard to select a strategy for the type of debt deleveraging reduces since much of debt reduction is non-voluntary due to foreclosure, default etc. I then assume deleveraging is made in proportion to the debt value of each type, one date earlier. The value of fixed rate debt is D_t^F and the value of variable rate debt is D_t^V. Without policy the dynamics of fixed, variable and total debts are

$$D_{t^0}^F = (0.49)\bar{Y} , \quad D_t^F = \frac{D_{t-1}^F}{(1+\pi_t)} - \left(\frac{D_{t-1}^F}{D_{t-1}}\right) d_\ell Y_t$$

$$D_{t^0}^V = (0.51)\bar{Y} , \quad D_t^V = D_{t-1}^V - \left(\frac{D_{t-1}^V}{D_{t-1}}\right) d_\ell Y_t .$$

(30a) Total debt without policy: $D_{t^0} = \bar{Y} \;,\; D_t = D_t^F + D_t^V \;,\quad t^0 \le t \le T.$

With policy it is

$$D_{t^0}^{FP} = (0.49)\bar{Y} , \quad D_t^{FP} = \frac{D_{t-1}^{FP}}{(1+\pi_t^P)} - \left(\frac{D_{t-1}^{FP}}{D_{t-1}^P}\right)[d_\ell Y_t^P + (0.20)(0.61)(Y_t^P - Y_t)]$$

$$D_{t^0}^{VP} = (0.51)\bar{Y} , \quad D_t^{VP} = D_{t-1}^{VP} - \left(\frac{D_{t-1}^{VP}}{D_{t-1}^P}\right)[d_\ell Y_t^P + (0.20)(0.61)(Y_t^P - Y_t)] .$$

(30b) Total debt with policy $D_{t^0}^P = \bar{Y} \;,\; D_t^P = D_t^{FP} + D_t^{VP} \;,\quad t^0 \le t \le T.$

Lower capital income due to higher wages has no effect on deleveraging.

The deleveraging condition requires debts to fall and household equity restored. T is the first date when deleveraging reaches $(0.24)\bar{Y}$ hence, it is the first date when

(31) $$D_T \le (0.76)\bar{Y}.$$

Since deflation increases real value of debts, a depression creates a race between deflation and deleveraging. Rebuilding of household equity is possible only if institutions permit rapid enough deleveraging and this will become clear in the analysis below. But then, who imposes these conditions and why an effective quarterly deleveraging rate of $d_\ell = 0.0175$? Deflation is endogenous, nominal savings is chosen by households but debt forgiveness, foreclosure and default requires policy effort and social norms regarding debt and financial obligations. Societies differ in their response to such problems but these social practices and norms determine the effective rate of deleveraging. Since all model features are deduced from actual data, $d_\ell = 0.0175$ is set so the implied ZLB duration is in a range of 40 - 50 quarters when ℂ = 1 is set at $\Delta\psi_{t^0} = 0.06$, $\Delta\psi_t = 0.026, t^0 < t \le T$. It is a social norm that has a strong effect on the social recovery rate. The record for the US is lower than $d_\ell = 0.0175$ but in the model lower deleveraging rates result in the economy never repairing household wealth spontaneously and never exiting the ZLB. This raises the problem of explaining the US experience which I discuss later.

Table 5: Depression's Duration with Spontaneous Recovery from the ZLB (quarters)

		$\Delta\psi_t$, $t^0 < t \le T$			
	$d_\ell \Downarrow$	0.026	0.0275	0.30	0.40
$\omega = 0.3$	$d_\ell < 0.023$	Never	Never	Never	Never
	2.300%	43	54	Never	Never
$\omega = 0.4$	1.400%	Never	Never	Never	Never
	1.575%	75	Never	Never	Never
	1.750%	44	48	62	Never
	1.925%	32	35	39	Never
$\omega = 0.5$	1.400%	56	60	70	Never
	1.575%	39	40	43	71
	1.750%	30	31	32	41
	1.925%	25	26	27	32

Table 5 reports depression's durations that result from varying d_ℓ, ω and $\Delta\psi_t, t^0 < t \le T$. First, the table shows that more intense deflation shocks $\Delta\psi_t, t^0 < t \le T$ require higher deleveraging rate d_ℓ for the depressions to have *finite duration*. Second, many parameter combinations result in depressions in which the stipulated deleveraging rates do not permit finite duration. That is, at these rather high deleveraging rates it is impossible to repay the rising real values of debts. Accelerated debt reduction requires more rapid debt elimination that cannot be attained through regular market outcomes. It requires new institutions to attain it. Failure to do can lead to the collapse of democratic market structures, giving rise to totalitarian societies or else, financial institutions holding on their balance sheets debt that is never actually paid. These were indeed the conditions in Japan when banks held on their books indefinitely non-performing loans without ever pressing for formal default.

Third, the question of price inflexibility has been discussed and I now observe that if $\omega = 0.3$ it is impossible to recover in finite time unless $d_\ell \geq 0.023$ and even for $d_\ell = 0.023$ the persistent shock cannot exceed $\Delta\psi_t = 0.0275$. High price flexibility imply shocks have stronger effect and if $\omega = 0.3$ deflation rates are high and deleveraging at lower than 2.3% cannot overcome a deflation-debt spiral.

Finally, Figure 2a traces the expected value of variables since the only active shock is ψ and all others are set at 0. Actual duration will vary due to the other random shocks and Figure 2b reports the result of activating all shocks at their regular levels but otherwise the models are the same. In Figures 2b random factors increase the depression's duration by 1 quarters. The joint effect of all shocks can be significant due to their Markov persistence. They can accidentally worsen the effect of the primary deflationary shock or they can improve it. All results presented here reflect a particular realization. The realization in Figure 2b actually reduces the effect of policy and increases the depression's duration under the policy by 2 quarters

Figure 2b
The Artificial Depression with All Shocks Active

Optimal or Constrained Household Allocations? Compared to direct shocks to the natural rate, Eggertson and Krugman (2013) assume that disregarding debt repayments required under debt contracts, *all* borrowers face an unanticipated shock by lenders who increase required debt payments. They are liquidity constrained who cannot borrow to make these added payments and a negative natural rate emerges from the corner conditions of their optimization. Although an elegant model device, there is no evidence borrowers were forced in 2007-2015 to increase debt payments above levels agreed. Also, liquidity constraints have a long history and, as in other cases, there is little evidence to support the view a large fraction of households could not borrow *even small amounts at any interest rate*. But then, the driving force of the Great Recession was massive unanticipated loss of asset values exactly when rising home values was a source of significant consumption demand. Home ownership is the most widespread asset held in the US and with a significant marginal propensity to consume out of $1 of wealth by low and middle income households, estimated by

Mian and Sufi (2014) to be around 12-13 cents, a major fall in demand was inevitable. The choice of which model device to use in order to generate a slide into the ZLB is then a matter of analytic convenience.

2.2.3 Redistribution Effects of a Wage Stabilization Policy

Since a stabilizing wage policy entails redistribution of profits to wages, one's thought focuses on changed aggregate demand that result from differences in propensities to consume. This is not the case in the integrated model since a stabilizing wage policy *has no redistribution effects*. To recall why, return to *Insurance Assumption 1* to observe that households receive equal share of profits or losses. Therefore, a wage increase that lowers profits is negated in the budget constraint because of its ownership of the firm. Inflation does redistribute wealth from lenders to borrowers but since all utility functions are the same, such redistribution results only from differences in beliefs (see (26e)). But these have a long run *symmetric* distribution since no one is *permanently* optimistic hence there is no *inflation induced* long run income or wealth redistribution. The main result, seen in the equilibrium map of Table 1B is that *without redistribution effects, a stabilizing wage policy under the ZLB has strong effects of combating deflation and increasing output*. I then simulate the model below to compute the dynamic paths of output, inflation and deleveraging and the resulting ability of the economy to attain full employment *without any redistribution effects*.

Since in reality a wage policy has redistribution effects, these can be estimated outside the model and added to the model's output and inflation effects. Let me illustrate this process. To estimate the policy's partial equilibrium income redistribution effects I use 2007 US GDP level of $14 Trillion with 61% share of wages and salaries. Now, equilibrium wage is a linear function

$$\hat{w}_t = A_w^0 + A_w^Z Z_t + A_w^{\zeta}\hat{\zeta}_t + A_w^{\zeta_{-1}}\hat{\zeta}_{t-1} + A_w^{\zeta_{-2}}\hat{\zeta}_{t-2} + A_w^{\zeta_{-3}}\hat{\zeta}_{t-3} + A_w^{\zeta_{-4}}\hat{\zeta}_{t-4} + A_w^{\psi}\psi_t .$$

A wage policy adds intensity $\hat{\vartheta}_t$ to $(\zeta_{t-1}, \zeta_{t-2}, \zeta_{t-3}, \zeta_{t-4})$ hence the change in the real wage is

(32) $$\Delta\hat{w}_t = [A_w^{\zeta_{-1}} + A_w^{\zeta_{-2}} + A_w^{\zeta_{-3}} + A_w^{\zeta_{-4}}]\hat{\vartheta}_t .$$

In the standard case the elasticity is $A_w^{\zeta_{-1}} + A_w^{\zeta_{-2}} + A_w^{\zeta_{-3}} + A_w^{\zeta_{-4}} = 0.55$ and with $\hat{\vartheta}_t = 0.10$, a 10% increased wage scales imply a 5.5% increase in mean equilibrium wage. To compute the redistribution effects use the 61% labor share to conclude that it will raise 2007 wage income by $470 Billion. This income of workers can be used for consumption or for savings and deleveraging. Hence, redistribution entails a conflict: higher consumption increases demand and output but higher savings speeds up time to recovery. The short policy duration suggests I assume again a 20% savings rate out of this income hence deleveraging increases by $94 Billion which is 0.67% of 2007 GDP and demand increases by $376 Billion. What about a lower demand of higher income households? Estimates during the Great Recession[5] reveal a rise in saving rates of wealthier households who receive most capital income. Some show a 51% savings rate by the top 1% and 37% by the top 5%, others show even higher rates. Most estimates of the savings rate of high income household are smaller but an average rate of 35% would not be far off the mark. Hence the demand of higher income households, due to lower capital income by $470 Billion, would fall by $306 Billion with a net increase in demand of $70 Billion. In sum, redistribution has two, rather modest, effects which are approximately

- increased direct deleveraging by $94 Billion or 0.67% of 2007 GDP
- increase demand by $70 Billion or 0.5% of 2007 GDP.

If one uses a multiplier of 1.5, output increases by 0.75%. These modest amounts can then be added to all simulation results reported later. If the savings rate of workers out of the added income is 10% the results are much stronger. With a multiplier of 1.5 direct deleveraging rises by $47 Billion (0.34% of GDP), direct demand is higher by $117 Billions (0.84% of GDP) and output rises by $176 Billion (1.26% of GDP).

2.3 Implementing a Stabilizing Wage Policy

The term "standard model" refers to the parameter specification in Section 1 with the standard parameters $\Delta\psi_{t^0} = 0.06, \Delta\psi_t = 0.026 \text{ for } t > t^0, d_\ell = 0.0175, \vartheta_{t^0} = 0.10, \vartheta_t = 0 \text{ for } t > t^0$. All simulations are based on the equilibrium map in Table 1B .

2.3.1 Active Stabilizing Wage Policy

Figure 3 reports the effect of a stabilizing wage policy in the standard model. The policy is called "active" since it raises all wage scales by 10% and keeps it with cost of living adjustments until termination. The bold discs in the figure appear at dates when the wage policy is applied and broken lines trace the paths of variables under the policy. As before, heavy bold lines trace time paths under the ZLB and without a policy. Note the four bold discs after the 9 quarters of active policy. These are the four transitional quarters after policy termination when each of the four new wage offers eliminate the residual effect of policy and the reported mean wage adjusts in four steps. Before evaluating the policy I note that since the depression is artificial, policy evaluation is done mostly *by comparing the effect of the policy relative to the reference economy in Figure 1a without policy and the simulations use the same seed.*

[5] See for example *http://www.businessinsider.com/chart-savings-rate-by-income-level-2013-3*

Figure 3
Active Stabilizing Wage Policy with Intensity 0.10

All reference paths without policy in Figure 3 are the same as in Figure 1a since it is the same standard model specifications. Now, disregarding the effect of the initial shock, the significant differences between the equilibrium paths with and without policy are as follows:

- duration of the depression under the policy is reduced to 9 quarters rather than 44;
- wage-inflation spiral results in a 6% annualized inflation rate instead of an 8% deflation rate;
- rising output gap that reaches -2% at policy termination instead of -5% output below potential;
- falling unemployment that reaches 6% at policy termination instead of a 14% unemployment rate in the reference economy for 45 quarters. Full employment is reached after one more year;
- profits decline by about -7% for 9 quarters instead of -1.3% for 45 quarters which are about equal.

The policy causes a recovery in 9 quarters for two reasons. First, it replaces the deflation's drag with a short duration wage-inflation spiral that reverses the negative deflation wealth effect and improves deleveraging. Second, unemployment falls and output rises, further assisting the deleveraging process. Higher real wages under the policy surely reduce profits for the policy's duration. However, profits fall sharply only for the 9 quarter policy's duration and recover during the policy termination phase. The subsequent sharp rise of the nominal rate to about +7% controls inflation and lowers output and wages back to the range of steady state values, allowing profits to return to steady state levels. Higher wage rates present firms with a trade-off between a 9 quarters of 7% profit decline instead of a 45 quarters of 1.3% lower profits. In the standard case of Figure 3 it appears that any *net* loss to firm owners is relatively small, if any.

The effect of the assumed 20% deleveraging out of $(0.61)[Y_t^P - Y_t]$ is significant but simple. If the rate is lower at 10%, the policy's duration increases to 10 quarters and if the rate is even lower at 0%, the duration rises to 11 quarters. For each 10 percentage points of lower deleveraging rate out of the added wage income of the newly employed, the depression is longer by 3 months.

As suggested earlier, I now examine the effect of the parameter $\mu = 1$, measuring wage flexibility with respect to unemployment. If $\mu = 2$ then $\lambda_u = 0.30$ instead of $\lambda_u = 0.15$ I assumed here. The results of Figure 3 change as follows: without policy the depression settles down to a -10% deflation, -5.5% output gap and about 12% unemployment rate but spontaneous recovery is impossible. With policy the effect of the change is minor: duration rises to 10 quarters, inflation is at 5% and unemployment of about 8.5% at termination. A one quarter longer duration and higher unemployment can be avoided by increased policy intensity $\tilde{\varpi}_{t^0} = 0.12$ that completely negates the effect of higher wage flexibility.

Focusing on the 5%-6% inflation rates caused by an active policy, some may object to 9 quarters of such rates with common political resistence to “currency debasement.” Political factors are essential for policy design and to lower inflation a policy-makers can reduce policy intensity. Figure 4 reports results for the policy $\tilde{\varpi}_{t^0} = 0.06$, $\tilde{\varpi}_t = 0$ for $t > t^0$. Lower policy intensity implies an equilibrium wage increase of only 3.3% hence endogenous variables change slower than in Figure 3. The important policy effect is to break the -8% deflationary spiral and attain price stability with about 0% inflation but with the *negative outcome that the depression’s duration rises from 9 quarters to 11.* Under this policy output gap is about -4% and unemployment fluctuates around 12%. Only when deleveraging is completed the deflation pressure abates and exit from the ZLB is possible. Loss of profits is smaller at -4.5% for 11 quarters instead of -7% for 9 quarters but relative to the more active $\tilde{\varpi}_{t^0}=0.10$ it is modest: profits are *lower* by -2.5% for 9 quarters and *higher* by 4.5% for 2 quarters. In short, a low intensity policy may be politically reasonable: it avoids the inflation outcomes in Figure 3 but the price society pays for that is very high, summarized as follows:

- output loss of about 2% of potential for 9 quarters and 4% of potential for 2 quarters;
- unemployment is higher by about 2% for 9 quarters and by 10% for 2 quarters.

Figure 4
Stabilizing Wage Policy in the Standard Model with Lower Intensity of 0.06

Preference for policy with low intensity calls for reflection on circumstances when such a policy would, in fact, be preferred. In search of effective ways to solve the problem of unemployment it is important to recognize *a depression is the outcome of massive human errors and a recovery imposes social cost.* The only questions are what cost we pay and who bears them. Political resistence to debt restructuring leaves inflation the only feasible tool for recovery to full employment. Hence, objections to a policy that utilizes temporary inflation is opposition to transfer of wealth from lenders to borrowers *when timely economic recovery is impossible without such transfers*. These can be attained by a slow grind of personal default, debt negotiations, foreclosure etc. while taking the risk of the collapse of our democratic market institutions if austerity is pushed to the limit. Or else, it can be attained much faster with much higher output, lower unemployment and higher profits once inflation is accepted as a policy tool. All political considerations reflect mostly conflicting economic interests.

2.3.2 Active vs. Gradualist Policies

Policies in Figures 3-4 set the increased wage scales at t^0 and maintain it throughout. Alternatively, one may raise wage scales slowly at a constant rate up to the termination date. Since cost of living adjustments are applied, these raises have a slow cumulative effect on endogenous variables. Figure 5 reports results of a gradualist policy $\tilde{\omega}_{t^0} = 0.01$ and $\tilde{\omega}_t = 0.01$ for $t > t^0$ that raises all wage scales by 1% each quarter until all policy goals are attained. For comparability with Figure 3 all model parameters, except for policy parameters, are the same. Such comparisons cannot be perfect since the policy duration is endogenous hence wage scale increases of 10% for 9 quarters cannot be exactly the same as a cumulative wage scale increases of 1% per quarter for a random duration.

Figure 5
Gradualist Stabilizing Wage Policy with Intensity 0.01

The gradualist policy's duration turns out to be 11 quarters hence it ends up raising the wage scales by a cumulative +11% compared to a one time increase of +10% in the active policy of Figure 3. As a result, the cumulative effects on endogenous variables are different. The key differences are:

- Active policy's gap is -2% in 6 quarters but a gradualist policy takes 9 quarters to reach -2%;
- Active policy settles at inflation rates of 5%-6% while a gradualist policy ends up at 8%;
- Active policy reduces unemployment rate to 8% in 7 quarters while a gradualist policy takes full 11 quarters to reach it;

Evaluation of different policies requires an objective but if the aim is a rapid recovery with modest values of the aggregates, then an active policy is a better policy. Political constraints may prevent an active policy since it requires a strong initial change and adjustment that lasts 2.25 years. In contrast, a gradualist policy requires a slow change which may be politically easier but typically results in recessions of longer durations and more extreme values taken by the aggregates.

2.3.3 Why is the Policy Effective Even without Redistribution Effects

Since the model does not permit redistribution, why does the policy accelerate recovery? The answer is that a rise in the real wage with cost of living adjustment raises inflation but since under the ZLB higher inflation lowers the real rate by exactly the same amount, increasing demand and output. The increased demand is enhanced by the size of nominal assets holdings of the private sector. Hence, the recovery and exit from the ZLB are built on *declaring the goals of fiscal policy to be an acceleration of inflation and the* *resulting higher output under a zero nominal rate.* A stabilizing wage policy also lowers the unemployment rate, a fact related to other Keynesian paradoxes under the ZLB such as Eggertsson's (2010) Paradox of Toil and Eggertsson and Krugman's (2012) Paradox of Flexibility that are often stated in terms of the slope of a static aggregate supply. The issue here is *the effect of higher real wage on employment and unemployment.* One may state it as a labor demand function that rises with real wage and this would be correct in the very narrow sense that in Tables 1b and 5 the reduced form equilibrium elasticity of output with respect to wage is $\eta_{ws}^{y} > 0$. But this is an insufficient insight since in this economy there is no static labor demand function. Firms commit to meet demand at current prices therefore they hire all required labor *regardless of the wage*. Actual employment is derived from dynamic optimal pricing that determines the *speed of the firm's price adjustment* given the current state. Therefore, a useful interpretation views employment as a function of the *speed of price adjustment*. The policy's positive unemployment effect results from the relative speeds of labor supply response and price adjustments to changed wage. Let me explain.

