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The Kesten-Stigum Model and the Treatment of Uncertainty in Equilibrium Theory

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CHAPTER 13

THE KESTEN-STIGUM MODEL AND THE TREATMENT OF UNCERTAINTY IN EQUILIBRIUM THEORY

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13.1. *Introduction*

The treatment of uncertainty in social theory has always been a subject for diverse viewpoints and the present time is no exception. The view of uncertainty adopted in the papers by Kesten-Stigum and Radner which were presented in this meeting raises some very sharp questions. I shall devote the first section below to reviewing the Kesten Stigum paper and then turn to the broader issues which this theory raises.

13.2. *Review of the Kesten-Stigum Theory*

We note first that the theory at hand has been developed in two papers. The first is an earlier paper by Stigum [8] entitled 'Balanced growth under uncertainty' and the second is the one by Kesten and Stigum entitled 'Balanced growth under uncertainty in decomposable economies'. These papers represent an extension of the earlier Samuelson-Solow [7] theory of balanced growth to the case in which the environment is random.

The model describes the random evolution of the aggregate stock vector $\boldsymbol{x}(t)$ in the economy, where the random variable $\boldsymbol{x}(t)$ must satisfy

$$E\{\boldsymbol{x}(t+1) \mid \boldsymbol{x}(0), \boldsymbol{x}(1), \ldots, \boldsymbol{x}(t)\} = \boldsymbol{H}(\boldsymbol{x}(t)),$$

* In preparing these remarks I benefited from conversations and correspondence with K. J. Arrow to whom I am indebted.

where $\boldsymbol{H}(\boldsymbol{x}(t))$ is a function which incorporates the technology, resources and rules of intertemporal allocations. It is to be understood that in the uncertainty case the economy has a vector $\boldsymbol{x}(t)$ of stocks which are used to produce output: part of this output is consumed and part reinvested. Thus the relation $\boldsymbol{x}(t+1) = \boldsymbol{H}(\boldsymbol{x}(t))$ means that the state of technology and resources as well as rules of production and consumption decisions are part of the function $\boldsymbol{H}(\boldsymbol{x}(t))$. Since in the uncertainty version of this model the conditional expectation of $\boldsymbol{x}(t+1)$ is $\boldsymbol{H}(\boldsymbol{x}(t))$ it is clear that the only uncertainty allowed here is that which arises in the exogenous environmental conditions provided they do not influence such structural elements which are part of $\boldsymbol{H}(\cdot)$. In particular, such random processes as improvements in the state of technology, or the increase in the availability of natural resources, are not allowed. Turning now to the papers themselves, it may appear that the results under the condition of indecomposability are different from those under the condition of decomposability. In fact this is not the case, and results are essentially the same provided the symbols and the concepts are properly interpreted. Let us review this issue now and indicate below in what way the results are different for the two cases.

(a) First we note that the model of 'decomposability' as proposed here is not as general as described by the authors and the concept of 'triangular' would probably be more appropriate. This follows from the fact that $\boldsymbol{H}(\cdot)$ has a 'two sector' structure of the form $\boldsymbol{H} = (\boldsymbol{H}^1, \boldsymbol{H}^2)$ where $H_i^1 = H_i$ if $1 \leq i \leq d_1$ and $H_i^2 = H_i$ if $d_1 + 1 < i < d$ and the two functions are defined over $\boldsymbol{x} = (\boldsymbol{x}^1, \boldsymbol{x}^2)$ as follows:

$$H_i = H_i(\boldsymbol{x}^1) \text{ for } 1 \leq i \leq d_1,\ \boldsymbol{x}^1 \text{ is } d_1\text{-dimensional},$$

$$H_i = H_i(\boldsymbol{x}^1, \boldsymbol{x}^2) \text{ for } d_1 + 1 \leq i \leq d,\ \ \boldsymbol{x}^2 \text{ is } (d - d_1)\text{-dimensional},$$

The sector 1 is an indecomposable sector composed of goods (industries) $1 \leq i \leq d_1$ and sector 2 is an indecomposable sector composed of goods (industries) $d_1 + 1 \leq i \leq d$.

