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Determinants of stock market volatility and risk premia*

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Summary. We show the dynamics of diverse beliefs is the primary propagation mechanism of volatility in asset markets. Hence, we treat the characteristics of the market beliefs as a primary, primitive, explanation of market volatility. We study an economy with stock and riskless bond markets and formulate a financial equilibrium model with diverse and time varying beliefs. Agents' states of belief play a key role in the market, requiring an endogenous expansion of the state space. To forecast prices agents must forecast market states of belief which are beliefs of "others" hence our equilibrium embodies the Keynes "Beauty Contest." A "market state of belief" is a vector which uniquely identifies the distribution of conditional probabilities of agents.

Restricting beliefs to satisfy the rationality principle of Rational Belief (see Kurz, 1994, 1997) our economy replicates well the empirical record of the (i) moments of the price/dividend ratio, risky stock return, riskless interest rate and the equity premium; (ii) Sharpe ratio and the correlation between risky returns and consumption growth; (iii) predictability of stock returns and price/dividend ratio as expressed by: (I) Variance Ratio statistic for long lags, (II) autocorrelation of these variables, and (III) mean reversion of the risky returns and the predictive power of the price/dividend ratio. Also, our model *explains the presence of stochastic volatility in asset prices and returns.* Two properties of beliefs drive market volatility: (i) rationalizable over confidence implying belief densities with fat tails, and (ii) rationalizable asymmetry in frequencies of bull or bear states.

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1 Introduction

The forces which determine equilibrium market volatility and risk premia are probably the most debated topics in the analysis of financial markets. The debate is driven, in part, by empirical evidence of market "anomalies" which have challenged students of the subject. Consumption based asset pricing theory has had a profound impact on our view of financial markets. Early work of Leroy (1973), Lucas (1978), Breeden (1979), Grossman and Shiller (1981), Mehra and Prescott (1985), Hansen and Singleton (1983) and others show how intertemporal optimization of investor incorporates a subtle relationship between consumption growth and asset returns. However, when examined empirically, this simple relationship fails to provide a correct *quantitative* measure of risk premia. The Equity Premium Puzzle (see, Mehra and Prescott, 1985) is a special case of the fact that assets prices are more volatile than can be explained by "fundamental" shocks.

Difficult to account for risk premia are not confined to the consumption based asset pricing theory; they arise in many asset pricing models. Three examples will illustrate. The Expectations Hypothesis is rejected by most studies of the term structure (e.g. Backus, Gregory and Zin, 1989; Campbell and Shiller, 1991) implying an excessive risk premium on longer maturity debt. Foreign exchange markets exhibit a "Forward Discount Bias" (see Froot and Frankel, 1989; Engel, 1996) implying an unaccounted for risk premium for holding foreign exchange. The pricing of derivatives generates excess implied volatility of underlying securities, resulting in unaccounted for risk premia. Some treat these as unusual anomalies and give them distinct labels, presuming standard models of asset pricing can account for all normal premia. Our perspective is different.

We suggest market volatility and risk premia *are primarily determined by the structure of agents' expectations* called "market state of belief." Diversity and dynamics of beliefs are then the root cause of price volatility and the key factor explaining risk premia. Agents may be "bulls" or "bears." A bull at date t expects the date $t+1$ rate of return on investments to be higher than normal, where "normal" is defined by the empirical distribution of past returns. Date t bears expect returns at $t+1$ to be lower than normal. Agents do not hold Rational Expectations (in short RE) since the environment is dynamically changing, non stationary, and true probabilities are unknown to anyone. In such complex environment agents use subjective models. Some consider these agents irrational, but one cannot require them to know what they cannot know: there is a wide gulf between an RE agent and irrational behavior. We explore the structure of economies with diverse beliefs and show they must have an expanded state space. Our computing models assume agents hold rational beliefs in accord with the theory of Rational Belief Equilibrium (in short, RBE) due to Kurz (1994) and explored in Kurz (1996, 1997). This rationality principle strikes a proper balance between RE and irrational behavior. To assist readers unfamiliar with this theory, we briefly explain it.

The Rational Belief Principle. "Rational Belief" (in short, RB) *is not a theory that demonstrates rational agents should adopt any particular belief.* Indeed, since the RB theory explains the observed belief heterogeneity, it would be a contradiction to propose that a particular belief is the "correct" belief agents should adopt. The RB theory starts by observing that the true stochastic law of motion of the economy is non-stationary with structural breaks and complex dynamics hence the probability law of the process is not known. Agents have a long history of past data generated by the process which they use to compute relative frequencies of finite dimensional events hence all finite moments. With this knowledge they compute the empirical distribution and use it to construct an empirical probability measure over sequences. Kurz (1994) shows the estimated probability model is stationary and hence it is called the "empirical measure" or the "stationary measure."