A New Keynesian theory needs to be understood as a theory about speeds at which variables adjust and this differentiates it from a theory of auction markets. By assumption, supplied labor adjusts to changed wage *at an infinite speed* hence actual employment and unemployment depend upon the speed of firms' response. If price adjust very slowly (large ω), higher wage results in higher demand but decreased profits reduce demand and nullify the effect of higher wage. The result is a small change of output and inflation, lower profits and *higher* unemployment. This is what the elasticities in Table 3 show. If prices adjust rapidly, higher real wage raises demand and inflation which, under a ZLB, further boosts demand, employment and profits. A *double effect of higher wage on demand and inflation* ensures profits do not fall as much and unemployment falls (see Table 3 for small ω). The outcome depends upon the actual speed of price adjustment and this highlights a basic property of a stabilization wage policy.

The policy goals and the incentives of firms are aligned. With inflexible wage the question returns to the effect of the speed of price adjustment. Table 3 shows elasticities with respect to wage are monotonic with ω. More specifically, at low ω the elasticity of unemployment with respect to wage is negative and large in absolute value (-2.29 at $\omega = 0.2$) but rises and *positive* at larger ω (0.35 at $\omega = 2/3$). Since a wage increase raises all firms' marginal cost, if the wage increase is significant, their survival hinges on their ability to raise prices without delay. When all firms respond by raising prices the deflation is halted and a wage-inflation spiral is initiated which is exactly the aim of policy hence the policy is more effective the more rapidly prices respond to wage increases. To explore this effect further I report in Figure 6 the result of the standard model of Figure 3 altered by ω = 0.3 but a changed degree of price flexibility alters the economy *and typically calls for a different policy, making comparisons difficult.* With this in mind I compare it with the economy under $\vartheta_{t^0} = 0.10$ in Figure 3. It shows that with higher price flexibility, the standard deleveraging rate is too low and an exit from the ZLB in finite time is infeasible. This conclusion, together with those in Table 5 confirm Eggertsson and Krugman's (2012) result that price flexibility magnify the negative effects of shocks and in many cases makes it impossible for the economy to recover spontaneously.

Figure 6
Stabilizing Wage Policy under Higher Price Flexibility (ω = 0.3)

Figure 6 shows under $\vartheta_{t^0} = 0.10$ deleveraging is completed in 7 quarters. If all wage scales rise by 10% then, with an elasticity of unemployment with respect to wage scales equal to -0.963 (see Table 3), equilibrium wage rises by only 4.8% and the unemployment rate falls. The increased real wage together with quarterly cost of living adjustments transforms a deflation rate of -15% into an inflation rate of 16%. This is surprising and rather extreme. Under ω = 0.4 the elasticity is -0.27 and a 10% higher wage scale increases wage by 5.4%. Figure 3 shows that it transforms a -10% deflation to the more plausible inflation rate of 6%.

2.3.4 Policy Robustness

Table 6 reports a final experiment that tests the policy robustness with respect to the size of the deflation shock and price flexibility. In this table the deleveraging rate is fixed at the high rate of 1.75% not achieved in the US during 2007-2015.

Table 6: Policy Robustness and Depression's Duration (quarters)

		$\Delta\psi_t \;,\; t^0 < t \le T$			
	$\vartheta_t \Downarrow$	0.026	0.0275	0.30	0.40
$\omega = 0.3$	No Policy	Never	Never	Never	Never
	$\vartheta_t = 0.06$	10	10	11	12
	$\vartheta_t = 0.10$	7	7	7	7
	$\vartheta_t = 0.14$	5	5	5	5
$\omega = 0.4$	No Policy	44	48	62	Never
	$\vartheta_t = 0.06$	13	13	13	15
	$\vartheta_t = 0.10$	9	9	10	10
	$\vartheta_t = 0.14$	8	8	8	8
$\omega = 0.5$	No Policy	30	31	32	41
	$\vartheta_t = 0.06$	16	16	16	18
	$\vartheta_t = 0.10$	13	13	13	14
	$\vartheta_t = 0.14$	11	11	11	11

The table shows that without policy, depression's duration rises with the size of the deflation shock and with degree of price flexibility. This contradicts the common view that price flexibility helps recovery and explains that higher price flexibility imply a stronger effect of the deflation's shock and a stronger debt-deflation cycle that retards recovery. Arguments that suggest price flexibility advances recovery simply ignore the effects of deflation on debts.

Table 6 also reveals the surprising result that the *effect of the policy* is very robust with respect to the deflation shock. Once a deflationary cycle is broken, the road to recovery is open and is relatively insensitive to the size of shocks. Higher price flexibility renders the policy more effective, as was demonstrated before, but Table 6 explains that *for any degree of price inflexibility the policy remains effective:* the degree of policy success is fully determined by the selection of policy intensity $\hat{\vartheta}_t$ that needs to be adjusted to ω.

3. Concluding Remarks with a Reflection on Policy During 2007-2015

To recover rapidly from deflationary shocks that cause transition to the ZLB, public policy must aim to generate an inflationary counter-force. Monetary policy cannot achieve it regardless of promises a central bank makes for future inflation since the market recognizes the political constraints of the bank. Yet, the US did not fall into a deflationary spiral during the eight years of the Great Recession and this appears in conflict with the theory presented here which shows that a primary force to be contained is a deflationary pressure. The US experience is explained by the fact that although the amount of US deleveraging has been relatively small and $\mathbf{d}_\ell = 1.75\%$ not attained, actual policy was much more active than appears on the surface. First, by experimenting with unconventional policies the Central Bank demonstrated that expansion of money base cannot produce inflation of commodity prices but, combined with the bank's success in reducing the public's perceived riskiness of financial assets, it can generate a massive rise in asset prices. The Federal Reserve reports that Between 2008 and 2014(2) the value of US assets owned by the private sector rose from $70.8 Trillions to $95.4 Trillion as liabilities remained constant in the $13.5-14.0 Trillion range, generating a small wealth effect, mostly by higher wealth households. Second, in contrast with budget battles and Congressional demands for austerity, cumulative federal deficit during 2007-2014 added up to $7.81 Trillion. Although much was negated by austerity at the state and local level, the net effect of the two factors was to generate a component of demand that arrested some of the deflationary pressures. A very slow rebuilding of household balanced sheets also benefitted from the rise, during 2008 - 2014(2), of other asset categories such as Pension Entitlements that rose by $6.12 Trillion and Mutual Fund Shares held that rose by $4.43 Trillion, small part of which benefitting middle and low income households. Some Fed officials have claimed QE works "in practice but not in theory." In fact it works both in theory and in practice to engineer rising stock and other asset prices but not commodity inflation with a small and slow effect on the balance sheets of middle and low income households and a small fall in the unemployment rate. A better US experience relative to the rest of the world cannot imply the policy "worked" when the Great Recession has lasted almost 8 years with cumulative GDP gap of over $6 Trillion in 2014 prices, most of which lost by the unemployed and by middle and low income households. This gap will have long term consequence. Apart from the pain and suffering of millions of people who lost homes jobs and hope, the damage is so deep that in some areas it will take a long time to heal such as the level and quality of education, the crumbling US infrastructure, the depreciated human capital of the unemployed and the rapidly rising income and wealth disparity. Much of this could have been avoided with a better stabilization policy that serves all. One lesson of the 2007-2015 Great Recession is that when cyclical factors raise public deficits, little political slack is left for initiating massive deficit financed public stabilization spending in order to restore full employment and the standard Keynesian tools are unavailable. This leaves little choice but more direct stabilizing public interventions.

Based on results of this paper I suggest that a stabilizing wage policy with *an active* constant 10% increased real wage scales could have been initiated after the collapse of Lehman Brothers in September of 2008. If an inflation rate of 4%-6% for the policy's short duration could have been politically tolerated, the US would have recovered from the Great Recession with an exit from the ZLB lower sub-economy in less than 9 quarters and full employment restored by 2011-2012. Under political pressure a policy intensity of 6% constant higher wage scales would have achieved recovery in 11 quarters with policy-induced inflation of less than 2%. The policy would have temporarily reduced business profits but subsequent profits would have been higher than without the policy. The policy stabilizing effects, as measured in this paper, are understated since two additional factors need to be taken into account. First, the direct redistribution effects of the policy must be added as they will boost demand and enhance recovery beyond levels estimated here. Second, the policy should be coordination with an active central bank QE policy to counter the short term business losses and raise asset prices so as to increase demand further above levels estimated here.

The proposed policy would not have required federal budget expenditures, it would have reduced public deficits after 2010 and the US would have reached 2015 with a much lower national debt. The policy attains it's goal by a strong temporary intervention in the labor market instead of generating demand with public expenditures. However, one must recognize that the policy is effective only under the ZLB; it is designed to avoid the massive human cost of a depression by promoting rapid rapid recovery. It does not solve long term structural problems that persist after exit from the ZLB. These require their own solution.

References

Abraham, K. and Haltiwanger, J.C. (1995): "Real Wages and the Business Cycle." *Journal of Economic Literature* **33**, 1215-1264.

Akerlof, G.A. and Yellen, J.L. (1985): "A Near-Rational Model of the Business Cycle, with Wage and Price Inertia," *Quarterly Journal of Economics,* **100**, 823-838

Autor, D. H. and Dorn, D. (2013): "The Growth of Low Skill Service Jobs and the Polarization of the US Labor Market." *American Economic Review,* **103(5)**: 1553–1597.

Autor, D. H. (2014): "Polanyi's Paradox and the Shape of Employment Growth." A paper presented to the Federal Reserve meeting in Jackson Hole, Wyoming. MIT, August 11, 2014.

Ashenfelter, O. and Card, D. (1982): "Time Series Representation of Economic Variables and Alternative Models of the Labor Market." *Review of Economic Studies* **49**, 761-781.

Basu, S. (1996): "Procyclical Productivity: Increasing Returns or Cyclical Utilization?" *The Quarterly Journal of Economics*, **111** , 719 -751.

Benigno, P., and Ricci, L., A. (2008): "The Inflation-Unemployment Trade-Off at Low Inflation." *NBER Working Paper 13986.*

Benigno, P., and Ricci, L., A. (2011): "The Inflation-Output Trade-Off with Downward Wage Rigidities," *American Economic Review* **101**, 1436-1466.

Benhabib, J., Schmitt-Grohé S., and Uribe, M. (2001) : "The Perils of Taylor Rules." *Journal of Economic Theory* **96**, 40-69.

Benhabib, J., Schmitt-Grohé S., and Uribe, M. (2002) : "Avoiding Liquidity Traps." *Journal of Political Economy* **110**, 535-563.

Bewley, T. (1999): *Why Wages Don't Fall During a Recession.* Cambridge, MA: Harvard University Press.

Bils, M. (1985): "Real Wages Over the Business Cycle: Evidence from Panel Data." *Journal of Political Economy* **93**, 666–89.

Bils, M., and Peter J. Klenow, P. J. (2004): Some evidence on the importance of sticky prices. *Journal of Political Economy* **112**, 947–985.

Blanchard, O. J., Fischer, S. (1989): *Lectures on Macroeconomics*. Cambridge, MA: MIT Press.

Blanchard, O. J., Kahn, C.M. (1980): "The Solution of Linear Difference Models under Rational Expectations." *Econometrica* **48**, 1305 - 1311.

Blanchard, O. J., Galí, J. (2010): "Labor Markets and Monetary Policy: A New Keynesian Model with Unemployment." *American Economic Journal: Macroeconomics* **2**, 1–30

Calvo, G. A., (1983): "Staggered Prices in a Utility Maximizing Framework." *Journal of Monetary Economics* **12**, 983 -998.

Card, D. and Hylsop, D. (1997): " Does Inflation "Grease" the Wheels of the Labor Market?" In Romer, C. and Romer, D. (eds) *Reducing Inflation*, 71-114. Chicago: University of Chicago Press.

Christiano, L. j., Eichenbaum, M. and Evans, C. L., (2005): "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy." *Journal of Political Economy* **113**, 1-45.

Christiano, L. j., Eichenbaum, M. and Rebelo, S. (2011): "When Is the Government Spending Multiplier Large?" *Journal of Political Economy* **119**, 78-121.

Cochrane, J. H. (2014) : " The New- Keynesian Liquidity Trap." Manuscript, University of Chicago.

De Menil, G. and Enzler G., J. (1970): "Prices and Wages in the FR-MIT-PENN Econometric Model." In *The Econometrics of Price Determination*, Board of Governors of the Federal Reserve, Washington. D.C. 277-308.

Del Negro, M., Giannoni, M. P., and Patterson, C. (2012): "The forward guidance puzzle." Staff Reports 574, Federal Reserve Bank of New York.

Del Negro, M., Giannoni, M. P., and Schorfheide, F. (2013), "Infation in the Great Recession and New Keynesian Models." Federal Reserve Bank of New York Staff Report #618.

Cecchetti, S. G. (1984): "Indexation and Income Policy: A Study of Wage Adjustment in Unionized Manufacturing." *Journal of Labor Economics* **5**, 391-412.

Correia, I., E. Farhi, J. P. Nicolini, and P. Teles (2013): “Unconventional Fiscal Policy at the Zero Bound,” *American Economic Review* **103**, 1172 -1211

Eggertsson, G. (2006): “The Deflation Bias and Committing to Being Irresponsible.” *Journal of Money, Credit, and Banking* **36**, no. 2:283–322.

Eggertsson, G. (2008): “Great Expectations and the End of the Depression.” *American Economic Review* **98**, no. 4:1476–1516.

Eggertsson, G. (2010): “The Paradox of Toil.” Staff Report no. 433, Federal Reserve Bank of New York.

Eggertsson, G. B. (2011): “What Fiscal Policy Is Effective at Zero Interest Rates?,” In *NBER Macroeconomics Annual*, **25**, 59-112, Chicago: University of Chicago Press.

Eggertsson, G. B., and P. Krugman (2012): “Debt, Deleveraging, and the Liquidity Trap: A Fisher-Minsky-Koo Approach,” The Quarterly Journal of Economics **127**, 1469–1513.

Eggertsson, G. and M. Woodford (2003): “The Zero Bound on Interest Rates and Optimal Monetary Policy.” *Brookings Papers on Economic Activity* No. 1, 139–211.

Eggertsson, G. and M. Woodford (2004): “Optimal Monetary and Fiscal Policy in a Liquidity Trap.” *International Seminar on Microeconomics* **1**, 75–131.

Eichenbaum, M. (1991): “Real Business-Cycle Theory, Wisdom or Whimsy?” Journal of Economic Dynamics and Control, **15**, 607 - 626.

Eichenbaum, M. S., Jaimovich, N., and Rebelo, S. (2011): “Reference prices, costs, and nominal rigidities. American Economic Review **101**, 234-262.

Fernández-Villaverde, J., Guerrón-Quintana, P. and Rubio-Ramírez J. F (2014): “Supply-Side Policies and the Zero Lower Bound.” Manuscript.

Fischer, S. (1977): “Long Term Contracts, Rational Expectations, and the Optimal Money Supply Rule.” *Journal of Political Economy* **85**, 163- 190.

Galí, J. (2008): *Monetary Policy, Inflation and the Business Cycle: An Introduction to the New Keynesian Framework*. Princeton: Princeton University Press.

Gertler, M., and Trigari, A. (2009) “Unemployment Dynamics with Staggered Nash Wage Bargaining.” *Journal of Political Economy* **117**, 38-86.

Hall, R. (2005a): “Employment Fluctuations with Equilibrium Wage Stickiness.” *American Economic Review* **95**, 50–65.

Hall, R. (2005b): “Employment Efficiency and Sticky Wages: Evidence from Flows in the Labor Market.” *Review of Economics and Statistics* **87**, 397–407.

Hall, R. (2005c): “Job Loss, Job Finding, and Unemployment in the U.S. Economy over the Past Fifty Years.” In *NBER Macroeconomics Annual,* Gertler, M. and Rogoff, K. (ed.) 101–37. Cambridge, MA: MIT Press.

Hayman, S. (1970): “Prices and Price Behavior in Three U.S. Econometric Models.” In *The Econometrics of Price Determination*, Board of Governors of the Federal Reserve, Washington. D.C. pp. 5-16

Hirsch, A., A. (1970): “Price Simulations with the OBE Econometric Model.” In *The Econometrics of Price Determination*, Board of Governors of the Federal Reserve, Washington. D.C. pp. 237-276.

Hosken, D., Reiffen, D. (2004): Patterns of retail price variation. *Rand Journal of Economics* **35** (1, Spring): 128–46.

Koo, R. C. (2008): *The Holy Grail of Macroeconomics: Lessons from Japan’s Great Recession.* New York: Wiley.

Kehoe, P. J., Midrigan, V. (2010): “Prices Are Sticky After All.” Federal Reserve Bank of Minneapolis Staff Report 413, September 2010.

Kiley, M. T. (2013) : “Policy Paradoxes in the New Keynesian Model.” Manuscript.

King, R.G., Rebelo, S.T. (1999): “Resuscitating Real Business Cycles.” In J. Taylor and M. Woodford (eds.), *Handbook of Macroeconomics, Volume 1B.* Amsterdam: North-Holland, Chapter **14**, 927 -1007.

Kean, M., Moffitt, R. and Runkle, D. (1988): “ Real Wages over the Business Cycle: Estimating the Impact of Heterogeneity with Micro Data.” *Journal of Political Economy* **96**, 1232-1266.

Kurz, M. (1994): “On the Structure and Diversity of Rational Beliefs.” *Economic Theory* **4**, 877 - 900 . (An edited version appears as Chapter **2** of Kurz, M. (ed.) (1997)).