(b) The basic theory proved by the authors of the two papers is the same for both the indecomposable and the decomposable cases. This theory can be very generally stated as follows: Given certain regularity conditions on $\boldsymbol{H}(\cdot)$ and the very strong 'bounded variances' conditions on the stochastic process $\boldsymbol{x}(t)$, there exist a positive real number λ and

a non-negative, non-zero vector $\boldsymbol{V}$ and a random variable λ such that

$$\text{(i)}\ \lim_{t\to\infty} \frac{\boldsymbol{x}(t)}{\lambda^t} = g\boldsymbol{V},$$

$$\text{(ii)}\ E\{g \mid \boldsymbol{x}(0)\} < \infty.$$

This means that asymptotically the uncertainty regarding the growth rate vanishes while the uncertainty regarding the 'level' of the growth path is present but not essential. Thus we have again the basic conclusion similar to those obtained in other cases that for any given economy the uncertainty regarding the growth path will vanish with the passage of time. These results extend the Samuelson Solow results to the case of uncertainty and represent a nice analytical achievement by the authors.

(c) The difference between the indecomposable and the decomposable cases are to be found in the values that λ and $\boldsymbol{V}$ take. In the indecomposable case, λ and $\boldsymbol{V}$ are simply the eigenvalue and vector of $\boldsymbol{H}(\cdot)$; thus $\boldsymbol{H}(\boldsymbol{V}) = \lambda\boldsymbol{V}$ and with the regularity conditions on $\boldsymbol{H}$ we have $\lambda > 0$ and $\boldsymbol{V} \gg 0$.

In the indecomposable case we let $\lambda_1, \lambda_2, \boldsymbol{V}^1, \bar{\boldsymbol{V}}^2$ and $\boldsymbol{V}^2$ be defined as follows:

$$\text{(i)}\ \ \boldsymbol{H}^1(\boldsymbol{V}^1) = \lambda_1\boldsymbol{V}^1,$$
$$\text{(ii)}\ \ \boldsymbol{H}^2(\boldsymbol{0}, \bar{\boldsymbol{V}}^2) = \lambda_2\bar{\boldsymbol{V}}^2,$$
$$\text{(iii)}\ \ \boldsymbol{H}^2(\boldsymbol{V}^1, \boldsymbol{V}^2) = \lambda_1\boldsymbol{V}^2,$$

where the existence of all these scalars and vectors is proved. It is clear that λ_1 and $\boldsymbol{W} = (\boldsymbol{V}^1, \boldsymbol{V}^2)$ are the eigenvalue and vector of $\boldsymbol{H}(\cdot)$; thus $\boldsymbol{H}(\boldsymbol{W}) = \lambda_1\boldsymbol{W}$. However, λ_1 is not necessarily the balanced growth rate and $\boldsymbol{W}$ is not necessarily the balanced proportions. The results depend upon the relations between λ_1 and λ_2. The Kesten–Stigum theory applies only to the cases where $\max[\lambda_1, \lambda_2] > 1$ and $\lambda_1 \neq \lambda_2$.

Now, if $\lambda_1 > \lambda_2$ then the results of the decomposable case are identically the same as those of the indecomposable case: λ_1 is the asymptotic growth rate and $\boldsymbol{V} = \boldsymbol{W} = (\boldsymbol{V}^1, \boldsymbol{V}^2)$ is the vector of asymptotic proportions.

If $\lambda_1 < \lambda_2$, then the interpretation is different: the asymptotic growth rate of the economy becomes λ_2 and the first sector becomes asymptotically insignificant. The theory states that the second sector is asymptotically self-sustaining with the vector of relative stock proportions of $V = (0, \bar{V}^2)$.