In contrast with a Rational Expectations Equilibrium (in short, REE) where the true law of motion is known, agents in an RBE who do not know the truth, form subjective beliefs based only on observed data. Hence, any principle on the basis of which agents can be judged as rational must be based *only on the data.* Since a "belief" is a model of the economy together with a probability over sequences of variables, it can be used to generate artificial data. With simulated data an agent can compute the empirical distribution of observed variables. The RB theory then proposes a simple Principle of Rationality. It says that if an agent's model generates an empirical distribution which is not the same as the one known for the economy, then the agent's model (i.e. "belief") is irrational. The RB rationality declares a belief to be rational only if it is a model which cannot be disproved with the empirical evidence. Since diverse theories are compatible with the evidence, this rationality principle allows diversity of beliefs among equally informed rational agents. For agents holding RB date t theoretical moments may deviate from the empirical moments but the RB rationality principle requires the *time average of all theoretical moments to equal the empirical moments*. In particular, date t forecasts may deviate from empirical forecasts but the time average of all forecasts must agree with the empirical forecasts. It follows *as a theorem* that agents who hold rational beliefs must have forecast functions which vary over time. The tool we use to describe the beliefs of agents is the "market state of belief." It is explained in details in this paper.

The RB rationality is compatible with several known theories. An REE is a special case of an RBE. Most models of Bayesian learning satisfy the RB rationality principle. Also, several models of Behavioral Economics satisfy this principle for some parameter values.

Earlier papers using the RBE rationality principle have also argued that agents' beliefs are central to explaining market volatility (e.g. Kurz, 1996, 1997a; Kurz and Schneider, 1996; Kurz and Beltratti, 1997; Kurz and Motolese, 2001; Kurz and Wu, 1996; Nielsen, 1996). These papers aimed to explain a list of financial "anomalies." (for a unified treatment see Kurz and Motolese, 2001). The RBE theory was used by Kurz (1997b) and Nielsen (2003) to explain the volatility of foreign exchange rates. These papers used OLG models where exogenous shocks are discrete and agents have a finite set of belief states. Wu and Guo (2003, 2004) study speculation and trading in the steady state of an infinite horizon model. This paper's contribution consists of four parts: (i) ours is an infinite horizon model, (ii) all random variables are continuous, (iii) AR(1) processes describe beliefs and exogenous shocks, and (iv) we explicitly model agents' beliefs about the market state of belief, which are beliefs about the beliefs of others. We argue that this is the crucial property needed for understanding the volatility and risk premia in financial markets.

The Main Results. First, "Belief states" are developed as a tool for equilibria with diverse belief. Next, our method is to use properties of market beliefs as primitive explanation of volatility. Two characteristics of beliefs fully account for all features of volatility and premia observed in markets:

(A) *high intensity of fat tails in the belief densities of agents;*
(B) *asymmetry in the proportion of bull and bear states in the market over time.*

High intensity means agents exhibit rationalizable over confidence with fat tails in their subjective densities. *Asymmetry* in frequency of belief is a characteristic which says that on average, at more than half of the time agents do not expect to make excess returns. Our model also *implies market returns must exhibit stochastic volatility* which is generated by the dynamics of market belief.

2 The economic environment

The economy has two types of agents and a large number of identical agents within each type. An agent is a member of one of the two types of infinitely lived dynasties identified by their endowment, utility (defined over consumption) and by their belief. A dynasty member lives a fixed short life and during his life makes decisions based on his own belief without knowing the states of belief of his predecessors. He is replaced by an identical member. There are two assets: a stock and a riskless, one period, bond. There is an aggregate output process $\{Y_t, t=1,2,\ldots\}$ which is divided between dividends $\{D_t, t=1,2,\ldots\}$ paid to owners of the common stocks and non - dividend endowment which is paid to the agents. The dividend process is described by

$$D_{t+1} = D_t e^{x_{t+1}} \tag{1}$$

where $\{x_t, t=1,2,\ldots\}$ is a stochastic process under a true probability *which is non-stationary with structural breaks and time dependent distribution*. This time varying probability is not known by any agent and is not specified. Instead, we assume $\{x_t, t=1,2,\ldots\}$ is a stable process[1] hence it has an *empirical distribution*