Kurz, M. (ed) (1997): *Endogenous Economic Fluctuations: Studies in the Theory of Rational Belief.* Studies in Economic Theory, No. **6**, Berlin and New York: Springer-Verlag.
Kurz, M. (2009): "Rational Diverse Beliefs and Economic Volatility." Chapter 8 in Hens, T. and Schenk-Hoppé, K.R. (ed.) *Handbook Of Financial Markets: Dynamics and Evolution*." Amsterdam: North-Holland, 440- 506.
Kurz, M., (2012): "A New Keynesian Model with Diverse Beliefs." Working Paper, Stanford University, February 27, 2012.
Kurz, M., (2015): "On Wage Inflexibility." Working Paper, Stanford University, May, 2015.
Kurz, M., Motolese, M. (2011): "Diverse Beliefs and Time Variability or Risk Premia." *Economic Theory* **47,** 293 - 335.
Kurz, M., Piccillo, G., Wu, H. (2013): "Modeling Diverse Expectations in an Aggregate New Keynesian Model." *Journal of Economic Dynamics and Control* **37** 1403 - 1433
Lebow, D.E., Stockton, D.J., Wascher, W. (1995): " Inflation, Nominal Wage Rigidity and the Efficiency of Labor Markets. Federal Reserve Board, Washington D.C.
Malcomson, J. M. (1999) "Individual Employment Contracts," in Ashenfelter, O. and Card, D. eds., *Handbook of labor economics,* Vol. 3B. New York: North-Holland, 2291–2372.
Mankiw, N.G. (1985) : "Menu Cost and Large Business Cycles: A Macroeconomic Model of Monopoly." *Quarterly Journal of Economics* **101**, 529 - 537.
McLaughlin, K.J. (1994) : "Rigid Wages?" *Journal of Monetary Economics* **34**, 383 - 414.
Mertens, K. and Ravn, M. O. (2014) : "Fiscal Policy in an Expectations Driven Liquidity Trap." *Review of Economic Studies*, Forthcoming.
Mian, A. and Sufi, A. (2014): *House of Debt*. Princeton: Princeton University Press
Nakamura, E., Steinsson, J. (2008): "Five facts about prices: A reevaluation of menu cost models." *Quarterly Journal of Economics* **123** , 1415–1464.
Pissarides, C. A. (1985) "Short-Run Equilibrium Dynamics of Unemployment, Vacancies and Real Wages." *American Economic Review,* **75**, 676–90.
Pissarides, C. A. (2000) *Equilibrium Unemployment Theory*. Cambridge, MA: MIT Press.
Pissarides, C. A. (2009): " The Unemployment Volatility Puzzle: Is Wage Stickiness the Answer?" *Econometrica* **77**, No. 5, 1339-1369.
Shimer, Robert. (2004) "The Consequences of Rigid Wages in Search Models." *Journal of the European Economic Association* **2**, 469–79.
Solon, G., R. Barsky, and J. A. Parker (1994): "Measuring the Cyclicality of Real Wages: How Important Is Composition Bias?" *Quarterly Journal of Economics* **109**, 1–25.
Summers, L. H. (1986): "Some Skeptical Observations on Real Business Cycle Theory." *Federal Reserve Bank of Minneapolis Quarterly Review* **10**, 23 - 27.
Taylor, J. B. (1979): "Staggered Price Setting in a Macro Model." *American Economic Review* **69**, 108-113.
Taylor, J. B. (1980): "Aggregate Dynamics and Staggered Contracts." *Journal of Political Economy* **88**, 1–23.
Taylor, J. B. (1983): "Union Wage Settlements During A Disinflation." *American Economic Review* **73**, 981–993.
Taylor, J. B. (1999) "Staggered Price and Wage Setting in Macroeconomics." In *Handbook of Macroeconomics,* Vol. 1, Part 2, edited by J.B. Taylor and M.Woodford, pp. 1009–1050. North-Holland: Elsevier.
Tobin, J. (1972): "The Wage-Price Mechanism: Overview of the Conference." In *The Econometrics of Price Determination*, Board of Governors of the Federal Reserve, Washington. D.C. pp. 5-16
Walsh, C.E. (2010): *Monetary Theory and Policy*, Third edition. Massachusetts: The MIT Press
Werning, I. (2012): "Managing a Liquidity Trap: Monetary and Fiscal Policy." Manuscript, MIT.
Wieland, J. (2014) : "Are Negative Supply Shocks Expansionary at the Zero Lower Bound" Manuscript
Woodford, M. (2003). *Interest and prices: Foundations of a theory of monetary policy*. Princeton: Princeton University Press.

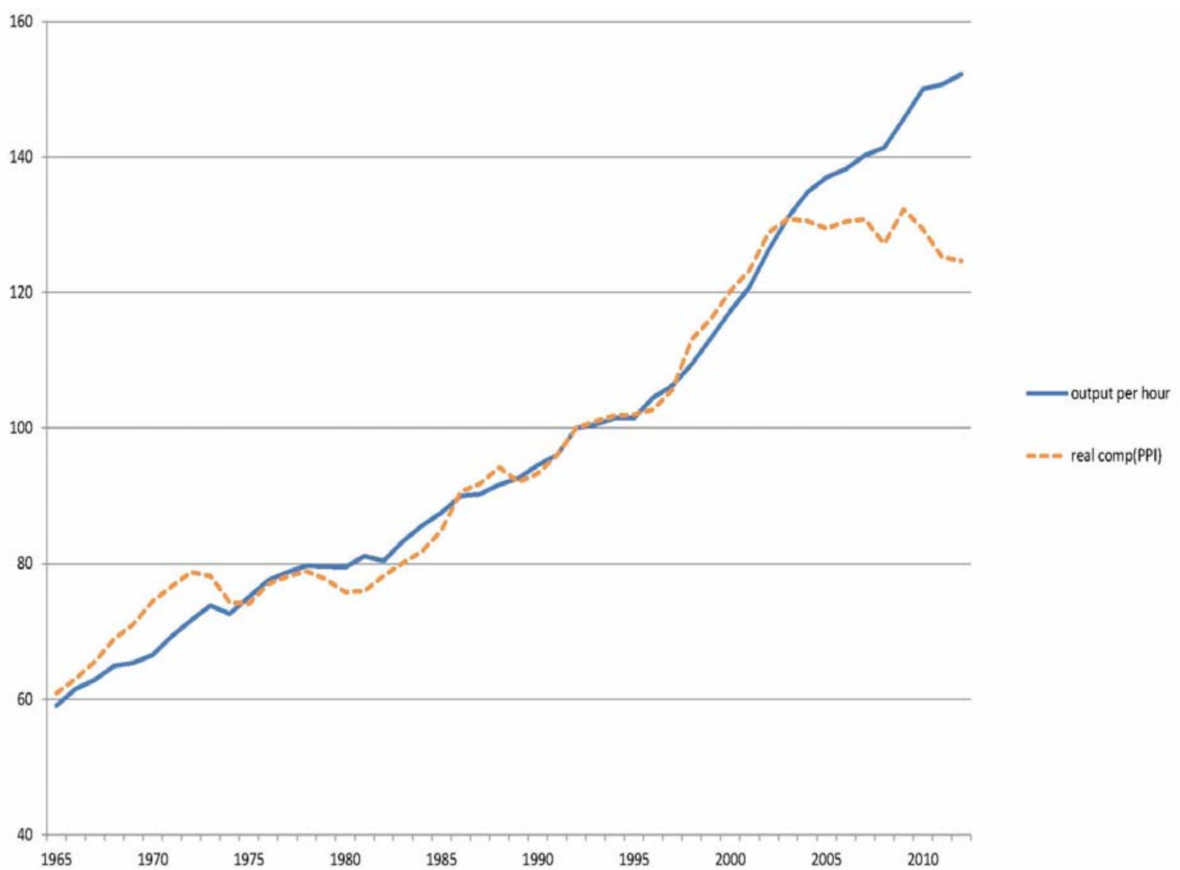

Figure 1: Output per man hour is for the US business sector, Table B-49 of the Economic Report of the President, 2014. Real Compensation (PPI) is total compensation per hour in the business sector, Table B-49 of the same report. Nominal total compensation is divided by the Producer Price Index of all finished goods, Table B-65 of the same report with 1992 = 100.

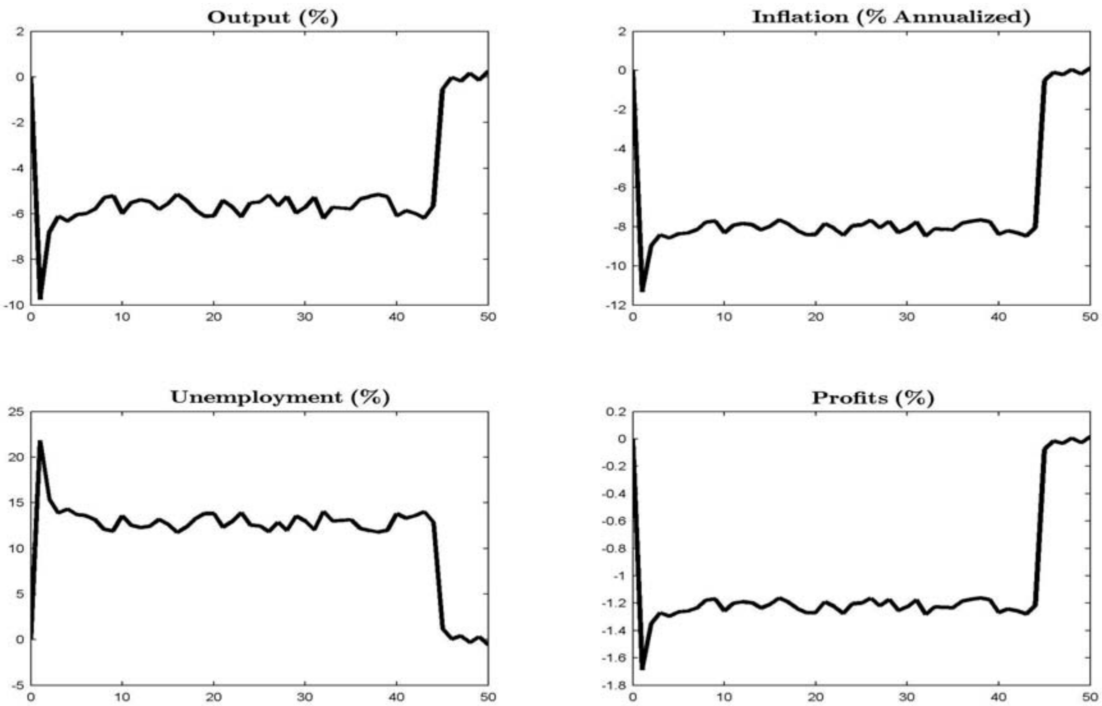

Figure 2a: Artificial depression with intensity (0.06, 0.026), deleveraging at the rate of 1.75% per quarter and only demand shock activated. Depression's duration is 44 quarters.

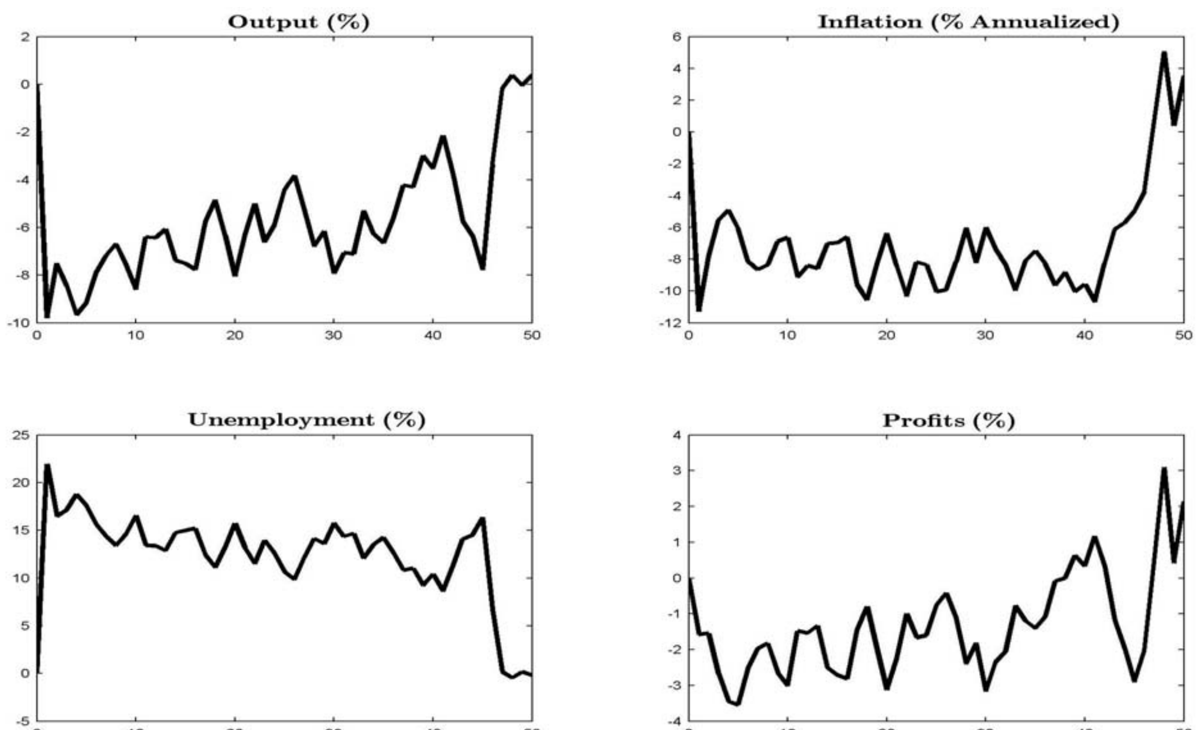

Figure 2b: Artificial depression with intensity (0.06, 0.026), deleveraging at the rate of 1.75% per quarter and all shocks activated. Depression's duration is 45 quarters.

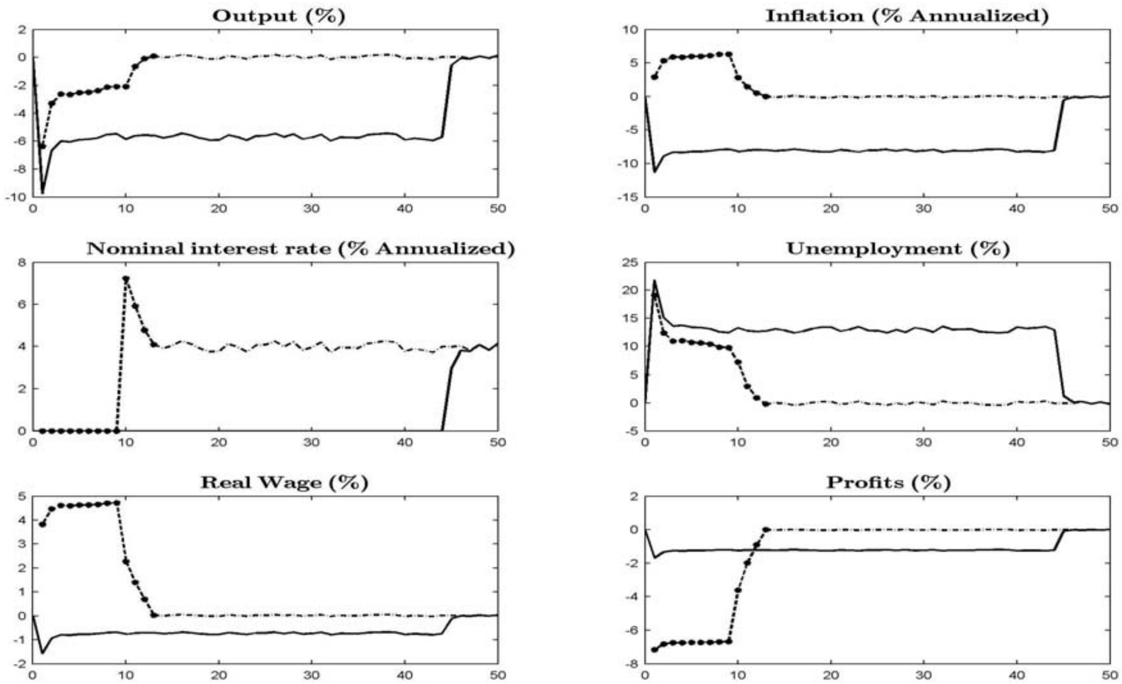

Figure 3: Comparison of standard economy under wage stabilization policy at intensity (0.10, 0.0) with the economy without policy. Depression's duration with policy is 9 quarters vs. 44 quarters without policy. Economy without policy in solid paths and active policy noted in discs except termination.

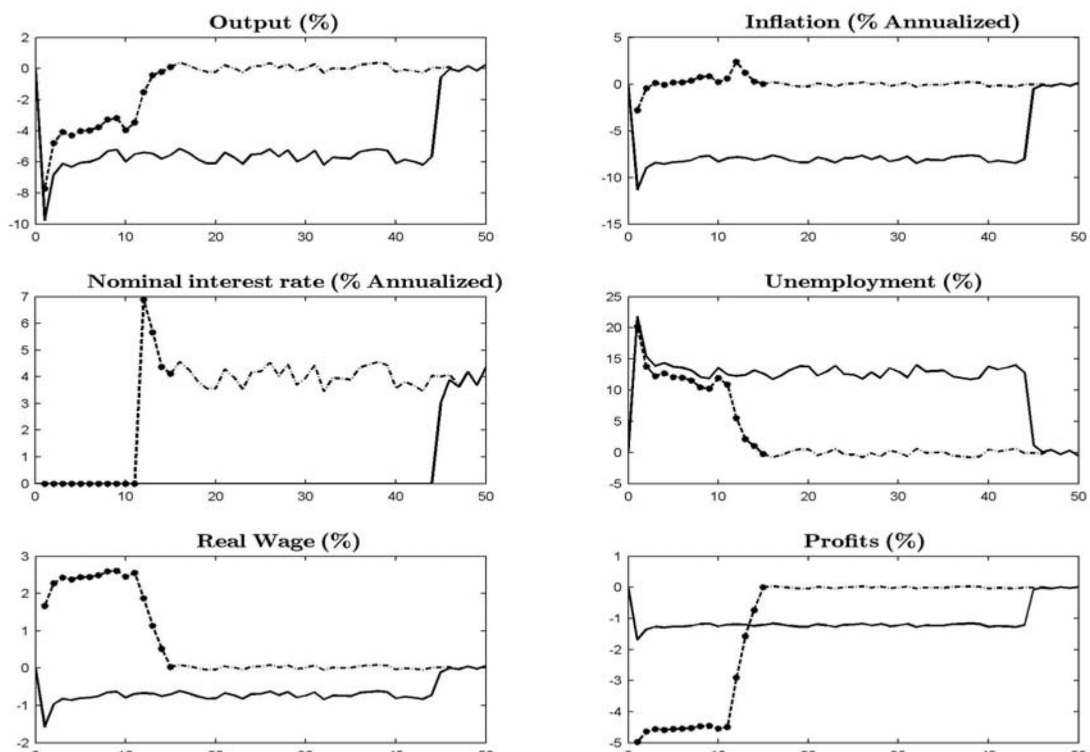

Figure 4: Comparison of standard economy under stabilizing wage policy at intensity (0.06, 0.0) with the economy without policy. Depression's duration with policy is 11 quarters vs. 44 quarters without policy. Economy without policy in solid paths.