(d) Finally, we note that the conditions in Professor Stigum's paper are essentially the same conditions used by Professors Kesten and Stigum in their paper:

(i) the critical bounded variances conditions and the regularity conditions on $H(\cdot)$ are essentially the same;

(ii) Professor Stigum's condition requiring the conditional probability of $x(t+1)$ to be smaller (larger) than $H(x(t))$ in case λ is smaller (larger) than 1 is replaced in the decomposable case by the conditions on the function M and the initial condition $x(0)$; the function of all these conditions in the proofs is the same; and

(iii) the concavity condition on $H(\cdot)$ assumed by Professor Stigum is slightly altered: for the proof of convergence it is assumed, in the second paper, that $|H(x) - H(y)| \leq K|x - y|$ while for the proof that $0 < E\{g \mid x(0)\} < \infty$ Professors Kesten and Stigum assume concavity as well.

Leaving additional technical matters aside, I note that the uncertainty underlying the Kesten-Stigum universe and the one operating in the Radner [6] model presented in this meeting are the same. It is the random process of the environment which generates the random consequences of economic decisions. This fact is true in spite of the fact that the Kesten–Stigum economy is supposed to describe a decentralized economy while the Radner theory deals with central planning processes.

It may be argued that for the theory of planning the only relevant uncertainty is the one regarding the environment. This is probably true for a completely centralized economy where all decisions, including investment and consumption decisions for each individual, are made in the center. Such economies are of little interest, mostly because they do not exist. If we restrict our attention to decentralized economies with price guided allocation mechanism, then it is not clear at all that the important uncertainty arises from the random nature of the environment. In fact, I suspect that in competitive economies the random nature of the environment which gives rise to 'exogenous uncertainty' is probably small compared to the all-important endogenous uncertainty which we shall discuss now.

13.3. Exogenous vs Endogenous Uncertainties

The traditional theory of equilibrium under uncertainty was developed by Arrow and Debreu. In this theory uncertainty exists with respect to 'the state of the world'. This is usually viewed as the availability of resources and the possibilities of production and consumption. In this theory a set of contingency markets are to be established in which, for each set of prices for contingency contracts, individuals will buy and sell future commodities and services contingent upon the state of the world. This is a very good description of the insurance world: the consumer pays a price today for a delivery of a new home by the insurance company if his house burns down. The traditional Arrow–Debreu model was criticized by many writers, including Radner [5], who proposed an extension of the traditional model to allow different individuals to have different information available to them. Radner's analysis leads to the conclusion that, because of such differences in information, many contingency markets may not function. This is a generalization of the phenomenon of 'moral hazard' (see ref. [1]). There are other reasons, such as transaction cost, which may prevent the achievement of a complete system of contingency markets. However, all the attempts at modifying the Arrow–Debreu view of uncertainty are only designed to allow the individual decision maker to cope with a rather incredible task of specifying his utility of consumption in every state of the world and establishing his probability distribution over all states of the world. Note that, if consumer preferences are random, then the configuration of individual preferences is part of 'the state of the world'. This means that each participant must establish his probability distribution of all participant's preferences; there must exist markets in which delivery is contingent upon the actual preferences of each individual participant, and these must be revealed after the actual selection takes place.

Radner's solution to the above is simply to assume that individuals do not and cannot know certain things, and thus they form their supply and demand correspondences under conditions of ignorance. In such circumstances it is safe to assume that no contract will be signed between agents 1 and 2 where delivery will be contingent upon the utility function of agent 3. But this obviously means that the utility function of individual 3 will remain an endogenous random variable which gives rise to risk that cannot be insured against. Also, in many instances, the cost of establishing the precise state of the world, once it has occurred, may be very high. If the marginal gain from the functioning of this contingency market is less than the cost of producing the information, the market will not function.

Thus, all in all, diversity of information, the phenomenon of 'moral hazard', transaction cost, and the cost of establishing 'the state of the world', all contribute to the failure of contingency markets to function. This in turn gives rise to risks and uncertainties which, in many cases, the individual agent can neither insure against nor purchase more.

The above are examples of what I would regard as 'endogenous uncertainty' in the sense that this uncertainty is either created within the system or is a reflection of the internal functioning of the economy. These are not the only cases of endogenous uncertainty: we turn now to discuss other forms of such uncertainty and their consequences.