[1] A *Stable Process* is defined in Kurz (1994). It is a stochastic process which has an empirical distribution of the observable variables defined by the limits of relative frequencies of finite dimensional events. These limits are used to define the empirical distribution which, in turn, induces a probability measure over infinite sequences of observables which we refer to as the "stationary measure" or the "empirical measure." A general definition and existence of this probability measure is given in Kurz (1994), (1997) or Kurz and Motolese (2001) where it is shown that this probability must be stationary. Statements in the text about "the stationary measure" or "the empirical distribution" is always a reference to this probability. Its centrality arises from the fact that it is derived from public information and hence *the stationary measure is known to all agents and agreed upon by all to reflect the empirical distribution of observable equilibrium variables.*

which is known to all agents who learn it from the data. This empirical distribution is represented by a stationary Markov process, with a year as a unit of time, defined by[2]

$$x_{t+1} = (1-\lambda_x)x^* + \lambda_x x_t + \rho^x_{t+1} \quad \text{with} \quad \rho^x_{t+1} \sim N(0, \sigma^2_x) \quad \text{i.i.d.} \tag{2}$$

The infinitely lived agents are enumerated j = 1, 2 and we use the following notation:

C^j_t - consumption of j at t;
θ^j_t - amount of stock purchased by j at t;
B^j_t - amount of one period bond purchased at discount by agent j at t;
$\tilde{q}^s_t$ - stock price at date t;
q^b_t - the discount price of a one period bond at t;
Λ^j_t - non capital income of agent j at date t;
H_t - information at t, recording the history of all observables up to t.

Given probability belief Q^j_t, agent j selects portfolio and consumption plans to solve the problem

$$\underset{(C^j, \theta^j, B^j)}{\text{Max}} \; E_{Q^j}\Big[\sum_{t=0}^{\infty} \beta^t \frac{1}{1-\gamma}(C^j_t)^{1-\gamma} | H_t\Big] \tag{3a}$$

subject to:

$$C^j_t + \tilde{q}^s_t \theta^j_t + q^b_t B^j_t = \Lambda^j_t + (\tilde{q}^s_t + D_t)\theta^j_{t-1} + B^j_{t-1}. \tag{3b}$$

We assume additively separable, power utility over consumption, a model that failed to generate premia in other studies (see Campbell and Cochrane, 2000). We focus on diverse beliefs hence assume the two utility functions are the same. Introduce the normalization for $j = 1, 2$

$$\omega^j_t \equiv \frac{\Lambda^j_t}{D_t}, \quad c^j_t \equiv \frac{C^j_t}{D_t}, \quad q^s_t \equiv \frac{\tilde{q}^s_t}{D_t}, \quad b^j_t \equiv \frac{B^j_t}{D_t}.$$

With this normalization the budget constraint becomes

$$c^j_t + q^s_t \theta^j_t + q^b_t b^j_t = \omega^j_t + (q^s_t + 1)\theta^j_{t-1} + b^j_{t-1} e^{-x_t}. \tag{3b$'$}$$

The Euler equations are

$$(c^j_t)^{-\gamma} q^s_t = \beta E_{Q^j_t}[(c^j_{t+1})^{-\gamma}(1 + q^s_{t+1}) e^{(1-\gamma)x_{t+1}} | H_t] \tag{4a}$$

$$(c^j_t)^{-\gamma} q^b_t = \beta E_{Q^j_t}[(c^j_{t+1})^{-\gamma} e^{-\gamma x_{t+1}} | H_t] \tag{4b}$$

[2] The key assumption is then that *agents do not know the true probability but have ample past data from which they deduce that (2) is implied by the empirical distribution.* Hence the data reveals a memory of length 1 and residuals which are i.i.d. normal. This assumption means that even if (2) is the true data generating process, agents do not know this fact. An agent may believe the true process is non-stationary and different from (2) and then build his subjective model of the market.

and the market clearing conditions are then

$$\theta_t^1 + \theta_t^2 = 1 \tag{4c}$$

$$b_t^1 + b_t^2 = 0. \tag{4d}$$

3 A Rational Expectations Equilibrium (REE)

Strictly speaking we cannot evaluate the REE since the true output process has not been specified. We thus define an REE to be *the economy in which all agents believe that (2) is the true output process*. To evaluate the volatility of this REE in an annual model we specify parameters of (2). Unfortunately, different estimates of the parameter values are available, depending upon time span of data, unit of time (annual vs. quarterly) and definition of terms (see, for a sample, Backus, Gregory and Zin, 1989; Campbell, 2000; Rodriguez, 2002 Appendix 2; based on Shiller in (http://www.econ.yale.edu/shiller)). We use the *annual* estimates in Campbell (2000), Table 3, which are consistent with Mehra and Prescott (1984). Hence, *for the rest of this paper* we set $\beta = 0.96$, $\gamma = 2.00, x^* = 0.01773$, $\lambda_x = -0.117$ and $\sigma_x = 0.03256$, all within the empirically estimated range. Ours is a theoretical paper aiming to draw qualitative conclusions. We use realistic parameter values since we wish our simulations to result in numerical values which are close to the observed data. For simplicity we assume $\omega_t^j = \omega$, a constant hence total resources, or GNP, equal $(1 + 2\omega)D_t$ and the Dividend/GNP ratio is $\frac{1}{1+2\omega}$. As we model only income from publically traded stocks, the Dividend/(Household Income) ratio should include corporate dividends but exclude self employed income and imputed income from other asset categories. This ratio is about 15% (see survey data in Heaton and Lucas, 1996) hence we select $\omega_t^j = 3$. This fact has very little effect on the results.