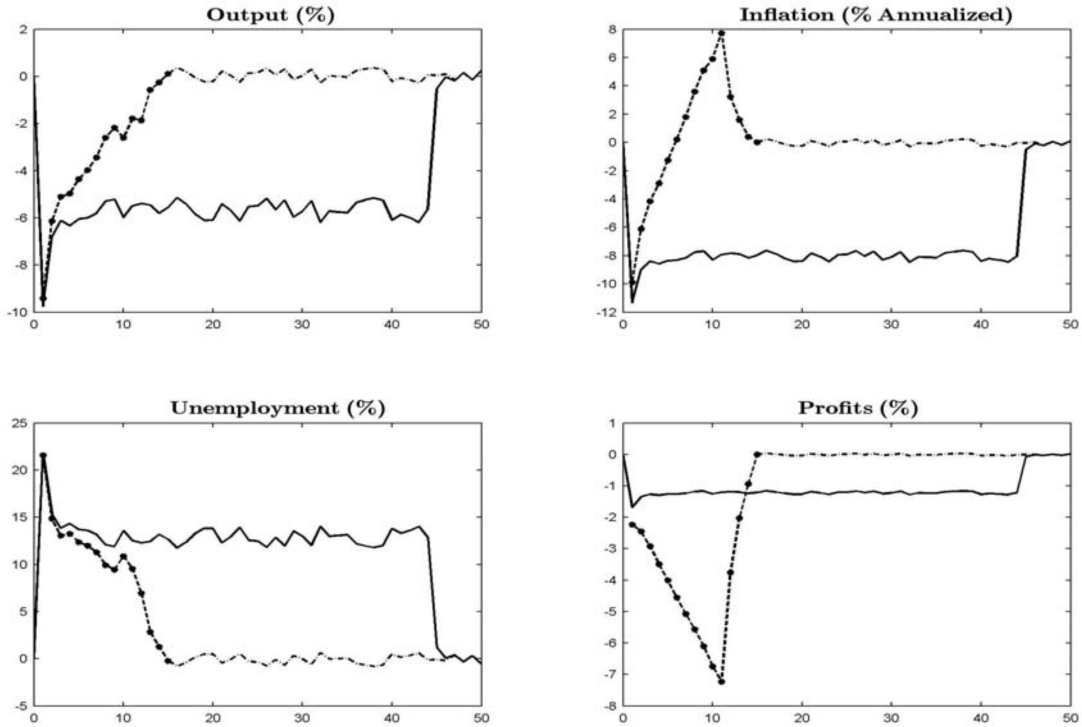

Figure 5: Comparison of four equilibrium paths in the standard economy under a gradualist stabilizing wage policy at intensity (0.01, 0.01) with those of same economy but without policy. Depression's duration with policy is 11 quarters vs. 45 quarters without policy. Economy without policy in solid paths.

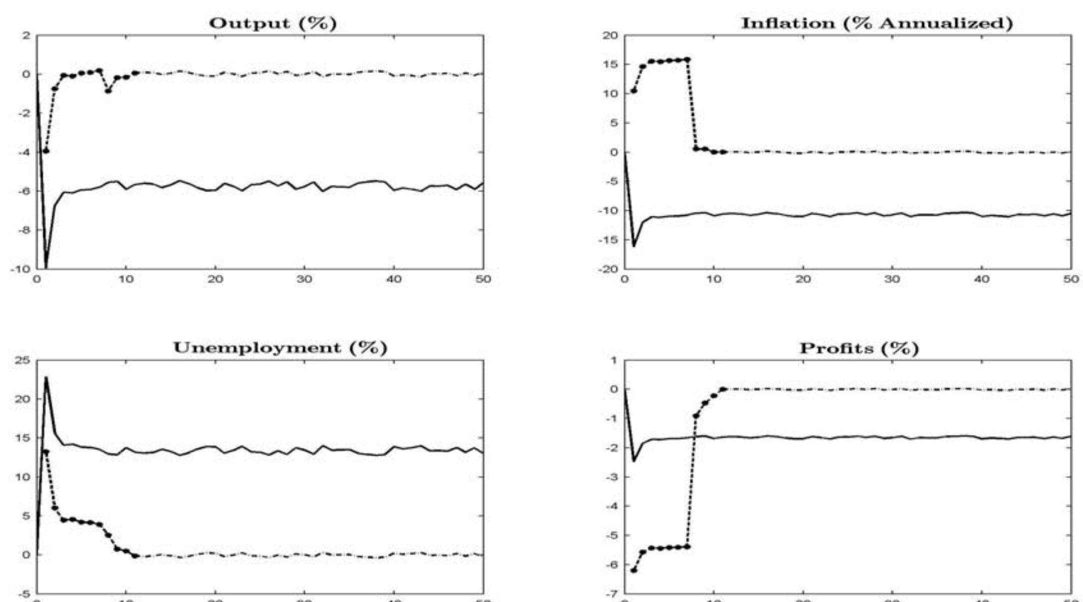

Figure 6: Comparison of four equilibrium paths in the standard economy under active stabilizing wage policy at intensity (0.10, 0.0) but ω = 0.30 with those of the same economy without policy. Depression's duration with policy is 7 quarters but without policy recovery is impossible in finite time. Economy without policy in solid paths.

Stabilizing Wage Policy: Mathematical Appendix A
The Structure of Individual Beliefs

by

Mordecai Kurz, May 27, 2015

For beliefs to be diverse there must be something agents do not know and on which they disagree. Here I stipulate it to be the distribution of the exogenous shocks (ζ_t, ψ_t). The true process of technology and demand shocks is not known. It is a non-stationary process, subject to structural changes and regime shifts due to diverse causes (see Kurz (2009)). Following the "Rational Belief" approach (see Kurz (1994), (1997)), agents have past data on these variables hence their empirical distribution is common knowledge. By "empirical distribution" I mean the distribution one deduces from a long data series by computing relative frequencies or moments and where such computations are made without judgment or estimating the effect of transitory short term events. The empirical distribution of a stochastic process leads to a deduction of a stationary probability on sequences which is then common knowledge. It plays a crucial role in the theory developed here. I denote this stationary probability with the letter m and refer to it as the "empirical probability." To simplify assume (ζ_t, ψ_t) have a distribution with an empirical transitions which are Markov of the form

(B.1a) $\zeta_{t+1} = \lambda_\zeta \zeta_t + \rho^\zeta_{t+1}$

(B.1b) $\psi_{t+1} = \lambda_\psi \psi_t + \rho^\psi_{t+1}$

$$\begin{pmatrix} \rho^\zeta_{t+1} \\ \rho^\psi_{t+1} \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \sigma^2_\zeta, & 0, \\ 0, & \sigma^2_\psi \end{bmatrix} \right), \quad \text{i.i.d.}$$

The truth is that both processes are subject to shifts in structure, taking the true form

(B.2a) $\zeta_{t+1} = \lambda_\zeta \zeta_t + \lambda^s_\zeta s_t + \tilde{\rho}^\zeta_{t+1}$

(B.2b) $\psi_{t+1} = \lambda_\psi \psi_t + \lambda^s_\psi s_t + \tilde{\rho}^\psi_{t+1}$

$$\begin{pmatrix} \tilde{\rho}^\zeta_{t+1} \\ \tilde{\rho}^\psi_{t+1} \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \tilde{\sigma}^2_\zeta, & 0, \\ 0, & \tilde{\sigma}^2_\psi \end{bmatrix} \right).$$

Regime parameters s_t are unobserved hence (B.1a)-(B.1b) are time averages of (B.2a)-(B.2b). To simplify it is assumed there is only one factor hence there will be one belief parameter that pins down an agent's belief about *all state variables*. More general models have multiple factors and belief variables.

B.1 Describing Belief with State Variables: Rationality and Belief Diversity Imply Dynamics

Agents may believe (B.1a)-(B.1b) are the true transitions, and some do, but typically they do not and form their own beliefs about these structural parameters. I introduce agent i's state variable *denoted by* g_t^i and used to describe i's belief. It is a perception variable which pins down his subjective transitions <u>of all state variables</u>. Agent i knows g_t^i but since forecast samples are taken, he also observes the *distribution* of g_t^j across j but not *specific* g_t^j of others. This entails a small measure of information asymmetry as each agent knows his g_t^i but only the distribution of the others. But this asymmetry does not matter since I also assume "anonymity" which means agent i is small and does not assume g_t^i impacts market belief. For an expression of anonymity suppose for a moment the economy has finite agents with a distribution $(g_t^1, g_t^2, \ldots, g_t^N)$ of individual beliefs. To impose anonymity use notation of $(v_t^1, v_t^2, \ldots, v_t^N)$ to describe the *market distribution of beliefs* which is observed and taken by agents as given. The condition $v_t^i = g_t^i$ is then an equilibrium condition. All observe past distributions $(v_\tau^1, v_\tau^2, \ldots, v_\tau^N)$ for $\tau < t$. I do not use this formalism here but only the assumption that the distribution of g_t^i and its mean Z_t are common knowledge.

The assumption that a belief is described by a single index is a mathematical simplification. In more complex models one introduces a vector of belief indices that permit more subtle distinction between such belief about different moments such as mean values and risk assessment.

How is g_t^i used by an agent? I use the notation $(\zeta_{t+1}^i, \psi_{t+1}^i)$ to express i's *perception* of t+1 shocks before they are observed. By convention $(E_t^i \zeta_{t+1}, E_t^i \psi_{t+1})$[1] *is the same as* $(E_t^i \zeta_{t+1}^i, E_t^i \psi_{t+1}^i)$ since individual expectation can be taken only with respect to perception. Agent i's date t *perceived* distribution of $(\zeta_{t+1}^i, \psi_{t+1}^i)$ is specified by .

$$\text{(B.3a)} \qquad \zeta_{t+1}^i = \lambda_\zeta \zeta_t + \lambda_\zeta^g g_t^i + \rho_{t+1}^{i\zeta}$$

$$\text{(B.3b)} \qquad \psi_{t+1}^i = \lambda_\psi \psi_t + \lambda_\psi^g g_t^i + \rho_{t+1}^{i\psi}$$

$$\begin{pmatrix} \rho_{t+1}^{i\zeta} \\ \rho_{t+1}^{i\psi} \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix} , \begin{bmatrix} \sigma_\zeta^2, & \sigma_{\zeta\psi}, \\ \sigma_{\psi\zeta}, & \sigma_\psi^2 \end{bmatrix} \right)$$

The assumption that $(\sigma_\zeta^2, \sigma_{\zeta\psi}^2)$ is the same for all agents is made for simplicity. It follows that given public information I_t at date t, individual perception g_t^i specifies the *difference* between date t forecast and the forecasts under the empirical probability m:

---

[1] The notation $(\zeta_{t+1}^i, \psi_{t+1}^i)$ is used to highlight *perception* of the macro variables $(\zeta_{t+1}, \psi_{t+1})$ by agent i before they are observed. In general, for an aggregate variable x_{t+1}, there is no difference between $E_t^i x_{t+1}^i$ and $E_t^i x_{t+1}$ since i's expectations can be taken only with respect to i's perception. However, it is important to keep in mind the context. If in a discussion the variable x_{t+1} is assumed to be observed at t+1, then it cannot be perceived at that date. Hence, the notation x_{t+1}^i expresses *perception* of x_{t+1} by agent i before the variable is observed and $E_t^i x_{t+1}$ expresses the expectations of x_{t+1} by i, *in accordance with his perception*. This procedure does not apply to i-specific variables such as $E_t^i \hat{c}_{t+1}^i$ which has a natural interpretation.

(B.4) $$E^{i}[(\zeta_{t+1},\psi_{t+1})|I_t,g_t^{i}] - E^{m}[(\zeta_{t+1},\psi_{t+1})|I_t] = (\lambda_{\zeta}^{g} g_t^{i}, \lambda_{\psi}^{g} g_t^{i}).$$

I adopt three rationality principles.

Rationality Principle 1: A belief cannot be a *constant transition* unless an agent believes the stationary transition (B.1a)-(B.1b) is the truth.

Rationality Principle 2: A belief does not deviate from (B.1a)-(B.1b) consistently and hence the belief index g_t^{i} must have an unconditional mean of zero.

Condition (B.4) shows how to measure g_t^{i} using forecast data since $E^{m}[(\zeta_{t+1},\psi_{t+1})|I_t]$ is a standard econometric forecast employing past data by making no judgment about any time interval. When $g_t^{i}=0$ for all t, agent i believes m is the truth.

The two rationality principles imply that if an economy has diverse beliefs and such diversity persists without opinions tending to merge, then a typical agent's belief g_t^{i} *must fluctuate over time.* This is the most important implication of rationality requirements: *rationality implies dynamics*. The reason is simple. Agents cannot hold constant, invariant, transitions unless they are (B.1a)-(B.1b). Since diversity persists, (B.1a)-(B.1b) are not the belief of most but since the time average of an agent's transitions must be (B.1a)-(B.1b), they must fluctuate. This relation between rationality and dynamics is central to the Rational Belief approach (e.g. Kurz (1994), (1997), (2009)). The next step is the treatment of belief dynamics as state variables. I postulate individual states of belief are random and unknown as they may depend upon assessments made and data observed in the future. The first two principles do not specify the dynamics of belief but the third addresses the issue.

Rationality Principle 3: The transition functions of g_t^{i} are Markov, taking a form which exhibit persistence and takes the form

(B.5) $$g_{t+1}^{i} = \lambda_Z g_t^{i} + \lambda_Z^{\zeta}[\zeta_{t+1} - \lambda_{\zeta}\zeta_t] + \lambda_Z^{\psi}[\psi_{t+1} - \lambda_{\psi}\psi_t] + \rho_{t+1}^{ig} \quad , \quad \rho_{t+1}^{ig} \sim N(0,\sigma_g^2)\ ,$$ ρ_{t+1}^{ig} are correlated across i[2].

Correlation of ρ_{t+1}^{ig} over i reflects correlated beliefs across agents and this correlation is central

[2] Condition (B.5) specifies the distribution of g_t^{i} hence it specifies values it will take at t+1 given the *observed* values of variables on the right hand side. For this reason one does not use perception notation here. However, in other contexts an agent takes the expectation $E_t^{i} g_{t+1}^{i}$ before g_{t+1}^{i} is known, at which point expectations of the perception v_{t+1}^{i} is taken on the right.

to the theory. Analogous law of motion applies if the shock is only ψ_t or both u_t and v_t.

Rationality Principle 3 says that agent belief state at date t+1 is unknown at t but has a known Markov transition. The uncertainty arises from two sources. The first is ρ^{ig}_{t+1} which is analogous to the concept of a "type" in games with incomplete information where an agent type is revealed only in the future. The second arises from $[\zeta_{t+1} - \lambda_\zeta \zeta_t]$ and $[\psi_{t+1} - \lambda_\psi \psi_t]$ which I call "learning feed-back" variables. These permit an agent to alter his own belief in view of large forecast errors. Such changes arise from the fact the agent knows much of the true non-stationarity cannot be assessed with large data and can only be deduced from small amount of recent data hence this short term learning has only a transitory effect. Note that in the text I treat ψ_t as a proxy to demand shocks and avoid learning feed-back from its realization hence $\lambda^\psi_Z = 0$. Also, I repeatedly use the term "forecasting belief" in the sense of taking expectations of objects like (B.5) or its aggregate and *uncertainty about future market belief state is central to this theory*. How can one justify (B.5) which plays such a key role in the theory? The first answer is that the data supports this specification (see Kurz and Motolese (2011)). Alternatively, in Kurz (2008) I prove (B.5) analytically as a result of Bayesian inference An updated proof is available in the Appendix of Kurz, Piccillo and Wu (2013). Such interpretation is not essential since Principle 3 can stand on its own.

B.2 Modeling Diverse Beliefs: Market Belief and the Central Role of Correlation

The fact that individual beliefs fluctuate implies market belief (i.e. the distribution of g_t^i) may also fluctuate and uncertainty about an agent future belief imply that future market belief is also uncertain. Indeed, market belief is a crucial macro economic uncertainty which needs to be explored.

Averaging (B.5), denote by Z_t the mean of the cross sectional distribution of g_t^i and refer to it as " average market belief." It is observable. Due to correlation across agents' ρ^{ig}_t, the law of large numbers does not apply and the average of ρ^{ig}_t over i does not vanish. I write it in the form

(B.6) $$Z_{t+1} = \lambda_Z Z_t + \lambda^\zeta_Z[\zeta_{t+1} - \lambda_\zeta \zeta_t] + \lambda^\psi_Z[\psi_{t+1} - \lambda_\psi \psi_t] + \tilde{\rho}^Z_{t+1}.$$

The distribution of $\tilde{\rho}^Z_{t+1}$ is unknown and may vary with time. The presence of this random term show the dynamics of Z_t depends upon *correlation* across agents' beliefs. I ρ^{ig}_t in (B.5) were independent across i, the law of large numbers would have implied $\tilde{\rho}^Z_t = 0$ hence correlation ensures market belief does not degenerate into a deterministic relation $Z_{t+1} = \lambda_Z Z_t + \lambda^\zeta_Z[\zeta_{t+1} - \lambda_\zeta \zeta_t] + \lambda^\psi_Z[\psi_{t+1} - \lambda_\psi \psi_t]$. Since correlation is not determined by individual rationality it becomes *a belief externality*. In sum, random individual belief translates into macro uncertainty about future market belief. This uncertainty plays a central role in the theory and correlation externality is the basis for such uncertainty.

Since Z_t are observable, market participants have data on $\{(\zeta_t, \psi_t, Z_t), t = 1, 2, \ldots\}$ and know the *joint empirical distribution* of these variables. I assume this is a Markov empirical probability with a transition function described by the system of equations of the form

(B.7a) $\zeta_{t+1} = \lambda_\zeta \zeta_t + \rho^\zeta_{t+1}$

(B.7b) $\psi_{t+1} = \lambda_\psi \psi_t + \rho^\psi_{t+1}$

(B.7c) $Z_{t+1} = \lambda_Z Z_t + \lambda^\zeta_Z[\zeta_{t+1} - \lambda_\zeta \zeta_t] + \lambda^\psi_Z[\psi_{t+1} - \lambda_\psi \psi_t] + \rho^Z_{t+1}$

$$\begin{pmatrix} \rho^\zeta_{t+1} \\ \rho^\psi_{t+1} \\ \rho^Z_{t+1} \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0, \\ 0 \end{matrix} \begin{bmatrix} \sigma^2_\zeta, & 0, & 0 \\ 0, & \sigma^2_\psi, & 0 \\ 0, & 0, & \sigma^2_Z \end{bmatrix} \right), \text{ i.i.d.}$$

This is a combination of technology and demand shocks.

Agents who do not believe (B.7a)-(B.7c) are the truth formulate subjective beliefs. A belief is described with state variables $(\zeta^i_{t+1}, \psi^i_{t+1}, Z^i_{t+1}, g^i_{t+1})$[3] and takes the form of a subjective perception model

(B.8a) $\zeta^i_{t+1} = \lambda_\zeta \zeta_t + \lambda^g_\zeta g^i_t + \rho^{i\zeta}_{t+1}$

(B.8b) $\psi^i_{t+1} = \lambda_\psi \psi_t + \lambda^g_\psi g^i_t + \rho^{i\psi}_{t+1}$

(B.8c) $Z^i_{t+1} = \lambda_Z Z_t + \lambda^\zeta_Z[\zeta_{t+1} - \lambda_\zeta \zeta_t] + \lambda^\psi_Z[\psi_{t+1} - \lambda_\psi \psi_t] + \lambda^g_Z g^i_t + \rho^{iZ}_{t+1}$

(B.8d) $g^i_{t+1} = \lambda_Z g^i_t + \lambda^\zeta_Z[\zeta_{t+1} - \lambda_\zeta \zeta_t] + \lambda^\psi_Z[\psi_{t+1} - \lambda_\psi \psi_t] + \rho^{ig}_{t+1}$

$$\begin{pmatrix} \rho^{i\zeta}_{t+1} \\ \rho^{i\psi}_{t+1} \\ \rho^{iZ}_{t+1} \\ \rho^{ig}_{t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \hat\sigma^2_\zeta, & \hat\sigma_{\zeta\psi}, & \hat\sigma_{\zeta Z}, & \hat\sigma_{\zeta g} \\ \hat\sigma_{\psi\zeta}, & \hat\sigma^2_\psi, & \hat\sigma_{\psi Z}, & \hat\sigma_{\psi g} \\ \hat\sigma_{Z\zeta}, & \hat\sigma_{Z\psi}, & \hat\sigma^2_Z, & \hat\sigma_{Zg} \\ \hat\sigma_{g\zeta}, & \hat\sigma_{g\psi}, & \hat\sigma^2_{Zg}, & \sigma_g \end{bmatrix} \right).$$

(B.8a)-(B.8d) show g^i_t pins down the transition of all state variables. This ensures one state variable pins down agent i's belief about how conditions at date t+1 are expected to be different from normal, where "normal" is represented by the empirical distribution. Comparing (B.7a)-(B.7c) with (B.8a)-(B.8d) shows that $E^i_t[Z_{t+1}] = \lambda_Z Z_t + \lambda^\zeta_Z \lambda^g_\zeta g^i_t + \lambda^\psi_Z \lambda^g_\psi g^i_t + \lambda^g_Z g^i_t$ and $E^m_t[Z_{t+1}] = \lambda_Z Z_t$ hence

(B.9)
$$E^i_t \begin{pmatrix} \zeta_{t+1} \\ \psi_{t+1} \\ Z_{t+1} \end{pmatrix} - E^m_t \begin{pmatrix} \zeta_{t+1} \\ \psi_{t+1} \\ Z_{t+1} \end{pmatrix} = \begin{pmatrix} \lambda^g_\zeta \\ \lambda^g_\psi \\ \lambda^\zeta_Z \lambda^g_\zeta + \lambda^\psi_Z \lambda^g_\psi + \lambda^g_Z \end{pmatrix} g^i_t.$$

which is the same as (B.4). g^i_t is then the subjective expected value of the unobserved state.