13.4. Endogenous Uncertainty and its Consequences

Uncertainty is related in an essential way to the sequential nature of economic activity. For this reason individual decision makers can discover from their own experience the value of knowledge and information. The Radner consumer who is assumed to have a fixed information structure will learn two fundamental facts: first, that he may produce or purchase information, and second, that what may be regarded by all people as inconceivable may indeed happen.

The above suggests that when information may be acquired or purchased, the amount of individual uncertainty becomes an endogenous element. This means that in his private decision making the consumer may elect to go to school or to engage the informational services of specialists in order to reduce his risk. Firms may select the amount of technological uncertainty by spending resources on research, and the risk regarding the availability of resources may depend on the resources spent on exploration. This endogenous uncertainty is amplified by the fundamental non-convexity in this economic activity: on the supply side, produced information has the character of public goods. On the demand side there is a problem in defining the demand for information when

one does not know the information one buys (see for example, ref. [1, chapter 12]).

On the other extreme, the Radner agent who operates under conditions of ignorance may discover that his basic probability space was 'wrong' in the sense that certain events which he did not conceive of did happen: perhaps the most spectacular events are precisely those which very few people had conceived of. This is inconsistent with the behavior of Radner's agent since, according to Radner, the basic probability space is known to each consumer and he has a well defined utility which depends upon each set of states of the world. This may lead an individual to assign subjective probability to all those unknown events that may happen but whose nature is unknown. If this is so, the typical individual will have a probability measure of less than 1 on the space of all events which he may conceive of. Individual reinforcement to this view arises from the occurrence of events which 'nobody conceived of'.

We finally arrive at the most fundamental endogenous uncertainty which is associated with any economic activity: the uncertainty regarding the capacity, managerial skills and qualities of other agents.

Owners of capital know that most of their capital is managed by other individuals, and primary among the sources of the return to their capital are the managerial skills and success of those who use the capital. Workers who sell their services to an employer will have a basic uncertainty regarding the nature, quality and duration of their employment which will depend upon the abilities of the employer. Further, in the provision of any human service, from the repair of your car to a medical diagnosis, there is a fundamental uncertainty due to the fundamental random nature of human decisions. And finally, in signing any contract there is always the uncertainty regarding the desire and ability of agents to deliver.

The essence of the above examples is that individual capacity for optimal action entails qualities which are basic random variables. The random variations in labor and managerial performance is a fundamental endogenous uncertainty which is probably the most important uncertainty in any social system. Note that some of the above uncertainties do give rise to some kind of 'contingency' markets: unemployment insurance and insurance of the functioning of private durable goods are examples[1]. On the other hand there is no insurance against a fall in price of a commodity caused by excessive capacity in an industry. Here a lag in market adjustment – common in almost all industries – may lead to excessive rate of entry or exit. This is a typical random variable resulting from the nature of production, the organization of the markets and the ability of management. It is obvious that the rate of entry and exit is a very important uninsurable random variable to any profit maximizing firm.

The fundamental consequence of the existence of endogenous uncertainty is the same as the consequences of any source of uncertainty for which there are no contingency markets functioning. These consequences are simple: individuals must bear their own uncertainty and they cannot capitalize their future endowments, assets or obligations. On the other hand, the markets will reopen in the next period and there are commodities, services or contracts that will be traded during the next market date and for which prices will be available in the next market date but for which no futures markets and prices are available today. The above leads to the simple observation that, in spite of the fact that many futures markets are not functioning today and many futures prices are not available today, the consumer forms expectations about prices tomorrow and those expectations become an important basis for decision making today! It is then clear that from the viewpoint of the individual consumer functioning in *today's* market, *prices tomorrow are random variables.* Apart from the consumer's ability to forecast prices tomorrow, it is perfectly reasonable for him to make contracts *which are contingent on market prices tomorrow*! Thus a bakery can make contracts for future delivery of bread contingent on the prices of flour. In this case the bakery can also purchase contracts for the delivery of wheat at a later date and thus make a firm sale of future bread. However, a travel agent might make a touring contract with his clients contingent upon the airline and hotel prices remaining unaltered. Most contracts which are made contingent upon future market prices are usually made contingent upon 'prices not changing' and then certain rules for renegotiation in case changes do take place. Thus all contracts with escalation clauses related to inflation are in fact examples of such contracts. Also all options, put and call contracts can be interpreted as