We simulated this REE and report in Table 1 the mean and standard deviations of (i) the price\dividend ratio q^s, (ii) the risky return R, (iii) the riskless rate r; the equity premium e_p, the Sharpe Ratio and the correlation between x and R. Moments of market data vary with sources reporting and methods of estimation. The market data reported in Table 1 are based on Shiller (http://www.econ.yale.edu/shiller) and others. The results are familiar. Note that apart from the low equity premium, the REE volatility measures are lower than market data by an order of magnitude. The "Equity Premium" is not the only puzzle; *the wider question is how to explain market volatility.*

Table 1. Simulated moments of key variables in REE (all moments are annualized)

	q^s	σ_{q^s}	R	σ_R	r	σ_r	e_p	ρ_{Rx}	shrp
Model Data	16.71	0.055	7.94%	3.78%	7.70%	0.79%	0.24%	0.995	0.064
Market Data	25	7.1	7%	18.00%	1.00%	5.70%	6.00%	0.100	0.333

Some have argued in favor of introducing habit formation in utility in order to generate time variability of risk aversion (e.g. see Abel, 1990, 1991;

Constantinides, 1990; Campbell and Cochrane, 1999, 2000)[3]. We are not persuaded by this model since habit formation can explain the equity premium only if it assumes unreasonably high degree of risk aversion[4]. Our alternative view proposes that risk premia are determined primarily by the structure of market expectations. Our argument is developed in Sections 4, 5 and 6. First, in Section 4 we develop the *general structure of equilibria where agents have time varying and diverse beliefs*. These ideas apply to any model with diverse beliefs, not only to a Rational Belief framework. We then explain the restrictions which the RB principle impose on the model and in Sections 5 and 6 we develop the main results.

4 The general structure of equilibria with diverse and time dependent beliefs

Although (2) represent moments of past data, agents believe the economy is not stationary and past data do not provide adequate guide to the future. With technology and institutional changes, agents do not believe a fixed stationary model captures the complexity of society. Hence, they may not agree on a "correct" model that generated this empirical evidence. Indeed, we would expect that different agents using the same evidence will come up with different theories to explain the data and hence with different models to forecast prices. Each investor may have his own model of market dynamics. But then, one may ask, what are the specific formal belief formation models agent use to deviate from the empirical forecasts and why do they select these models? Since agents do not hold RE, how do they rationalize their beliefs? These are questions which we cannot fully address here. Our methodology is *to use the distribution of beliefs to explain market volatility* hence we need to determine a level of detail at which agents "justify" their beliefs. If we aim at a complete specification of such modeling, our study is doomed to be bogged down in details

[3] Other approaches to the equity premium puzzle were reported by Brennan and Xia (1998), Epstein and Zin (1990), Cecchetti, Lam and Mark (1990, 1993), Heaton and Lucas (1996), Mankiw (1986), Reitz (1988), Weil (1989), and others. For more details see Kocherlakota (1996). Some degree of excess volatility can also be explained with explicit learning mechanism which does not die out (see for example Timmermann, 1996).

[4] Campbell and Cochrane (1999, 2000) assume that at habit the marginal utility of consumption and degree of risk aversion rise without bound. Hence, when consumption declines to habit, risk aversion increases, stock prices decline and risk premium rises. Although the model generates moments which are closer to those observed in the market, the theory is unsatisfactory. First, with X_t = habit, utility is $\frac{1}{1-\gamma}(C_t - X_t)^{1-\gamma}$. But why should the marginal utility and risk aversion *explode* when C_t approaches the mean of past consumption? Campbell and Cochrane (1999, p. 244) show that for the model to generate the desired moments, the degree of risk aversion is 80 at steady state and exceeds 300 frequently along any time path. If instead we use Abel's (1990, 1999) formulation $\frac{1}{1-\gamma}(\frac{C_t}{X_t})^{1-\gamma}$, marginal utility is normalized to be 1 at habit but the model does not generate volatility. Second, big fluctuations of stock prices are observed during long periods when consumption grows smoothly as was the case during the volatile period of 1992–2002. Finally, the model predicts *perfect* correlation between consumption growth and stock returns but the record (see Table 1) shows date t stock returns have low correlation with date t consumption growth. Above all, the habit formation model proposes a theory which claims that asset premia are caused by an unreasonably high degree of risk aversion. This is not credible.

of inference from small samples and information processing. Although interesting, from a general equilibrium perspective it is not needed. To study volatility we focus on a narrow but operational question. Since (4a)–(4b) require specification of conditional probabilities, we need only a tractable way to *describe* differences among agents' beliefs and time variability of their conditional probabilities, without fully specified models to justify them. From our point of view what matters is the fact that market beliefs are diverse and time dependent; the reasoning which lead agents to the subjective models are secondary. The tool we developed for this goal is the individual and the market "*state of belief*" which we now explain.