---

[3] Recall that the notation $(v^i_{t+1}, u^i_{t+1}, Z^i_{t+1})$ indicates agent i's *perception* of $(v_{t+1}, u_{t+1}, Z_{t+1})$. Since there is no difference between $E^i_t v^i_{t+1}$ and $E^i_t v_{t+1}$, I write $E^i_t v_{t+1}$ to express expectations of v_{t+1} by i, *in accordance with his perception*.

B.3 Some A-Priori Parameter Restrictions

The Rational Belief principle (see Kurz (1994)) restricts parameters of perception models by requiring the agent's belief, viewed as a dynamical system, to reproduce the empirical distribution which corresponds to the perception model. To illustrate consider the RB perception model in (8a)-(8d) relative to the empirical distribution in (B.7a)-(B.7c). Given the unconditional variance in (B.5), among the restrictions imposed by the Rational Belief principle is the requirement the variance-covariance matrix in (B.8a)-(B.8d) is positive definite. Other conditions include

$$\mathrm{Var}[\lambda_\zeta^g g_t^{\,i} + \rho_{t+1}^{i\zeta}] = \mathrm{Var}[\rho_{t+1}^{\zeta}] \quad \Rightarrow \quad (\lambda_\zeta^g)^2 \mathrm{Var}(g) + \hat{\sigma}_\zeta^2 = \sigma_\zeta^2$$
$$\mathrm{Var}[\lambda_u^g g_t^{\,i} + \rho_{t+1}^{i\psi}] = \mathrm{Var}[\rho_{t+1}^{\psi}] \quad \Rightarrow \quad (\lambda_\psi^g)^2 \mathrm{Var}(g) + \hat{\sigma}_\psi^2 = \sigma_\psi^2$$
$$\mathrm{Var}[\lambda_Z^g g_t^{\,i} + \rho_{t+1}^{iZ}] = \mathrm{Var}[\rho_{t+1}^{Z}] \quad \Rightarrow \quad (\lambda_Z^g)^2 \mathrm{Var}(g) + \hat{\sigma}_Z^2 = \sigma_Z^2 .$$

Selecting a normalization I set $\lambda_\zeta^g = 1$ hence the above rationality conditions imply

(B.10a) $$\mathrm{Var}(g) < \sigma_\zeta^2 \;,\; (\lambda_\psi^g)^2 \mathrm{Var}(g) < \sigma_\psi^2 \;,\; (\lambda_Z^g)^2 \mathrm{Var}(g) < \sigma_Z^2 \;,\; \tilde{\sigma}_\zeta < \sigma_\zeta \;,\; \tilde{\sigma}_\psi < \sigma_\psi \;,\; \tilde{\sigma}_Z < \sigma_Z$$

In addition, it can be shown that the variance of ρ_{t+1}^Z is restricted by σ_g^2 and is specified as

(B.10b) $$\sigma_Z^2 < \sigma_g^2 .$$

I have usually set these at $\sigma_Z = 0.9\sigma_g$. The unconditional variance of $g_t^{\,i}$ can be calculated from the empirical distribution to be

(B.11a) $$\mathrm{Var}[g] = \frac{1}{(1-\lambda_Z^2)}[(\lambda_Z^\zeta)^2\sigma_\zeta^2 + (\lambda_Z^u)^2\sigma_u^2 + \sigma_g^2] .$$

Without a learning feed-back it would be

(B.11b) $$\mathrm{Var}[g] = \frac{\sigma_g^2}{1-\lambda_Z^2} .$$

Hence, Bayesian learning feed-back causes $g_t^{\,i}$ to exhibit increased variance. Comparing the empirical distribution (B.7a)-(B.7c) with the perception model (B.8a)- (B.8d) shows that learning feed back causes the belief variable $g_t^{\,i}$ to introduce into (B.8a)- (B.8d) correlation with observed data which does not exist in the empirical distribution (B.7a)-(B.7c). For example, the empirical distribution shows that in the long run $\mathrm{Cov}(\zeta_{t+1}, \zeta_t) = \lambda_\zeta \mathrm{Var}(\zeta)$. This relation is not preserved in (B.8a) due to learning feed back. A rational agent who updates in real time recognizes that his perceived model exhibits higher variance than the empirical distribution and this fact raises two questions. First, how should the rationality restrictions account for the added variance due to learning feed-back? Second, what is a *reasonable* increase of variance due to a learning feed back that should be permitted by a rational agent?

My reply to the first question is to impose the restrictions (B.10a)-(B.10b) subject to the estimated variance in (B.11b), ignoring the added variance caused by updating and which the agent know is not in the data. This implies that I can write (B.10a)-(B.10b) as

(B.12) $$\sigma_g \leq \sigma_\zeta \sqrt{(1-\lambda_Z^2)} \quad , \quad |\lambda_\psi^g| \leq \frac{\sigma_\psi}{\sigma_\zeta} \quad , \quad |\lambda_Z^g| \leq \frac{\sigma_Z}{\sigma_\zeta} \leq 0.9\sqrt{(1-\lambda_Z^2)}.$$

Turning to the second question, note that most learning literature do not restrict this higher variance. Kurz, Piccillo and Wu (2013, Appendix A) show that I place a-priori restrictions on the learning feed-back parameters $(\lambda_Z^\zeta, \lambda_Z^\psi)$ by deducing them from the theory itself or from empirical evidence. They show $(\lambda_Z^\zeta, \lambda_Z^\psi)$ are in [0 , 0.70]. As to λ_Z, the empirical evidence reveals (see Kurz and Motolese (2011)) high persistence of mean market belief with λ_Z estimated in range of [0.6 , 0.8]. In the paper I set $\lambda_Z \simeq 0.7$, $\lambda_Z^\zeta \simeq 0.70$, $\lambda_Z^\psi \simeq 0$, $\lambda_\psi^g = \frac{\sigma_\psi}{\sigma_\zeta}$ and conditions (B.12) is satisfied

References

Kurz, M. (1994): "On the Structure and Diversity of Rational Beliefs." *Economic Theory* **4**, 877- 900. (An edited version appears as Chapter **2** of Kurz, M. (ed.) (1997)).

Kurz, M. (ed) (1997): *Endogenous Economic Fluctuations: Studies in the Theory of Rational Belief.* Studies in Economic Theory, No. **6**, Berlin and New York: Springer-Verlag.

Kurz, M. (2008): "Beauty Contests Under Private Information and Diverse Beliefs: How different?" *Journal of Mathematical Economics* **44,** 762-784.

Kurz, M. (2009): "Rational Diverse Beliefs and Economic Volatility." Chapter 8 in Hens, T. and Schenk-Hoppé, K.R. (ed.) *Handbook Of Financial Markets: Dynamics and Evolution*." Amsterdam: North-Holland, 440- 506.

Kurz, M., (2012): "A New Keynesian Model with Diverse Beliefs." Working Paper, Stanford University, February 27, 2012.

Kurz, M., Motolese, M. (2011): "Diverse Beliefs and Time Variability or Risk Premia." *Economic Theory* **47,** 293 - 335.

Kurz, M., Piccillo, G., Wu, H. (2013) "Modeling Diverse Expectations in an Aggregate New Keynesian Model." *Journal of Economic Dynamics and Control* **37,** 1403 - 1433.

Stabilizing Wage Policy: Mathematical Appendix B
Multiple Equilibria and Excluding the Ponzi Solution

This Draft: May 27, 2015

This Appendix explains why the Standard equilibrium concept is adopted in the text. It defines endogenous variables as linear function of the applicable date t state variables but this choice requires a more extensive exploration of the question of multiple equilibria and criteria for selecting any one equilibrium concept. We start by addressing a unique feature of models with heterogenous agents where a Ponzi solution always exists since individual borrowing is a feasible method of financing consumption. The approach is a direct equilibrium selection by excluding the Ponzi solution from being an equilibrium.

1. Excluding the Ponzi Solution

Individual decisions are linear in *individual states* $(1, Z_t, \zeta_t, \zeta_{t-1}, \zeta_{t-2}, \zeta_{t-3}, \zeta_{t-4}, \psi_t, \hat{b}^j_{t-1}, g^j_t)$ and the aggregates are linear in *aggregate states* $(1, Z_t, \zeta_t, \zeta_{t-1}, \zeta_{t-2}, \zeta_{t-3}, \zeta_{t-4}, \psi_t, 0, Z_t)$. Now recall that all equilibrium functions are actually *pairs of functions* for the upper and lower sub-economies. For example, the consumption decision function is

$$\hat{c}^{jU}_t = A_y^{U} \bullet (1, Z_t, \zeta_t, \zeta_{t-1}, \zeta_{t-2}, \zeta_{t-3}, \zeta_{t-4}, \psi_t, \hat{b}^j_{t-1}, g^j_t) \quad , \quad \hat{c}^{jL}_t = A_y^{L} \bullet (1, Z_t, \zeta_t, \zeta_{t-1}, \zeta_{t-2}, \zeta_{t-3}, \zeta_{t-4}, \psi_t, \hat{b}^j_{t-1}, g^j_t)$$

With this notational convention I state the individual decision functions

$$\text{(C.1a)} \quad \hat{c}^j_t = A_y^0 + A_y^Z Z_t + A_y^\zeta \hat{\zeta}_t + A_y^{\zeta_{-1}} \hat{\zeta}_{t-1} + A_y^{\zeta_{-2}} \hat{\zeta}_{t-2} + A_y^{\zeta_{-3}} \hat{\zeta}_{t-3} + A_y^{\zeta_{-4}} \hat{\zeta}_{t-4} + A_y^\psi \psi_t + A_y^b \hat{b}^j_{t-1} + A_y^g g^j_t$$

$$\text{(C.1b)} \quad \hat{q}^\star_{jt} = \frac{\omega}{1-\omega}[A_\pi^0 + A_\pi^Z Z_t + A_\pi^\zeta \hat{\zeta}_t + A_\pi^{\zeta_{-1}} \hat{\zeta}_{t-1} + A_\pi^{\zeta_{-2}} \hat{\zeta}_{t-2} + A_\pi^{\zeta_{-3}} \hat{\zeta}_{t-3} + A_\pi^{\zeta_{-4}} \hat{\zeta}_{t-4} + A_\pi^\psi \psi_t + A_\pi^b \hat{b}^j_{t-1} + A_\pi^g g^j_t]$$

$$\text{(C.1c)} \quad \hat{b}^j_t = A_b^0 + A_b^Z Z_t + A_b^\zeta \hat{\zeta}_t + A_b^u u_t + A_b^{\zeta_{-1}} \hat{\zeta}_{t-1} + A_b^{\zeta_{-2}} \hat{\zeta}_{t-2} + A_b^{\zeta_{-3}} \hat{\zeta}_{t-3} + A_b^{\zeta_{-4}} \hat{\zeta}_{t-4} + A_b^\psi \psi_t + A_b^b \hat{b}^j_{t-1} + A_b^g g^j_t .$$

The equilibrium conditions $\int_{[0,1]} \hat{c}^j_t dj = \hat{y}_t$, $\int_{[0,1]} \hat{b}^j_t dj = 0$, $Z_t = \int_{[0,1]} g^j_t dj$ and $\int_{[0,1]} \hat{q}^\star_{jt} = \frac{\omega}{1-\omega}\pi_t$ imply that the aggregates must satisfy

$$\text{(C.1d)} \quad \hat{y}_t = A_y^0 + A_y^Z Z_t + A_y^\zeta \hat{\zeta}_t + A_y^{\zeta_{-1}} \hat{\zeta}_{t-1} + A_y^{\zeta_{-2}} \hat{\zeta}_{t-2} + A_y^{\zeta_{-3}} \hat{\zeta}_{t-3} + A_y^{\zeta_{-4}} \hat{\zeta}_{t-4} + A_y^\psi \psi_t + A_y^b \times 0 + A_y^g Z_t$$

$$\text{(C.1e)} \quad \hat{\pi}_t = A_\pi^0 + A_\pi^Z Z_t + A_\pi^\zeta \hat{\zeta}_t + A_\pi^{\zeta_{-1}} \hat{\zeta}_{t-1} + A_\pi^{\zeta_{-2}} \hat{\zeta}_{t-2} + A_\pi^{\zeta_{-3}} \hat{\zeta}_{t-3} + A_\pi^{\zeta_{-4}} \hat{\zeta}_{t-4} + A_\pi^\psi \psi_t + A_\pi^b \times 0 + A_\pi^g Z_t$$

$$\text{(C.1f)} \quad \hat{q}_t = \frac{\omega}{1-\omega}[A_\pi^0 + A_\pi^Z Z_t + A_\pi^\zeta \hat{\zeta}_t + A_\pi^{\zeta_{-1}} \hat{\zeta}_{t-1} + A_\pi^{\zeta_{-2}} \hat{\zeta}_{t-2} + A_\pi^{\zeta_{-3}} \hat{\zeta}_{t-3} + A_\pi^{\zeta_{-4}} \hat{\zeta}_{t-4} + A_\pi^\psi \psi_t + A_\pi^b \times 0 + A_\pi^g Z_t]$$

$$\text{(C.1g)} \quad 0 = A_b^0 + A_b^Z Z_t + A_b^\zeta \hat{\zeta}_t + A_b^u u_t + A_b^{\zeta_{-1}} \hat{\zeta}_{t-1} + A_b^{\zeta_{-2}} \hat{\zeta}_{t-2} + A_b^{\zeta_{-3}} \hat{\zeta}_{t-3} + A_b^{\zeta_{-4}} \hat{\zeta}_{t-4} + A_b^\psi \psi_t + A_b^b \times 0 + A_b^g Z_t$$

(C.1h) $\hat{u}_t = A_u^0 + A_u^Z Z_t + A_u^{\zeta}\hat{\zeta}_t + A_u^{\zeta_{-1}}\hat{\zeta}_{t-1} + A_u^{\zeta_{-2}}\hat{\zeta}_{t-2} + A_u^{\zeta_{-3}}\hat{\zeta}_{t-3} + A_u^{\zeta_{-4}}\hat{\zeta}_{t-4} + A_u^{\psi}\psi_t$

(C.1i) $\hat{w}_t = A_w^0 + A_w^Z Z_t + A_w^{\zeta}\hat{\zeta}_t + A_w^{\zeta_{-1}}\hat{\zeta}_{t-1} + A_w^{\zeta_{-2}}\hat{\zeta}_{t-2} + A_w^{\zeta_{-3}}\hat{\zeta}_{t-3} + A_w^{\zeta_{-4}}\hat{\zeta}_{t-4} + A_w^{\psi}\psi_t$

Now, the standard transversality conditions are satisfied if debts are bounded. To ensure such boundedness, note that considering (C.1c) and taking into account all general equilibrium effects, sufficient conditions to ensure that $\hat{b}_{jt}$ remains bounded are $|A_b^{bU}|<1$, $|A_b^{bL}|<1$.

It follows from Taylor's Theorem that equilibrium parameters satisfy a decomposition principle. It states that, apart from belief variables, equilibrium elasticities of all endogenous variables relative to an exogenous variable are solved separately for that variable, on its own. Using this principle I solve the parameters of the state variable b_{t-1}^j, aiming to examine the parameters (A_b^{bU}, A_b^{bL}). To do that I insert the equilibrium maps (C.1a)-(C.1i) into the two pairs of structural equations (40a)-(40e)[1] in the text and deduce solutions of $(A_y^{bU}, A_\pi^{bU}, A_b^{bU}), (A_y^{bL}, A_\pi^{bL}, A_b^{bL})$ by matching coefficients. This process leads to the following set of six equations:

(C.2a) $$A_y^{bU} = [Q_U A_y^{bU} A_b^{bU} + (1-Q_U) A_y^{bL} A_b^{bL}] + \tau_b A_b^{bU}$$

(C.2b) $$A_y^{bL} = [Q_L A_y^{bL} A_b^{bL} + (1-Q_L) A_y^{bU} A_b^{bU}] + \tau_b A_b^{bL}$$

(C.2c) $$A_\pi^{bU} = \beta\omega[Q_U A_\pi^{bU} A_b^{bU} + (1-Q_U) A_\pi^{bL} A_b^{bL}]$$

(C.2d) $$A_\pi^{bL} = \beta\omega[Q_L A_\pi^{bL} A_b^{bL} + (1-Q_L) A_\pi^{bU} A_b^{bU}]$$

(C.2e) $$A_b^{bU} = (1/\beta) - A_y^{bU}[1 + (1-\alpha)\frac{\theta-1}{\theta}\frac{\sigma}{\eta}]$$

(C.2f) $$A_b^{bL} = (1/\beta) - A_y^{bL}[1 + (1-\alpha)\frac{\theta-1}{\theta}\frac{\sigma}{\eta}].$$

(C.2a)-(C.2f) imply bond parameters are symmetric: $A_y^{bU} = A_y^{bL} \equiv A_y^b, A_\pi^{bU} = A_\pi^{bL} \equiv A_\pi^b, A_b^{bU} = A_b^{bL} \equiv A_b^b$. To first solve A_y^b note that $A_y^b = 0 \Rightarrow A_b^b = 0$ due to (C.2a)-(C.2b) and $A_b^b = \frac{1}{\beta} > 0$ due to (C.2e)-(C.2f). This is a contradiction, hence $A_y^b \neq 0$. Let $\Xi = 1 + (1-\alpha)\frac{\theta-1}{\theta}\frac{\sigma}{\eta}$ then (C.2a) - (C.2b) imply

(C.3a) $$A_b^b = \frac{A_y^b}{A_y^b + \tau_b}$$

while (C.2e) - (C.2f) imply

[1] We use the convention of identifying each Appendix equation number with the capital letter of that Appendix so that all equation numbers without letters refer to equations in the text.