[1] See below for further discussion.

contracts to buy or sell something in the *future* contingent upon the market price prevailing at that time. Thus consider my purchasing a six-month option to buy a certain asset at a fixed price p_0. This contract is *equivalent* to a contract to deliver the same asset at the future date, delivery being contingent upon $p > p_0$. If $p < p_0$ then the buyer does not want the delivery to take place. In general it appears that contracts for future delivery contingent upon future prices are much more common than is generally assumed.

13.5. Uncertainty and Expectations: a Search for a New Equilibrium Concept

One simple fact must be stated at this point: given the fact that the Arrow–Debreu contingency markets deal with a relatively small amount of uncertainty, the bulk of social uncertainty is left to be dealt with in a different manner. There are first of all those risks which must be born by the individual consumer since neither contingency markets nor their social substitutes can function. This leads to a fundamental endogenous uncertainty. Then there are all the social substitutes for contingency markets which are markets in which uncertainty is defined with respect to events, i.e. set of states. Thus there may not exist insurance against the detailed configuration of other agents' conduct, but there exists insurance against malpractice suits; there may not exist contingency contracts which may specify the possible events that may happen to a firm and its employees, but there exists an insurance against unemployment. These are examples of the diverse methods of transferring risk through markets which are not contingency markets in the Arrow–Debreu sense; risk is defined in these markets in terms of events.

The most important consequence of the failure of contingency markets to function is the translation of a great deal of this form of untraded risk into uncertainty about prices which will prevail in future markets. This is clearly not included in the notion of uncertainty regarding the 'state of the world'. However, from the point of view of the individual agent operating in a world of incomplete markets, this is a real endogenous risk and gives rise to markets in which this uncertainty is traded. This trading follows from the fact that the presence of this uncertainty creates expectations and diversity of expectations, endowments and tastes create an extensive market for trade in this uncertainty. This is one way of looking at securities markets: in this context an investing individual who purchases a position in a mutual fund is purchasing essentially the same service of reducing his risk as the insurance policy which insures his home.

The extensive market for financial intermediaries contains important segments in which the risk of future variations in prices is traded. Moreover, given the difficulties which we have enumerated with regard to the 'state of the world' model of risk *it is probably the simplest and most efficient procedure to allow individuals to trade their expectations of future market prices*. We are thus led to the inevitable rejection of the Arrow–Debreu model of equilibrium under uncertainty. A proper theory of equilibrium under uncertainty will have to be developed in the context of an economy:

(1) with a sequence of markets;

(2) where individuals are allowed to make contracts contingent upon prices tomorrow; and

(3) where individual price expectations are formed endogenously and influence the allocation today.

It is important to note that in principle what is emphasized here is that we should extend the notion of the 'state of the world' to include future prices. Since Hicks, it has been well understood that a theory of plans and expectations must be integrated into the theory of general equilibrium. Recent work in the theory of temporary equilibrium has incorporated some of the ideas presented here (see for example, refs. [2] and [3]). In all of these contributions it has been assumed that there exists a *fixed* expectational function which assigns to any vector $\boldsymbol{p}$ of today's prices a probability distribution of tomorrow's prices. With this device it is then possible to define equilibrium relative to the fixed expectation function. This is most unsatisfactory, since there are reasons why expectations are formed, and some of these reasons were discussed above; to assume a fixed expectation function does not resolve any issue at all. Since it is clearly understood that the structure of expectations today can have dramatic effects on the allocation today, the assumption of an arbitrary individual expectation function is almost equivalent to the statement that any allocation may be an equilibrium. The fundamental issue is the endogenous formation of expectations. This is an open question.

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