4.1 Market states of belief and anonymity: expansion of the state space

The usual state space for agent j is denoted by S^j but when beliefs change over time we introduce an additional state variable called "*agent j state of belief.*" It is a variable generated by agent j, expressing his date t subjective view of the future and denoted by $g_t^j \in G^j$. It has the property that once specified, the conditional probability function of an agent is uniquely specified and hence has the form $Pr(s_{t+1}^j, g_{t+1}^j | s_t^j, g_t^j)$. Changes in j's conditional probability function are pinned down by j's state of belief; g_t^j is actually a proxy for j's conditional probability function. We note that g_t^j are *privately perceived* by agent j and have meaning only to him. Since a dynasty consists of a sequence of decision makers, g_t^j used by j has no impact on the description of beliefs by other dynasty members. In the model of this paper agents forecast dividend or profit growth rate x_{t+1} (i.e. the exogenous variable) hence $g_t^j \in \mathbb{R}$ describes agent j conditional probability of profit growth at $t+1$. We shall permit rational agents to be "bulls" who are optimistic about future excess returns or "bears" who are pessimistic about future excess returns. To understand the role of g_t^j we introduce later a reference parameter a and then interpret g_t^j in the following way:

- If $g_t^j = a$ agent j agrees with the empirical distribution and makes profit growth forecasts in accord with (2);
- If $g_t^j \neq a$ agent j disagrees with the empirical distribution. If $g_t^j > a$ he is a bear and makes *lower* profit growth forecasts than the ones implied by (2); if $g_t^j < a$ he is a bull and makes *higher* profit growth forecasts than the ones implied by (2).[5]

[5] Note that larger values of g imply a more bearish perspective. In the applications below larger g will express an agent's reduced probability belief in making excess returns. This may appear unnatural but we study equilibria with asymmetry measured by the frequency at which an agent is a bull or a bear. One of our main results says that the data supports a model where, on average, agents expect to make excess returns less than 50% of the time or, equivalently, that in a large market a majority of agents are pessimistic about making excess returns. We thus focus on the market pessimists. Since in the computational model we use a logistic function to express this asymmetry, it turns out that the use of a logistic function necessitates the condition that a larger g means more bearishness. Without the desired asymmetry g could have an opposite interpretation. We discuss this point further in Appendix A.

As indicated, we do not explain the reasoning used by agents to deviate from the empirical forecast. It is a common practice among forecasters to use the strict econometric forecast only as a benchmark. Given such benchmark, a forecaster uses his own model to add a component reflecting an evaluation of circumstances at a date t that call for a deviation at t from the benchmark. In short, g_t^j is a description of *how the model of agent j deviates from the statistical forecast implied by (2).* In this paper we assume that at any date the state of belief is a realization of a process of the form

$$g_{t+1}^j = \lambda_z g_t^j + \lambda_x^{zj}(x_t - x^*) + \tilde{\rho}_{t+1}^{g^j} \quad , \quad \tilde{\rho}_{t+1}^{g^j} \sim N(0, \tilde{\sigma}_{g^j}^2). \tag{5}$$

Persistent states of belief which depends upon current market data fit different cases of economies with diverse beliefs. We consider three examples to illustrate how one may think about them.

(i) *Measure of Animal Spirit*. "Animal Spirit" expresses intensity at which agents carry out investments and this, in turn, is based upon expected rewards. g_t^j identifies the probability an agent assigns to high or low rates of return hence g_t^j can be interpreted as a measure of "animal spirit."

(ii) *Learning Unknown Parameters*. In a learning context agents use prior distributions on unknown parameters. Since g_t^j defines an agent's belief about the profit growth process we can identify g_t^j as a posterior parameter of its mean value function. A posterior as a linear function of the prior and current data is familiar. We add $\tilde{\rho}_{t+1}^{g^j}$ to reflect changes in priors over time due to regime shifts and changes in structure. Among such changes are decision makers in the dynasty who are replaced by successors who select new priors. Infinite horizon is a proxy for a sequence of decision makers in a changing economy. With diverse beliefs, $\tilde{\rho}_{t+1}^{g^j}$ models a diversity of beliefs over time.