(C.3b) $$A_b^{\,b} = \frac{1}{\beta} - A_y^{\,b}\Xi$$

hence

$$A_y^{\,b} = [\frac{1}{\beta} - \frac{A_y^{\,b}}{A_y^{\,b} + \tau_b}]\frac{1}{\Xi}.$$

If $A_\pi^{\,b} \neq 0$ it follows from (C.2c)-(C.2d) that $A_b^{\,b} = \frac{1}{\beta\omega}$ which contradicts (C.3a) hence $A_\pi^{\,b} = 0$. Now return to (40a)-(40e) in the text to deduce

(C.4a) $$A_b^{ZU} = A_y^{gU}\Xi \ , \ A_b^{gU} = -A_y^{gU}\Xi \ , \ A_b^{\zeta U} = A_b^{\zeta_{-1}U} = A_b^{\zeta_{-2}U} = A_b^{\zeta_{-3}U} = A_b^{\zeta_{-4}U} = 0$$

(C.4b) $$A_b^{ZL} = A_y^{gL}\Xi \ , \ A_b^{gL} = -A_y^{gL}\Xi \ , \ A_b^{\zeta L} = A_b^{\zeta_{-1}L} = A_b^{\zeta_{-2}L} = A_b^{\zeta_{-3}L} = A_b^{\zeta_{-4}L} = 0.$$

Using (C.3a)-(C.3b) one concludes the following two opposite signs

$$A_b^{ZU} = -A_b^{gU} \quad , \quad A_b^{ZL} = -A_b^{gL} \ .$$

Next, we note that $\Xi = 1 + (1-\alpha)\frac{\theta-1}{\theta}\frac{\sigma}{\eta} > 0$ since $\theta > 1$ therefore (C.3a)-(C.3b) imply

$$(A_y^{\,b})^2 - \frac{1}{\Xi}(\frac{1-\beta}{\beta} - \tau_b)A_y^{\,b} - \frac{\tau_b}{\beta\Xi} = 0$$

with exactly two solutions

(C.5) $$A_y^{\,b} = \frac{\frac{1}{\Xi}(\frac{1-\beta}{\beta} - \tau_b) \pm \sqrt{\frac{1}{\Xi^2}(\frac{1-\beta}{\beta} - \tau_b)^2 + 4\frac{\tau_b}{\beta\Xi}}}{2}$$

one positive and one negative. Indeed, they are approximately

(C.6) $$A_y^{\,b} \cong \frac{1}{\Xi}(\frac{1-\beta}{\beta}) > 0 \quad , \quad A_y^{\,b} \cong -\frac{\tau_b}{|\sqrt{\beta\Xi}|} < 0.$$

A solution with $A_y^{\,b} < 0$ implies a dynamically unstable "Ponzi" solutions with $A_b^{\,b} > 1$. In such solution individual debts are unbounded hence it is not admissible as equilibrium.

2. On Multiple Equilibria and the Adopted Equilibrium Concept

The restriction $A_b^b < 1$ to exclude Ponzi solutions arises from borrowing by heterogeneous agents but multiple equilibria also arise when all agents are identical and without borrowing. Indeed, the problem of multiple solutions for NK models under the ZLB has recently been examined by a growing number of papers. All assume Rational Expectations with a single household or homogenous agents, some study the model without uncertainty and some in continuous time. However, none of these models integrates the upper and lower sub-economies. Without exploring fine differences among these papers we now explain the equilibrium concept adopted for the microeconomic equilibrium of (40a)-(40g) .

Theorem 2 of KPW (2013) implies that every linear microeconomic equilibrium discussed in the text implies a pair of parameters (B_y, B_π) and $\kappa = \frac{(1-\beta\omega)(1-\omega)}{\omega}$, $\Lambda = \frac{1-\alpha}{1-\alpha+\alpha\theta}$ such that the economy aggregates to

(C.7a) The IS Curve $\quad \hat{y}_t = E_t(\hat{y}_{t+1}) + B_y Z_t - (\frac{1}{\sigma})[\hat{r}_t - E_t(\hat{\pi}_{t+1}) + \hat{\psi}_t]$;

(C.7b) Phillips Curve $\quad \hat{\pi}_t = \kappa\Lambda[\hat{w}_t - \frac{1}{1-\alpha}(\hat{\zeta}_t - \alpha\hat{y}_t)] + \beta E_t\hat{\pi}_{t+1} + \beta B_\pi Z_t$;

(C.7c) $\hat{w}_t = \hat{w}_t^\star - \lambda_\pi\hat{\pi}_t - \lambda_u u_t$, $\hat{w}_t^\star = B_w^\zeta(\hat{\zeta}_{t-1} + \hat{\zeta}_{t-2} + \hat{\zeta}_{t-3} + \hat{\zeta}_{t-4})$, $u_t = -(\frac{1}{1-\alpha} + \frac{\sigma}{\eta})\hat{y}_t + \frac{1}{\eta}\hat{w}_t + \frac{1}{1-\alpha}\hat{\zeta}_t$

In the upper sub-economy $\hat{r}_t = \xi_\pi(\hat{\pi}_t) + \xi_y(\hat{y}_t - \hat{y}_t^F)$, $\hat{y}_t^F = \frac{(1+\eta)}{[\alpha+\eta+\sigma(1-\alpha)]}\hat{\zeta}_t$. By (C.7c) the wage rate can be written as

$$\hat{w}_t = \frac{\eta}{\eta+\lambda_u}[\hat{w}_t^\star - \lambda_\pi\hat{\pi}_t + \lambda_u(\frac{1}{1-\alpha} + \frac{\sigma}{\eta})\hat{y}_t] - \frac{\eta}{\eta+\lambda_u}\frac{\lambda_u}{1-\alpha}\hat{\zeta}_t .$$

Recall the expectation notation which, for the case of output, is

$$E_t^j(\hat{y}_{t+1}) \equiv \begin{cases} E_t(\hat{y}_{t+1}|U) \equiv \Omega_U E_t^j(\hat{y}_{t+1}^U) + (1-\Omega_U)E_t^j(\hat{y}_{t+1}^L) & \text{if current state is U} \\ E_t(\hat{y}_{t+1}|L) \equiv \Omega_L E_t^j(\hat{y}_{t+1}^L) + (1-\Omega_L)E_t^j(\hat{y}_{t+1}^U) & \text{if current state is L .} \end{cases}$$

Introduce

(C.8) $\quad H_1 = -\kappa\Lambda[\frac{\alpha}{1-\alpha} + \frac{\lambda_u\eta}{\eta+\lambda_u}\left(\frac{1}{1-\alpha} + \frac{\sigma}{\eta}\right)]$, $H_2 = 1 + \kappa\Lambda\left(\frac{\eta\lambda_\pi}{\eta+\lambda_u}\right)$, $\wp_t^1 = \zeta_{t-1}, \wp_t^2 = \zeta_{t-2}, \wp_t^3 = \zeta_{t-3}, \wp_t^4 = \zeta_{t-4}$.

Defining $(\wp_t^1 = \zeta_{t-1}, \wp_t^2 = \zeta_{t-2}, \wp_t^3 = \zeta_{t-3}, \wp_t^4 = \zeta_{t-4})$, the full dynamical system can then be written

$$
\begin{aligned}
&E_t\zeta_{t+1} = \lambda_\zeta \zeta_t \\
&E_t\psi_{t+1} = \lambda_\psi \psi_t \\
&E_t Z_{t+1} = \lambda_Z Z_t \\
&\wp_{t+1}^1 = \zeta_t \\
&\wp_{t+1}^2 = \wp_t^1 \\
&\wp_{t+1}^3 = \wp_t^1 \\
&\wp_{t+1}^4 = \wp_t^3
\end{aligned}
$$

.

(C.9)
$$
\begin{aligned}
&E_t[\hat{y}_{t+1}|U] + \frac{1}{\sigma}E_t[\hat{\pi}_{t+1}|U] = (1+\frac{\xi_y}{\sigma(1+\bar{r})})\hat{y}_t^U + \frac{\xi_\pi}{\sigma(1+\bar{r})}\hat{\pi}_t^U - \frac{\xi_y}{\sigma(1+\bar{r})}\frac{(1+\eta)}{[\alpha+\eta+\sigma(1-\alpha)]}\hat{\zeta}_t + \frac{1}{\sigma}\hat{\psi}_t - B_y Z_t \\
&\beta E_t[\hat{\pi}_{t+1}|U] = H_1\hat{y}_t^U + H_2\hat{\pi}_t^U + \frac{\kappa\Lambda}{1-\alpha}[1+\frac{\eta\lambda_u}{\eta+\lambda_u}]\hat{\zeta}_t - \beta B_\pi Z_t - \kappa\Lambda\frac{\eta}{\eta+\lambda_u}\hat{w}_t^\star \\
&E_t[\hat{y}_{t+1}|L] + \frac{1}{\sigma}E_t(\hat{\pi}_{t+1}|L) = \hat{y}_t^U + \frac{1}{\sigma}\hat{\psi}_t - B_y Z_t - \frac{\bar{r}}{\sigma(1+\bar{r})} \\
&\beta E_t[\hat{\pi}_{t+1}|L] = H_1\hat{y}_t^L + H_2\hat{\pi}_t^L + \frac{\kappa\Lambda}{1-\alpha}[1+\frac{\eta\lambda_u}{\eta+\lambda_u}]\hat{\zeta}_t - \beta B_\pi Z_t - \kappa\Lambda\frac{\eta}{\eta+\lambda_u}\hat{w}_t^\star
\end{aligned}
$$

together with policy rule (37) and transitions (31a)-(31b),(34). The Macro model is then a standard dynamical system consisting of *two pairs of equations* for the upper and lower sub-economies and we ask if it has a unique solution. To that end use (39) to write expected t+1 variables on the left and rearrange terms on the right to conclude that denoting by

$$F_t' = (\hat{y}_t^U, \hat{\pi}_t^U, \hat{y}_t^L, \hat{\pi}_t^L) \quad , \quad \hat{s}_t' = (\hat{\zeta}_t, \hat{\psi}_t, \hat{Z}_t, \wp_t^1, \wp_t^2, \wp_t^3, \wp_t^4)$$

then (C.9) is written in the form

(C.10)
$$B_1(E_t[F_{t+1}]) = a + B_2(F_t) + A\hat{s}_t.$$

$\hat{s}_t$ is a vector of zero mean state variables and (B_1, B_2, a, A) are defined by

(C.11a)
$$
B_1 = \begin{bmatrix} \Omega_U & \frac{1}{\sigma}\Omega_U & (1-\Omega_U) & \frac{1}{\sigma}(1-\Omega_U) \\ 0 & \beta\Omega_U & 0 & \beta(1-\Omega_U) \\ (1-\Omega_L) & \frac{1}{\sigma}(1-\Omega_L) & \Omega_L & \frac{1}{\sigma}\Omega_L \\ 0 & \beta(1-\Omega_L) & 0 & \beta\Omega_L \end{bmatrix}, \quad B_2 = \begin{bmatrix} [1+\frac{\xi_y}{\sigma(1+\bar{r})}] & \frac{\xi_\pi}{\sigma(1+\bar{r})} & 0 & 0 \\ H_1 & H_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & H_1 & H_2 \end{bmatrix}, \quad a = \begin{bmatrix} 0 \\ 0 \\ -\frac{\bar{r}}{\sigma(1+\bar{r})} \\ 0 \end{bmatrix}
$$

$$\text{(C.11b)}\qquad A = \begin{bmatrix} \frac{\xi_y}{\sigma(1+\bar{r})}\frac{(1+\eta)}{[\alpha+\eta+\sigma(1-\alpha)]} & \frac{1}{\sigma} & -B_y & 0 & 0 & 0 & 0 \\ \frac{\kappa\Lambda}{1-\alpha}[1+\frac{\eta\lambda_u}{\eta+\lambda_u}] & 0 & -\beta B_\pi & \kappa\Lambda\frac{\eta}{\eta+\lambda_u}B_w^\zeta & \kappa\Lambda\frac{\eta}{\eta+\lambda_u}B_w^\zeta & \kappa\Lambda\frac{\eta}{\eta+\lambda_u}B_w^\zeta & \kappa\Lambda\frac{\eta}{\eta+\lambda_u}B_w^\zeta \\ 0 & \frac{1}{\sigma} & -B_y & 0 & 0 & 0 & 0 \\ -\frac{\kappa\Lambda}{1-\alpha}[1+\frac{\eta\lambda_u}{\eta+\lambda_u}] & 0 & -\beta B_\pi & \kappa\Lambda\frac{\eta}{\eta+\lambda_u}B_w^\zeta & \kappa\Lambda\frac{\eta}{\eta+\lambda_u}B_w^\zeta & \kappa\Lambda\frac{\eta}{\eta+\lambda_u}B_w^\zeta & \kappa\Lambda\frac{\eta}{\eta+\lambda_u}B_w^\zeta \end{bmatrix}$$

Finally, the shocks have a Markov property hence

$$\text{(C.11c)}\qquad E_t[s_{t+1}] = \begin{bmatrix} \lambda_\zeta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_\psi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_Z & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \hat{s}_t \equiv S \bullet \hat{s}_t .$$

With this notation (C.10) is expressed by

$$E_t[F_{t+1}] = B_1^{-1}a + (B_1^{-1}B_2)F_t + (B_1^{-1}A)\hat{s}_t$$

and we will shortly show it has the same mathematical structure as any aggregate New Keynesian Model in which the only dynamics studied is the dynamics of the aggregates.

2.1 Transformations: Steady State and Eigenvalue Decomposition

Step 1 requires computation of the steady state defined by setting $s_t = 0$, $F_{t+1} = F_t = \bar{F}^S$ to have

$$\text{(C.12)}\qquad \bar{F}^s = (B_1 - B_2)^{-1}a \quad \text{with steady state values} \quad \begin{bmatrix} \bar{Y}^U \\ \bar{\pi}^U \end{bmatrix} \neq \begin{bmatrix} \bar{Y}^L \\ \bar{\pi}^L \end{bmatrix}.$$

Under the assumption that $(B_1 - B_2)$ is non-singular, insert $a = (B_1 - B_2)\bar{F}^s$ into (C.10)

$$\text{(C.13)}\qquad B_1(E_t F_{t+1}) = (B_1 - B_2)\bar{F}^s + B_2(F_t) + A\hat{s}_t$$

which means the new system is

$$B_1 E_t[F_{t+1} - \overline{F}^{\,s}] \;=\; + \, B_2[F_t - \overline{F}^{\,s}] \;+\; A\hat{s}_t .$$

This shows the integrated system fluctuates around a steady state which is endogenous, not equal to the steady states of either the upper or the lower sub-economies. Denoting the deviation of equilibrium values from steady state by $\hat{F}_t = F_t - \overline{F}^{\,s}$ the system is now

(C.14) $$E_t[\hat{F}_{t+1}] = (B_1^{-1}B_2)\hat{F}_t + (B_1^{-1}A)\hat{s}_t .$$

Use Jordan decomposition to deduce an invertible matrix C and a diagonal matrix J such that

(C.15) $$B_1^{-1}B_2 = CJC^{-1} \Rightarrow E_t[\hat{F}_{t+1}] = CJC^{-1}[\hat{F}_t] + B_1^{-1}A\hat{s}_t$$

hence

$$E_t[C^{-1}\hat{F}_{t+1}] = J[C^{-1}\hat{F}_t] + C^{-1}B_1^{-1}A\hat{s}_t .$$

Now define $V_t = C^{-1}\hat{F}_t$, $C^{-1}B_1^{-1}A = M$ and conclude that

(C.16) $$E_t[V_{t+1}] = JV_t + M\hat{s}_t .$$

The fact is that the Integrated model of the text has the property that there exists a critical value $\Omega_2^{\star}$ of the probability Ω_2 such that if $\Omega_2 < \Omega_2^{\star}$ all four eigenvalues of J are larger in absolute value than 1 while if $\Omega_2 > \Omega_2^{\star}$ then the matrix J has the form

$$J = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \quad , \quad |\lambda_1| < 1 \;,\; |\lambda_2| > 1 \;,\; |\lambda_3| > 1 \;,\; |\lambda_4| > 1 .$$

Computations used in the text employ the value of $\Omega_2 = 0.96$ and in most cases J has the form (C.16) which we now assume to be the standard case. We thus start by decomposing the matrices as follows. Note that V is 4x1, and M is 4x7 and I decompose these as follows

(C.17)
$$V = \begin{bmatrix} V^1 \\ V^2 \\ V^3 \\ V^4 \end{bmatrix} \quad , \quad M = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_2 \end{bmatrix}.$$

where V^j is 1×1 while M_j is 1×7. (C.16) is then reduced to a system of 4 independent equations

(C.18)
$$\begin{bmatrix} E_t V^1_{t+1} \\ E_t V^2_{t+1} \\ E_t V^3_{t+1} \\ E_t V^4_{t+1} \end{bmatrix} = \begin{bmatrix} \lambda_1 V^1_t \\ \lambda_2 V^2_t \\ \lambda_3 V^3_t \\ \lambda_4 V^4_t \end{bmatrix} + \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} \hat{s}_t .$$

Although each equation in (C.18) is solved on its own, the final solution is $\hat{F}_t = CV_t$ hence each component of $\hat{F}_t$ is a linear combinations of solutions to (C.18). But then, in principle, we are reduced to seeking a solution to an equation whose simplest one dimensional form is

(C.19)
$$E_t(x_{t+1}) = \lambda_x x_t + \vartheta\zeta_t \quad , \quad \zeta_{t+1} = \lambda_\zeta \zeta_t + \rho_{t+1} \quad , \quad |\lambda_\zeta| < 1 \quad , \quad \vartheta > 0 .$$

The Markov Assumption is standard in Economics.

2.2 Generic Multiplicity of Solutions: Empirical Evidence as a Guide to Selection

In a vast literature on expectational difference equations it has been established that many mathematical solutions exist to equations like (C.19) or (C.18)[2]. The term "mathematical" solutions needs to be stressed. We thus list the possible mathematical solutions to (C.19):

Solution 1: Standard. $x_t = A^\zeta \zeta_t \quad \Rightarrow \quad A^\zeta = \vartheta/(\lambda_\zeta - \lambda_x) \quad , \quad \lambda_\zeta \neq \lambda_x .$

Solution 2: Forward.
$$x_t = \begin{cases} -\vartheta[\sum_{k=0}^{\infty} (\lambda_x^{-1})^{k+1} \lambda_\zeta^k] \zeta_t = \dfrac{\vartheta}{\lambda_\zeta - \lambda_x}\zeta_t & \text{if } \lambda_x > \lambda_\zeta \\ \infty & \text{if } \quad \lambda_x < \lambda_\zeta . \end{cases}$$

When $\lambda_x > \lambda_\zeta$, solutions 1 and 2 coincide and

(C.20)
$$x_t = \frac{\vartheta}{\lambda_\zeta - \lambda_x}\zeta_t .$$

---

[2] Some early references include Sargent and Wallace (1973),(1975), Taylor (1977), Blanchard (1979), Blanchard and Kahn (1980). Blanchard and Fischer (1989) Chapter 5 offer a survey with many additional references.

Solution 3: Backward. $x_t = \lambda_x^t x_0 + \vartheta[\lambda_x^{t-1}\zeta_0 + \lambda_x^{t-2}\zeta_1 + \ldots + \zeta_{t-1}]$

(C.21) here x_t *is a deterministic solution* hence $x_{t+1} = \lambda_x x_t + \vartheta\zeta_t$.

It is a well known mathematical fact that (C.21) requires a boundary condition x_0.