(iii) *Privately generated subjective sunspot to depend upon real variables.* g_t^j may play the role of a private sunspot with three properties (a) an agent generates his own g_t^j under a marginal distribution known only to him, (b) it is not observed by other agents, and (c) it's distribution may depend upon real variables. Also, the correlation across agents is a market externality, not known to anyone. Under this interpretation g_t^j is a major extension of the common concept of a "sunspot" variable.

In equilibria with diverse beliefs agents' decision rules are functions of g_t^j hence equilibrium prices depend upon $g_t = (g_t^1, g_t^2, \dots, g_t^N)$, the agents' conditional probabilities. But then, should j be allowed to recognize his g_t^j is the j^{th} coordinate of g_t and thus give him some market power? The principle of *anonymity* introduced in Kurz, Jin and Motolese (2003a,b) requires competitive agents to assume they cannot affect endogenous variables. It is analogous to requiring a competitive firm to assume it has no effect on prices. The issue here is the specification of how agents forecast prices. To that end we define the "*market state of belief*" as a vector $z_t = (z_t^1, z_t^2, \dots, z_t^N)$, keeping in mind *the model consistency condition $z_t = g_t$ which is not recognized by agents*. This makes market state of belief a macroeconomic state variable and equilibrium prices actually become functions of z_t and not functions of g_t. Agent j views z_t as "market belief" and as unrelated to him since it is the belief of *other agents*. In small economies prices depend upon the distribution $z_t = (z_t^1, z_t^2, \ldots, z_t^N)$ but in many applications only a few moments matter. In some models of diverse beliefs writers focus only on the average, and define the market state of belief by the mean belief [6] $z_t = \frac{1}{N}\sum_{j=1}^{N} z_t^j$. Anonymity is so central to our approach that we use three notational devices to highlight it:

(i) g_t^j denotes the state of belief of j as known by the agent only.
(ii) $z_t = (z_t^1, z_t^2, \ldots, z_t^N)$ denotes market state of belief, observed by all. Competitive behavior means j does not associate g_t^j with z_t although $g_t^j = z_t^j$ is a model consistency condition.
(iii) $z_{t+1}^j = (z_{t+1}^{j1}, z_{t+1}^{j2}, \ldots, z_{t+1}^{jN})$ is agent j's forecast of the market state of belief at future date $t+1$.

The introduction of individual and market states of belief has two central implications:

(A) The economy has an *expanded state space*, including market belief z_t. $z_t = (z_t^1, z_t^2) \in \mathbb{R}^2$ in this paper. Hence diverse beliefs create new uncertainty which is the uncertainty of *what others may do*. This adds a component of volatility which cannot be explained by "fundamental" shocks. Denoting usual state variables by s_t, the price process $\{(q_t^s, q_t^b), t = 1, 2, \ldots\}$ is defined by a map like

$$\begin{bmatrix} q_t^s \\ q_t^b \end{bmatrix} = \Xi(s_t, z_t^1, z_t^2, \ldots, z_t^N). \tag{6}$$

Our equilibrium is thus an incomplete Radner (1972) equilibrium with *an expanded state space.*

(B) To forecast prices agents must forecast market beliefs. Although all use (6) to forecast prices, agents' forecasts are different since each forecasts (s_{t+1}, z_{t+1}) *given his own state* g_t^j. This is a feature of the Keynes Beauty Contest: to forecast equilibrium prices you must forecast beliefs of other agents. A Beauty Contest *does not entail higher order of beliefs*: at t you form belief about market belief z_{t+1} but the date $t+1$ market belief *is not* a probability about your date t belief state.[7]

We now return to the economy with two agent types and simplify by assuming the market belief (z_t^1, z_t^2) *is observable*. This assumption is entirely reasonable since there is a vast amount of public data on the distribution of forecasts in the market and on the dynamics of this distribution. Indeed, using forecast data obtained from the Blue Chip Economic Indicators and the Survey of Professional Forecasters we constructed various measures of market states of belief. Since (z_t^1, z_t^2) is

[6] See Woodford (2003), Morris and Shin (2002), Allen, Morris and Shin (2003), and others.

[7] Allen, Morris and Shin (2003) seem to suggest the Beauty Contest is associated with the failure of the market belief to satisfy the law of iterated expectations (see title of their paper). It is clear the average market probability beliefs does not satisfy the law of iterated expectations since the average conditional probability is not a proper conditional probability. However, this fact is independent of the problem defined by the Keynes Beauty Contest which requires agents in an economy with diverse beliefs to forecast the future average market state of belief.

observable we need to modify the empirical distribution (2) and include (z_t^1, z_t^2) in it. We assume the empirical distribution of profit growth and the states of belief is an AR process of the form

$$x_{t+1} = (1-\lambda_x)x^* + \lambda_x x_t + \rho_{t+1}^x \tag{7a}$$

$$z_{t+1}^1 = \lambda_{z^1} z_t^1 + \lambda_x^{z^1}(x_t - x^*) + \rho_{t+1}^{z^1} \tag{7b}$$

$$z_{t+1}^2 = \lambda_{z^2} z_t^2 + \lambda_x^{z^2}(x_t - x^*) + \rho_{t+1}^{z^2} \tag{7c}$$

$$\begin{pmatrix} \rho_{t+1}^x \\ \rho_{t+1}^{z^1} \\ \rho_{t+1}^{z^2} \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0, \\ 0, \end{matrix} \begin{bmatrix} \sigma_x^2, & 0, & 0 \\ 0, & 1, & \sigma_{z^1 z^2} \\ 0, & \sigma_{z^1 z^2}, & 1 \end{bmatrix} = \Sigma \right), \text{i.i.d.}$$

In any application one assumes the parameters of (7a)–(7c) are known by all. With regard to x, we have set values for $(\lambda_x, x^*, \sigma_x)$ in Section 2. We normalize by setting variances of $\rho_{t+1}^{z^j}$ equal to 1. To specify parameters of the z^j equations recall that z measures how *optimistic agents are about future returns on investment*. With this in mind we used forecasts reported by the Blue Chip Economic Indicators and the Survey of Professional Forecasters, and "purged" them of observables. We then estimated principal components to handle multitude of forecasted variables (for details, see Fan, 2003). The extracted indexes of beliefs imply regression coefficients around 0.5–0.8 hence we set $\lambda_{z^1} = \lambda_{z^2} = 0.7$. Investors' forecasts of financial variables such as corporate profits, are highly correlated and a value of $\sigma_{z^1 z^2} = 0.90$ is realistic. In this paper we study only symmetric economies where agents differ only in their beliefs hence we assume $\lambda_x^{z^1} = \lambda_x^{z^2} = \lambda_x^z$. The evidence shows that positive profit shocks lead agents to revise upward their economic growth forecasts implying $\lambda_x^z > 0$. Our best guess of this parameter leads us to set $\lambda_x^z = 0.9$ but we discuss it again later.

To write (7a)–(7c) in a more compact notation let $w_t = (x_t - x^*, z_t^1, z_t^2)$, $\rho_t = (\rho_t^x, \rho_t^{z^1}, \rho_t^{z^2})$ and denote by A the 3×3 matrix of parameters in (7a)–(7c). We then write (7a)–(7c) as

$$w_{t+1} = Aw_t + \rho_{t+1} \quad , \quad \rho_{t+1} \sim N(0, \Sigma). \tag{8}$$

Denote by V the 3×3 *unconditional* covariance of w defined by $V = E_m(ww')$. We use the value of V and compute it here as a solution of the equation

$$V = AVA' + \Sigma. \tag{9}$$

Finally, denote by m the probability measure on infinite sequences implied by (8) with the invariant distribution as the initial distribution. We then write $E_m(w_{t+1}|H_t) = Aw_t$ where H_t is the history at t. To complete the description of an equilibrium we need to specify the individual beliefs. However, we stress that *the description to follow is general*, applying to any model with diverse beliefs.

4.2 The general structure of beliefs and the problem of parameters

A *perception model* is a set of transition functions of state variables, expressing an agent's belief about date $t+1$ conditional probability. We first explain the general form of a perception model, and provide details later. Let $w^j_{t+1} = (x^j_{t+1}, z^{1j}_{t+1}, z^{2j}_{t+1})$ be date $t+1$ variables *as perceived by j* and let $\Psi_{t+1}(g^j_t)$ be a 3 dimensional vector of date $t+1$ random variables *conditional upon* g^j_t.

Definition 1. *A perception model in the economy under study has the general form*

$$w^j_{t+1} = Aw_t + \Psi_{t+1}(g^j_t) \tag{9a}$$

together with (5).

Since $E_m(w_{t+1}|H_t) = Aw_t$, we write (9a) in the simpler form

$$w^j_{t+1} - E_m(w_{t+1}|H_t) = \Psi_{t+1}(g^j_t). \tag{9b}$$

(9b) reveals that $E^j[\Psi_{t+1}|g^j_t]$ is j's deviation in forecasting w^j_{t+1} from $E_m(w_{t+1}|H_t)$. In general we have $E^j[\Psi_{t+1}|g^j_t] \neq 0$ and the mean of the agent's forecast changes with g^j_t. If $\Psi_{t+1}(g^j_t) = \rho_{t+1}$ as in (8), j uses the empirical probability m as his belief. Condition (9b) shows that we model $\Psi_{t+1}(g^j_t)$ so that agents may be over-confident by being optimistic or pessimistic *relative to the empirical forecasts*. We now postulate a random variable $\eta^j_{t+1}(g^j_t)$ with which we model $\Psi_{t+1}(g^j_t)$ simply by