Solution 4: Mixture. Any convex combination of the above solutions is also a solution.

Solution 5: Bubble. For any above add a "bubble" b_t with $E_t(b_{t+1}) = \lambda_x b_t$.

It is then obvious that from a mathematical point of view it is a generic property of (C.19) that it has multiple solutions of the order of the continuum, *regardless of the values taken by the underlying parameters*. But this means that from an economic point of view just specifying an equation like (C.19), even with x_0 given, is *an incomplete economic theory*. One cannot expect the mathematics of the problem to provide a substitute for a more complete economic specification of the theory. This is an important and well recognized fact (e.g. Sargent and Wallace (1975), Blanchard (1979)).

The standard approach to the problem is the Blanchard-Kahn (1980) theory that seeks a "unique" solution by requiring $\lambda_x > 1$ when Solutions 1 and 2 coincide. It is referred to as the "fundamental" solution, reflecting the application of similar equations to asset pricing theory where it implies an asset price equals the expected present value of future dividends. It is common to find in the literature the claim that $\lambda_x > 1$ is a *necessary and sufficient* condition for a unique solution, but this is misleading. When $\lambda_x > 1$ solutions 3, 4, 5 remain viable mathematical solutions of (C.19). What eliminates them is the more fundamental condition imposed by the *economics of the problem which requires a solution to be bounded*. When $\lambda_x > 1$ solutions 3 is unbounded although it is a legitimate mathematical solution. To satisfy boundedness a bubble solution must also either decay or burst with probability 1 in finite time (e.g. Blanchard and Watson (1982)). Hence, subject to any bubble being finite lived, among all bounded solutions the fundamental solution (C.20) is unique. But why reject unbounded solutions when models of economic growth have unbounded solutions? The answer is that the linear model (C.19) typically describes dynamics around steady state growth and *empirical evidence* shows random fluctuations around a deterministic path are bounded. In addition, for variables such as output, boundedness is imposed by limited resources. Hence, the crucial restriction that an acceptable economic solution must satisfy is ultimately based on *empirical facts* about the economy being represented by (C.19) but such evidence and the restrictions it imposes on solutions are not specified in the equation itself. Once we recognize that restrictions on a mathematical solution must emerge from the economy described by (C.19), we can ask what other evidence can be brought to bear. Such evidence would naturally vary from one case to the other, hence we explore the question in the context of the New Keynesian Model of the text.

Before continuing it is useful to consider two semantic issues that help the way one may look at the question at hand. First, an agent's expectation of an endogenous variable $E_t(x_{t+1}|I_t)$ has a meaning only if the agent perceives some equilibrium map $x_t = g(I_t)$ and computes the expected value of x_{t+1} by using the probability distribution on I_{t+1} conditional on I_t. Hence, in an economy with heterogenous agents a "solution" is an "equilibrium map" which must then be common knowledge among all market participants together with knowledge of the domain of the probability distribution. From this perspective one cannot just write $E_t(x_{t+1}) = \lambda_x x_t + \vartheta\zeta_t$ since there is a difference between a stochastic economy where $x_{t+1} = \lambda_x x_t + \vartheta\zeta_t + \rho_{t+1}$ with $E_t(\rho_{t+1}) = 0$ and where expectations matter and an economy with $x_{t+1} = \lambda_x x_t + \vartheta\zeta_t$ where at date t x_{t+1} is known to be deterministic and expectations do not matter. The economics of the problem specifies the economic environment where (C.19) is defined: if it is a stochastic environment then a deterministic solution $x_{t+1} = \lambda_x x_t + \vartheta\zeta_t$ is to be rejected since it is not applicable to that economy. For example, in all stock market models or macroeconomic models of output and inflation expectations matter hence a deterministic solution is not applicable since it is empirically irrelevant.

The second semantic issue is related to the term "fundamental" solution used in asset pricing. In our view this interpretation is not applicable to models where the endogenous variables are aggregates such as output, inflation or wage rates. To explain let us return to the textbook Walrasian system where decisions depend upon state variables and equilibrium prices are functions of these state variables. This is exactly the "fundamental" or standard solution 1 without the need for any forward looking present value calculations. Prices provide the economy with an adjustment mechanism to changes in state variables. To the extent that lagged variables impact the equilibrium conditions, they are state variables and equilibrium prices become functions of these state variables. The new state variables in Solutions 3 and 5 are not original variables but are added endogenously by the solution. By analogy, an equilibrium map that satisfied the equilibrium condition (C.19) is also an adjustment mechanism of endogenous variables responding to changed state variables. The difference is that multiple solutions offer diverse adjustment mechanisms *because the economy underlying the equilibrium condition (C.19) has incomplete markets*. It follows that we must not think of solution 1 as reflecting some present value calculations but rather as an *economic adjustment mechanism* to shocks and therefore there is a difference between Solution 1 and Solution 2 since Solution 1 remains valid even when $\lambda_x < \lambda_\zeta$. It thus appears that the term "standard solution" or "standard equilibrium map" is a more appropriate name for this solution than the term "fundamental." Under this standard Solution 1 aggregate economic variables adjust at date t to date t state variables, just like a price system that responds to state variables in a Walrasian system. This does not preclude other adjustment mechanisms that reflect the underlying economic reality from being established but the empirical record would then reveal which adjustment process is the one used in reality and why this equilibrium map is acceptable while others are not. This is particularly important for a New Keynesian Model since, as argued in the text, this theory is all about speed of adjustment hence different adjustment processes can lead to drastically different model implications and predictions. *Gross failures of such implications or predictions is as viable empirical evidence against an equilibrium map as the empirical criterion that rejects an unbounded solution as an equilibrium.* Putting it differently, rejection of a mathematical solution to (C.19) cannot be deduced from abstract mathematical conditions but from economic implications of the solution when such implications contradict empirical economic reality not explicitly stated in (C.19).

2.3 Bubbles in a New Keynesian Model

Bubbles or claims of bubbles have been associated with asset pricing. As to standard economic aggregates such as income and inflation, data for such variables are not known to exhibit any bubbles and no claims for the existence of such bubbles have been credibly made. One may conjecture that the reason for this fact is that a market bubble requires agents to take some risks since the value of the bubble in the future is unknown. But in order to take risk there must be a positive gain to speculators who trade on such bubbles and it is not clear how there could be a gain derived from the existence of a bubble in output. Some proposed to explain rare hyper-inflations as bubbles in the inflation process (e.g. Cagan (1956), Sargent and Wallace (1973),(1975)). This cannot happen without the central bank allowing it to form by expanding base money thus being part of the bubble. Absent decisive evidence for regular bubbles in economic aggregates, incorporating a bubble in the equilibrium map will serve no analytical purpose nor will it have important policy implications except to require the central bank to be disciplined.

Rich asset markets should be included in viable New Keynesian Models hence it is useful to address the question of bubbles in expanded models that include real assets like stocks. There are three points to be made here. First, since a bubble is a highly structured phenomenon, theoretical studies of bubbles in asset pricing show they cannot form in efficient infinite horizon economies but can form in inefficient OLG economies (e.g. Tirole (1982), (1985), Blanchard and Fischer (1989) Chapter 5). Second, recognizing historical events such as the Tulip Mania or the South Sea Bubble, no credible empirical study has conclusively demonstrated the existence of regular bubbles in asset prices of advanced economies where a bubble in (C.19) must take the highly disciplined form of $E_t(b_{t+1}) = \lambda_x b_t$ with a growth rate λ_x which, in a typical asset pricing model, is equal to the interest rate. The bubble itself must be common knowledge and observable by all market participants. Although "bubbles" are now in commonly usage in the press the fact is that all empirical discussion of bubbles has been conducted by assuming that *any excess asset price volatility above level predicted by Rational Expectations must be due to bubbles*. This view is in conflict by the vast empirical evidence for diverse beliefs in asset markets. Diverse beliefs contradict the "common knowledge" requirement of endogenous bubbles, but diverse beliefs are also the natural mechanism for generating excess volatility above the Rational Expectations level. Indeed, the model in the text has diverse beliefs where individual j's belief index g_t^j and market belief index Z_t are included in the state vector $\hat{s}_t$ and account for substantial fraction of the predicted volatility of individual variables such as consumption of j or aggregate volatility of output, inflation or interest rates. Third, in (C.18) a bubble is formed in the solutions for the vector V_t whereas the endogenous variables are solved under the equation $\hat{F}_t = CV_t$. Hence, a bubble formed endogenously in the solution V_t *must migrate to the entire vector of endogenous variables* $\hat{F}_t$ which are linear combinations of the components of V_t. Hence, in a General Equilibrium data of this economy an exact bubble must be observed and be common knowledge in all markets for all goods and not confined to one asset or one aggregate variable. This requires informational coordination of the highest order and universal agreement among market participants about the meaning of market data. Again, diversity of market beliefs makes the formation of such endogenous bubbles most unlikely.

If we reject bubbles as a model of excess market volatility in economic aggregates, does it mean we insist the hyperbolic rise of real estate prices during 1995 - 2007 was not a bubble? Yes, this price rise can be explained by a highly optimistic market belief in the rise of real estate prices and all it takes to observe a long period of a rising asset prices is a high degree of persistence in the indices of individual and market belief and the empirical evidence supports this requirement (e.g. see Kurz and Motolese (2001), (2011)). The empirical evidence also shows that this high degree of persistence of belief is a property of both private sector expectations and of the Fed's expectations.

2.3 *Longer Memory in a New Keynesian Model*

We now return to the end of Section 2.1 with the view to investigate not only the different solutions but also their empirical implications. Start by noting the obvious fact that any formulated theory specifies the memory in the structural equations. In (C.9) agents recall $(\zeta_{t-1}, \zeta_{t-2}, \zeta_{t-3}, \zeta_{t-4})$ which are state variables and in other models the equilibrium conditions (C.19) may contain other lagged variables as state variables. Hence, the existence of memory is not in question! The point is that the Backward Solution 3 introduces new memory via the equilibrium map and the question at hand is whether the added memory required by such proposed solution is compatible with the economics of the initial model. To that end let us return to (C.18).

Since $|\lambda_1| < 1$, $|\lambda_2| > 1$, $|\lambda_3| > 1$, $|\lambda_4| > 1$ the components V^1 and (V^2, V^3, V^4) of V behave differently: the forward solution of V^1 diverges while the requirement of boundedness and no bubbles ensure that (V^2, V^3, V^4) has a uniquely forward fundamental solution

(C.22)
$$\begin{bmatrix} V_t^2 \\ V_t^3 \\ V_t^4 \end{bmatrix} = -\begin{bmatrix} \sum_{k=0}^{\infty} (\lambda_2^{-1})^k M_2 S^{k-1} \\ \sum_{k=0}^{\infty} (\lambda_3^{-1})^k M_3 S^{k-1} \\ \sum_{k=0}^{\infty} (\lambda_4^{-1})^k M_4 S^{k-1} \end{bmatrix} \hat{s}_t \equiv \begin{bmatrix} A_V^2 \\ A_V^3 \\ A_V^4 \end{bmatrix} \hat{s}_t .$$

Convergence to an expression equivalent to (C.20) results from $|\lambda_2|>1, |\lambda_3|>1, |\lambda_4|>1$ and from a stable structure (C.11c) of the Markov matrix S. All this implies (V_t^2, V_t^3, V_t^4) is a linear function of the date t state vector which conforms to the Blanchard Kahn (1980) theory.

The situation is different for V^1. A solution satisfies the *single difference equation*

(C.23)
$$E_t V_{t+1}^1 = \lambda_1 V_t^1 + M_1 \hat{s}_t \quad , \quad |\lambda_1| < 1 \quad , \quad M_1 \text{ is a } 1\times 7 \text{ vector}.$$

Iterating (C.23) forward leads to an equation which is analogous to (C.22) but results in an infinite sum which diverges hence the boundedness restriction eliminates it as a solution.

To find a bounded V^1 solution observe first that contemporaneous Solution 1 exists and is

(C.24) $$V_t^1 = K\hat{s}_t \quad , \text{K a 1x7 vector.}$$

where

(C.25) $$K = [S - \lambda_1 I]^{-1} M_1 .$$

A solution (C.25) exists if λ_1 is not an eigenvalue of S, which is analogous to a requirement $\lambda_\zeta \neq \lambda_x$ that is violated only by chance. Sine $\hat{F}_t = CV_t$ the *standard solution* of the original problem is

(C.26a) $$\hat{F}_t = C \bullet \begin{bmatrix} [S - \lambda_1 I]^{-1} M_1 \\ A_V^2 \\ A_V^3 \\ A_V^4 \end{bmatrix} \hat{s}_t .$$

Solution 3: Backward. $$x_t = \lambda_x^t x_0 + \vartheta[\lambda_x^{t-1}\zeta_0 + \lambda_x^{t-2}\zeta_1 + \ldots + \zeta_{t-1}]$$

Turn now to the backward Solution 3 defined by

(C.27) $$V_t^1 = \lambda_1^t V_0^1 + M_1 \sum_{k=0}^{t-1} \lambda_1^k \hat{s}_{t-k-1} .$$

This history dependence *excludes date t states* and we examine this issue below. Since at date t+1 the quantity V_{t+1} is known, it means that

(C.28) $$V_{t+1}^1 = \lambda_1^t V_0^1 + M_1 \sum_{k=0}^{t} \lambda_1^k \hat{s}_{t-k} \quad \Rightarrow \quad E_t V_{t+1}^1 = V_{t+1}^1 .$$

Can the backward solution (C.28) include *current state* or *sunspot variables* hence it take the more general form $V_t^1 = \lambda_1^t V_0^1 + M_1 \sum_{k=0}^{t} \lambda_1^k \hat{s}_{t-k} + \rho_t^V$ where ρ_t^V is white noise? To test if such a solution is possible, insert it into (C.23), using the fact that

(C.29) $$E_t[\rho_{t+1}^V] = 0 \ , \ E_t[\hat{s}_{t+1}] = S \bullet \hat{s}_t$$

and ask if the following equation holds:

$$\text{(C.30)}\quad \lambda_1^{t+1}V_0^1 + M_1[\lambda_1^{t+1}\hat{s}_0 + \lambda_1^t\hat{s}_1 + \ldots + \lambda_1\hat{s}_t + S\bullet\hat{s}_t] = \lambda_1\left\{\lambda_1^t V_0^1 + M_1[\lambda_1^t\hat{s}_0 + \lambda_1^{t-1}\hat{s}_1 + \ldots + \hat{s}_t] + \rho_t^V\right\} + M_1\hat{s}_t\ ?$$

It fails due to the two terms $(\rho_t^V, \hat{s}_t)$ in the proposed solution, confirming (C.27). This solution also requires an initial value of V_0^1 but since $V_t = C^{-1}\hat{F}_t$ without loss of generality one can set $\hat{F}_0 = 0$ hence $V_0 = 0$. Under this specification the second possible solution of the original variables is

$$\text{(C.26b)}\qquad \hat{F}_t = C \bullet \begin{bmatrix} M_1\sum_{k=0}^{t-1}\lambda_1^k\hat{s}_{t-k-1} \\ A_V^2\hat{s}_t \\ A_V^3\hat{s}_t \\ A_V^4\hat{s}_t \end{bmatrix}.$$

2.4 Equilibrium Implications

Two conclusions emerge. First, the standard equilibrium map where endogenous variables are linear functions of the current state is a bounded solution. Second, the presence of one eigenvalue inside the unit circle imply that other bounded equilibrium maps may exist in the form (C.26b). The problem is that it implies an infinite dimensional state space in which the entire history is relevant. Moreover, the equilibrium map itself changes from date t to date t+1 since date t+1 state space is larger by one than the state space at date t. The first task is then to reduce the dimensionality. To that end the general solution (C.26b) is written in the simpler form

$$\text{(C.31)}\qquad \hat{F}_t = A^s\hat{s}_t + A^V V_t^1 \quad , \quad A^s \text{ is } 4\times 7 \ , \ A^V \text{ is } 4\times 1$$

where the matrix C is written as a set of four column vectors

$$C = [C_1 \ , \ C_2 \ , \ C_3 \ , \ C_4]$$

and

$$A^s = C_2 A_V^2 + C_3 A_V^3 + C_4 A_V^4$$
$$A^V = C_1 .$$

Now recall that

$$\text{(C.32)}\qquad V_t^1 = \lambda_1 V_{t-1}^1 + M_1\hat{s}_{t-1} \quad , \quad M_1 \text{ is } 1\times 7 \ , \ \hat{s} \text{ is } 7\times 1 .$$

Hence (C.31)-(C.32) imply

$$\hat{F}_t = A^{s}\hat{s}_t + \lambda_1[A^{V}V^{1}_{t-1}] + A^{V}M_1\hat{s}_{t-1} = A^{s}\hat{s}_t + \lambda_1[\hat{F}_{t-1} - A^{s}\hat{s}_{t-1}] + A^{V}M_1\hat{s}_{t-1}$$

hence

(C.33a) $$\begin{bmatrix} \hat{y}^{U}_t \\ \hat{\pi}^{U}_t \\ \hat{y}^{L}_t \\ \hat{\pi}^{L}_t \end{bmatrix} = A^{s}\hat{s}_t + \lambda_1 \begin{bmatrix} \hat{y}^{U}_{t-1} \\ \hat{\pi}^{U}_{t-1} \\ \hat{y}^{L}_{t-1} \\ \hat{\pi}^{L}_{t-1} \end{bmatrix} + [A^{V}M_1 - \lambda_1 A^{s}]\hat{s}_{t-1}.$$

The family of equilibrium maps is then written as

$$\hat{F}_t = A^{s}\hat{s}_t + \lambda_1\hat{F}_{t-1} + A^{s_1}\hat{s}_{t-1}$$

and now history is represented by $(\hat{F}_{t-1}, \hat{s}_{t-1})$.

In the case of the single shock the proposed equilibrium is

(C.33b) $$\hat{F}_t = A^{\zeta}\hat{\zeta}_t + \lambda_1\hat{F}_{t-1} + [A^{\zeta_1} - \lambda_1 A^{\zeta}]\hat{\zeta}_{t-1} \quad , \quad A^{\zeta} \text{ is } 4\times 1 \quad , \quad A^{\zeta_1} \text{ is } 4\times 1.$$

The matching coefficient conditions for the case of a general state space are

$$B_1[A^{s}S + \lambda_1 A^{s} + A^{s_1}] = B_2 A^{s} + A$$
$$\lambda_1 A^{s_1} = B_1^{-1}B_2 A^{s_1}$$

which imply

$$A^{s}S + A^{S_1} = B_1^{-1}B_2 A^{S} + B_1^{-1}A$$

(C.33c) $$\lambda_1 A^{s_1} = B_1^{-1}B_2 A^{s_1}$$

This proves A^{s_1} is the first eigenvector of $B_1^{-1}B_2$ (see (C.15)) which is not unique and is written as

$$A^{s_1} = \mu C_1.$$

Hence the equilibrium map finally becomes

(C.34a) $$\hat{F}_t = A^{s}\hat{s}_t + \lambda_1\hat{F}_{t-1} + \mu C_1\hat{s}_{t-1}$$

where C_1 is the standardized column vector of C with $C_{1,4} = 1$ associated with $|\lambda_1| < 1$. Since μ is arbitrary, the equilibrium family defined in (C.33a) is of the order of the continuum. That is, since any value of μ generates a solution of (C.33c), the expression in (C.34a) is an equilibrium map for any value of μ, including $\mu = 0$. For this special case of an equilibrium with $\mu = 0$

(C.34b) $$\hat{F}_t = A^{s}\hat{s}_t + \lambda_1\hat{F}_{t-1}.$$

Inspection of (C.34a) show the deterministic component in (C.26b) is represented by (λ_1, μ) where one can choose any μ as long as λ_1 is fixed and equal to the smallest diagonal of J. One cannot set in (C.34a) the values of $\lambda_1 = 0$ and $\mu \neq 0$ since no such equilibrium exists. In other words, if there are compelling reasons to set $\lambda_1 = 0$ in (C.34a) then automatically one must also set $\mu = 0$. For this reason, to investigate which properties of (C.34a) are relevant, one studies first the properties of the proposed equilibrium $\hat{F}_t = A^s \hat{s}_t + \lambda_1 \hat{F}_{t-1}$. If a longer memory is rejected as a possible solution then one sets $\mu = 0$ automatically.

The issue at hand is the difference between the adjustment mechanisms of endogenous variables to shocks offered by the two equilibria proposed:

- The Standard Equilibrium Map $\hat{F}_t = A^s \hat{s}_t$
- Equilibrium with Endogenous Memory $\hat{F}_t = \tilde{A}^s \hat{s}_t + \lambda_1 \hat{F}_{t-1}$

The new $\tilde{A}^s$ emphasizes these are two different adjustment mechanisms to shocks: A^s for the standard equilibrium and $\tilde{A}^s$ for the one with extra memory.

In the first equilibrium endogenous variables adjust directly to shocks and in the typical case where $(\hat{F}_t, \hat{s}_t)$ are percentage deviations from steady state the parameters are exactly the elasticities of the corresponding shocks. In the second equilibrium the response is very different. Since $|\lambda_1| < 1$, forces of mean reversion are strong and $\hat{F}_t$ partly adjusts to the memory $\hat{F}_{t-1}$. Hence, since $\hat{s}_t$ and $\hat{s}_{t-1}$ are highly correlated $\hat{F}_t$ does not adjust to $\hat{s}_t$ but rather to $(\hat{s}_t - \hat{s}_{t-1})$. In short, these are two very different adjustment mechanisms and one can expect them to offer very different predictions of the behavior of endogenous variables. This suggests that the principle advocated at the end of Section 2.1 would be a practical way of asking which solution is more suitable. It states that

- *Gross failure of such implications or predictions is as viable empirical evidence against an equilibrium map as the empirical criterion that rejects an unbounded solution as an equilibrium.*

Since the model developed in the text has many complex features caused by the assumptions of diverse beliefs and inflexible wages, a more effective way of testing for the difference between the two equilibrium maps is to consider the simplest integrated model under Rational Expectations, flexible wages and a single technological shock. This is a well known model whose predictions are well known and hence it will offer a clear examination of the issues involved.

2.5 Endogenous Memory in a Simple New Keynesian Model

Under the specified assumptions

$$\hat{w}_t = \left(\frac{\eta}{1-\alpha} + \sigma\right)\hat{y}_t - \frac{\eta}{1-\alpha}\hat{\zeta}_t$$

and Phillips Curve

$$\hat{\pi}_t = \kappa\Lambda\left[\left(\frac{\eta}{1-\alpha} + \sigma\right)\hat{y}_t - \frac{\eta}{1-\alpha}\hat{\zeta}_t - \frac{1}{1-\alpha}(\hat{\zeta}_t - \alpha\hat{y}_t)\right] + \beta E_t\hat{\pi}_{t+1}$$

hence

$$\hat{\pi}_t = \kappa\Lambda\left(\frac{\alpha+\eta+\sigma(1-\alpha)}{1-\alpha}\right)\hat{y}_t + \beta E_t\hat{\pi}_{t+1} - \kappa\Lambda\frac{\eta+1}{1-\alpha}\hat{\zeta}_t.$$

Under this modification the equation system changes as follows:

$$\hat{F}_t = (\hat{\zeta}_t, \hat{y}_t^{U}, \hat{\pi}_t^{U}, \hat{y}_t^{L}, \hat{\pi}_t^{L})$$

The dynamical system is then written as

(C.35a) $$E_t[\hat{y}_{t+1}|U] + \frac{1}{\sigma}E_t(\hat{\pi}_{t+1}|U) = \left(1 + \frac{\xi_y}{\sigma}\right)\hat{y}_t^{U} + \frac{1}{\sigma}\xi_\pi(\hat{\pi}_t^{U}) - \frac{\xi_y}{\sigma}\frac{(1+\eta)}{[\alpha+\eta+\sigma(1-\alpha)]}\hat{\zeta}_t$$

(C.35b) $$\beta E_t[\hat{\pi}_{t+1}|U] = -\kappa\Lambda\left(\frac{\alpha+\eta+\sigma(1-\alpha)}{1-\alpha}\right)\hat{y}_t^{U} + \hat{\pi}_t^{U} + \frac{\kappa\Lambda}{1-\alpha}[1+\eta]\hat{\zeta}_t$$

(C.35c) $$E_t[\hat{y}_{t+1}|L] + \frac{1}{\sigma}E_t(\hat{\pi}_{t+1}|L) = \hat{y}_t^{L} - \frac{\bar{r}}{\sigma(1+\bar{r})}$$

(C.35d) $$\beta E_t[\hat{\pi}_{t+1}|L] = -\kappa\Lambda\left(\frac{\alpha+\eta+\sigma(1-\alpha)}{1-\alpha}\right)\hat{y}_t^{L} + \hat{\pi}_t^{L} + \frac{\kappa\Lambda}{1-\alpha}[1+\eta]\hat{\zeta}_t$$

which is written in the form

$$B_1(E_t\hat{F}_{t+1}) = B_2(\hat{F}_t) + A\hat{\zeta}_t$$

where

$$B_1 = \begin{bmatrix} \Omega_U & \frac{1}{\sigma}\Omega_U & (1-\Omega_U) & \frac{1}{\sigma}(1-\Omega_U) \\ 0 & \beta\Omega_U & 0 & \beta(1-\Omega_U) \\ (1-\Omega_L) & \frac{1}{\sigma}(1-\Omega_L) & \Omega_L & \frac{1}{\sigma}\Omega_L \\ 0 & \beta(1-\Omega_L) & 0 & \beta\Omega_L \end{bmatrix}$$

$$B_2 = \begin{bmatrix} [1+\frac{\xi_y}{\sigma(1+\bar{r})}] & \frac{\xi_\pi}{\sigma(1+\bar{r})} & 0 & 0 \\ -\kappa\Lambda(\frac{\alpha+\eta+\sigma(1-\alpha)}{1-\alpha}) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\kappa\Lambda(\frac{\alpha+\eta+\sigma(1-\alpha)}{1-\alpha}) & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -(\frac{\xi_y}{\sigma(1+\bar{r})}\frac{(1+\eta)}{[\alpha+\eta+\sigma(1-\alpha)]}) \\ \frac{\kappa\Lambda}{1-\alpha}(1+\eta) \\ 0 \\ \frac{\kappa\Lambda}{1-\alpha}(1+\eta) \end{bmatrix}$$

where $B_1^{-1}B_2 = CJC^{-1} \Rightarrow \hat{F}_t = CV_t$ and the matrix J has three eigenvalues outside the unit circle and one inside. In this case the transition for V_t^1 is

$$V_{t+1}^1 = \lambda_1 V_t^1 + M_1\hat{\zeta}_t$$

and the equilibrium with endogenous memory is then

$$\hat{F}_t = A^\zeta\hat{\zeta}_t + A^V V_t^1$$

Carrying out the transformation to lagged endogenous variables one finds that

$$\hat{F}_t = A^\zeta\hat{\zeta}_t + \lambda_1\hat{F}_{t-1} + [A^V M_1 - \lambda_1 A^\zeta]\hat{\zeta}_{t-1}.$$

Defining $A^{\zeta_1} = [A^V M_1 - \lambda_1 A^\zeta]$, the proposed equilibrium is written as

$$\hat{F}_t = A^\zeta\hat{\zeta}_t + \lambda_1\hat{F}_{t-1} + A^{\zeta_1}\hat{\zeta}_{t-1}.$$

Since by (C.34a) $A^{\zeta_1} = \mu C_1$ where C_1 is the first standardized column vector of C. The final equation is

$$\hat{F}_t = A^\zeta\hat{\zeta}_t + \lambda_1\hat{F}_{t-1} + \mu C_1\hat{\zeta}_{t-1}.$$

If $\mu = 0$ then

(C.36) $$\hat{F}_t = A^\zeta\hat{\zeta}_t + \lambda_1\hat{F}_{t-1}.$$

where

$$A^\zeta = [(\lambda_\zeta + \lambda_1)I - B_1^{-1}B_2]^{-1}(B_1^{-1}A).$$

Note that the standard equilibrium parameter vector A^ζ without the added memory is

$$[(\lambda_\zeta I - B_1^{-1}B_2]^{-1}(B_1^{-1}A),$$

which is different.

We next compute the two equilibria and compare them under the same parameters used in the text: β=0.99, $\alpha=\frac{1}{3}$, σ=0.90, η=1.0, θ=6, λ_ζ=0.90, $\Omega_1 = 0.9964$, $\Omega_2 = 0.96$, ξ_y=0.50, ξ_π=1.5. The value of ω in the text was ω = 0.40 due to inflexible wages but typical values used in the literature without inflexible wages are higher. We therefore report results for $0.40 \le \omega \le 2/3$.

The table below reports results of the two equilibrium maps for $\mu = 0$. Standard maps are designated by $\lambda_1 = 0$ while maps with endogenous memory are designated by $\lambda_1 > 0$.The first column reports the real part of the eigenvalue outside the unit circle and the next two columns report the constants defining the endogenous steady state, dominated by the force of 4.4% annual deflation in the lower sub-economy. All constants in the upper sub-economy's maps are not zero but are less in absolute values than 10^{-3}hence they are not reported.

- Columns 5-7 report parameters of the upper sub-economy's aggregates in the equilibrium with $\lambda_1 > 0$
- Columns 8-10 report parameters of the lower sub-economy aggregates in the equilibrium with $\lambda_1 > 0$
- Columns 11-13 report parameters of the upper sub-economy aggregates in the standard equilibrium
- Columns 14-16 report parameters of the upper sub-economy aggregates in the standard equilibrium.

In neither equilibrium do we introduce shocks that send the economy into the lower sub-economy but by decomposition, the effect of such shocks do not change the elasticity with respect to ζ. For the same reason the introduction of other shock will not change the parameters with respect to ζ.

Testing Standard Equilibria vs. Equilibria with Endogenous Memory

				$\lambda_1 > 0$						$\lambda_1 = 0$					
ω	λ_1	A_y^{0L}	A_π^{0L}	$A_y^{\zeta U}$	$A_\pi^{\zeta U}$	$A_w^{\zeta U}$	$A_y^{\zeta L}$	$A_\pi^{\zeta L}$	$A_w^{\zeta L}$	$A_y^{\zeta U}$	$A_\pi^{\zeta U}$	$A_w^{\zeta U}$	$A_y^{\zeta L}$	$A_\pi^{\zeta L}$	$A_w^{\zeta L}$
2/3	.72	-.004	-.011	-.37	.28	-.88	-2.08	.67	-5.00	.93	-.12	.72	1.24	.15	1.47
0.60	.66	-.003	-.011	-.47	.56	-1.14	3.24	-.97	7.78	.96	-.13	.80	1.16	.13	1.27
0.50	.56	-.001	-.010	-.33	1.15	-.80	1.22	-.35	2.93	.99	-.14	.88	1.10	.12	1.13
0.40	.46	-.001	-.010	.31	1.42	.74	1.05	-.29	2.51	1.01	-.15	.92	1.07	.12	1.07

The results for the standard equilibrium are as expected: for all values of ω, positive productivity increases output and wages in the upper and lower sub-economies. It lowers inflation in the upper sub-economy and reduces the deflationary pressure in the lower sub-economy. Positive productivity causes a positive adjustment of the wage rate which is stronger in the lower sub-economy than in the upper sub-economy because the zero interest rate under which adjustment of wages is the only mechanism available. Negative demand shocks that send the economy into the ZLB cause output, wages and inflation to fall, as seen in the constants in the table, but these declines do not depend upon the memory in the equilibrium map.

Results for the equilibrium with longer endogenous memory describe an entirely different economic environment . For almost all values of ω, a positive technology shock

- *lowers* output and wages in the upper sub-economy and increases output and wages in the lower sub-economy
- *increases* inflation and interest rates in the upper sub-economy and intensifies deflation in the lower sub-economy

This economy spends most of its time in the upper sub-economy for which ample empirical evidence is available to bolster one's expectation that the standard results of this economy should to be approximately replicated in equilibrium. Since we have only limited empirical evidence on the behavior of the lower sub-economy and the aim of the integrated model at hand is to study this sub-economy, the above statement is made without any judgment on how this lower sub-economy should behave. With this in mind one would expect that in this regular upper sub-economy higher productivity should increase output and wages. This is not the case in the proposed equilibrium adjustment with endogenous memory which exhibits rather perverse behavior and the introduction of $\mu > 0$ leaves these results the same hence this case does not need to be assessed separately.

We thus consider a direct question. What is the implied dynamic equilibrium adjustment under which an increased productivity in the upper sub-economy lowers output and wage? Does it correspond to any conceivable economy?

To explore the question recall that in the standard equilibrium with $\lambda_1 = 0$ the economy adjusts at date t only to $\hat{\zeta}_t$. Equilibrium adjustment of output, inflation and the wage rate respond to productivity in the familiar manner: output and wages rise while inflation falls. The interest rate typically fall since the output gap does not rise hence the lower inflation due to higher productivity dominate. Also recall our earlier observation that in a New Keynesian Model equilibrium must be viewed as an adjustment mechanism to external factors and in a solution with endogenous memory the key endogenous variables do not adjust primarily to $\hat{\zeta}_t$ since they must adjust relative to the history $\hat{F}_{t-1}$ which operates on these variables with persistence $\lambda_1 > 0$. The dominant effect on the dynamic adjustment is then the memory since by iterating (C.36) we can solve, say, for output by using the transition $\hat{\zeta}_{t+1} = \lambda_{\zeta}\hat{\zeta}_t + \rho^{\zeta}_{t+1}$ and deducing the two forms the solution takes:

Form 1: $$y^{t} = A^{\zeta}_{y}[\hat{\zeta}_t + \lambda_1\hat{\zeta}_{t-1} + \lambda_1^2\hat{\zeta}_{t-2} + \ldots + \lambda_1^t\hat{\zeta}_0]$$

Form 2: $$y_t^{t} - \lambda_1\hat{y}_{t-1} = A^{\zeta}_{y}\zeta_t = A^{\zeta}_{y}[\hat{\rho}^{\zeta}_t + \lambda_{\zeta}\hat{\rho}^{\zeta}_{t-1} + \lambda_{\zeta}^2\hat{\rho}^{\zeta}_{t-2} + \ldots + \lambda_{\zeta}^t\hat{\zeta}^{\zeta}_0]$$

Form 1 shows that date t output adjusts to the weighted average of past productivity with weights of λ_1^k while solution form 2 shows date t output actually adjusts to date t-1 output when $y_t - \lambda_1\hat{y}_{t-1}$ changes in response to a weighted average of past zero mean i.i.d. productivity innovations.

The crucial market mechanism that generates this adjustment is the *effect of productivity on market expectations*. $\hat{\zeta}_t > 0$ generates a sharp rise in inflation expectations, a sharp rise in the interest rate and a sharp decline in aggregate demand, followed by a fall in output employment and wages. The central cause of this outcome is that *current inflation is more sensitive to inflation expectations than to actual cost of production* which are, in fact, lower in response to $\hat{\zeta}_t > 0$ and such higher inflation raises the interest rate by so much that aggregate demand and employment fall even while wages are lower. The equilibrium map in the table above records the reduced form net effect that $\hat{\zeta}_t > 0$ cause a decline in output, employment and wages.The structure of expectations is completely reversed under the ZLB when the interest rate cannot play the adjusting role it plays in the above discussion. Now a $\hat{\zeta}_t > 0$ generates a sharp fall in inflation expectations which exacerbates the deflationary effects of the ZLB. Hence, equilibrium with endogenous memory implies that a factor that reduces the ZLB's deflationary effects are large declines in labor productivity. In sum, in this equilibrium inflation expectation dominate the market adjustment and the long memory is the mechanism for transmitting the market signals necessary for this adjustment. With this clarified, we can now turn to the second question, whether this is a plausible equilibrium mechanism.

The equilibrium in question is drastically inefficient as it generates large negative output gaps whose size grows with $\hat{\zeta}_t > 0$. To see why recall that the gap is

$$\text{Gap} = \hat{y}_t - \hat{y}_t^{F} = \hat{y}_t - \frac{(1+\eta)}{[\alpha+\eta+\sigma(1-\alpha)]}\hat{\zeta}_t \,.$$

Since given $\hat{\zeta}_{t-1}$, the value of $\hat{y}_t$ falls with $\hat{\zeta}_t$ while $\hat{y}_t^{F}$ declines with $\hat{\zeta}_t$, large $\hat{\zeta}_t > 0$ often generate large negative output gaps. Due to the slow adjustment to the memory, this economy has deep cycles where typical recessions last 5-7 years.

This examination leads to the observation that the solution (C.26b) is a proper mathematical solution but any in considering the relevance of any solution one consider additional economic reasoning or empirical evidence not specified in the original formulation. The condition requiring a solution to be bounded is an empirically based restriction which is not present at the formulation of the initial expectation difference equation. Rejecting a solution from being an equilibrium on the ground that it calls for output and wages to fall under the normal conditions of the upper sub-economy in response to an increased productivity is as justified as a rejection of a solution on the ground that it requires the equilibrium endogenous variables to become unbounded.

References

Blanchard, O. J. (1979): "Backward and Forward Solutions for Economies with Rational Expectations." *American Economic Review* **69**, 114-118.

Blanchard, O. J., Watson, M. (1982): "Bubbles, Rational Expectations and Financial Markets." In *Crises in the Economics and Financial Structure*, ed. P. Wachtel, Lexington, MA: Lexington Books.

Blanchard, O. J., Fischer, S. (1989): *Lectures on Macroeconomics*. Cambridge, MA: MIT Press.

Blanchard, O. J., Kahn, C.M. (1980): "The Solution of Linear Difference Models under Rational Expectations." *Econometrica* **48**, 1305 - 1311.

Cagan, P. (1956): "The Monetary Dynamics of Hyperinflation." In *Studies in the Quantity Theory of Money*, ed. M. Friedman, Chicago: University of Chicago Press.

Taylor, J. B. (1977): "Conditions for Unique Solutions in Stochastic Macroeconomic Models with Rational Expectations." *Econometrica* **45**, 1-44.

Tirole, J. (1982): "On the Possibility of Speculation under Rational Expectations." *Econometrica* **50**, 1163-1181.

Tirole, J. (1985): "Asset Bubbles and Overlapping Generations." *Econometrica* **53**, 1499-1528.

Sargent, T., Wallace, N. (1973): "The Stability of Models of Monetary Growth with Perfect Foresight." *Econometrica* **41**, 1043-1048

Sargent, T., Wallace, N. (1975): "Rational Expectations, the Optimal Monetary Instrument and the Optimal Money Supply Rule." *Journal of Political Economy* **83**, 241-255