$$\Psi_{t+1}(g^j_t) = \begin{pmatrix} \lambda^x_g \eta^j_{t+1}(g^j_t) + \tilde{\rho}^{xj}_{t+1} \\ \lambda^{z1}_g \eta^j_{t+1}(g^j_t) + \tilde{\rho}^{zj1}_{t+1} \\ \lambda^{z2}_g \eta^j_{t+1}(g^j_t) + \tilde{\rho}^{zj2}_{t+1} \end{pmatrix}, \quad \tilde{\rho}^j_{t+1} \sim N(0, \Omega^j_{\rho\rho}), \text{i.i.d.} \tag{10}$$

where $\tilde{\rho}^j_{t+1} = (\tilde{\rho}^{xj}_{t+1}, \tilde{\rho}^{zj1}_{t+1}, \tilde{\rho}^{zj2}_{t+1})$. By (9a) a perception model includes g^j_{t+1} as a fourth dimension with an innovation $\tilde{\rho}^j_{t+1}$ and a covariance matrix denoted by Ω^j, reflecting the vector $r^i_j = Cov(w^i, g^j)$ for $i = 1, 2, 3$. For simplicity we use only one random variable $\eta^j_{t+1}(g^j_t)$ to define all components of $\Psi_{t+1}(g^j_t)$. In addition, we study only symmetric markets where agents differ at any t only in two respects: they may have different date t states of belief and different portfolios due to difference in histories. Hence, we assume $\lambda_g = (\lambda^x_g, \lambda^{z1}_g, \lambda^{z2}_g)$ are common to both agents. These describe how an agent's forecasts vary with his beliefs. To specify agents' beliefs in any particular model one needs to specify $\Psi_{t+1}(g^j_t)$, $\lambda_g = (\lambda^x_g, \lambda^{z1}_g, \lambda^{z2}_g)$ and $r^i_j = Cov(w^i, g^j)$ for i = 1, 2, 3. Combining all these parts we can formulate the final form of the perception models of agent j:

$$x_{t+1}^j = (1-\lambda_x)x^* + \lambda_x x_t + \lambda_g^x \eta_{t+1}^j(g_t^j) + \tilde{\rho}_{t+1}^{xj} \tag{11a}$$

$$z_{t+1}^{j1} = \lambda_z z_t^1 + \lambda_x^z(x_t - x^*) + \lambda_g^z \eta_{t+1}^j(g_t^j) + \tilde{\rho}_{t+1}^{zj1} \tag{11b}$$

$$z_{t+1}^{j2} = \lambda_z z_t^2 + \lambda_x^z(x_t - x^*) + \lambda_g^z \eta_{t+1}^j(g_t^j) + \tilde{\rho}_{t+1}^{zj2} \tag{11c}$$

$$g_{t+1}^j = \lambda_z g_t^j + \lambda_x^z(x_t - x^*) + \tilde{\rho}_{t+1}^{gj} \tag{11d}$$

$\tilde{\rho}_{t+1}^j = (\tilde{\rho}_{t+1}^{xj}, \tilde{\rho}_{t+1}^{zj1}, \tilde{\rho}_{t+1}^{zj2}, \tilde{\rho}_{t+1}^{gj})$ is i.i.d. Normal with mean zero and covariance matrix Ω^j of the form

$$\Omega^j = \begin{pmatrix} \Omega_{\rho\rho}^j, & \Omega_{wg^j} \\ \Omega'_{wg^j}, & \sigma_{g^j}^2 \end{pmatrix} \tag{11e}$$

where $\Omega_{wg^j} = [Cov(\tilde{\rho}_{t+1}^x, \tilde{\rho}_{t+1}^{g^j}), Cov(\tilde{\rho}_{t+1}^{z^1}, \tilde{\rho}_{t+1}^{g^j}), Cov(\tilde{\rho}_{t+1}^{z^2}, \tilde{\rho}_{t+1}^{g^j})]$. Note two facts. First, $\lambda_g^x \eta_{t+1}^j(g_t^j)$ in the first equation reflects diverse beliefs about profit shocks. The agent's forecast of x_{t+1}^j is $E_{Q_t^j}(x_{t+1}^j - x^*) = \lambda_x(x_t - x^*) + \lambda_g^x E_{Q_t^j} \eta_{t+1}^j(g_t^j)$, not $\lambda_x(x_t - x^*)$ as in (7a). The term $\lambda_g^x E_{Q_t^j} \eta_{t+1}^j(g_t^j)$ measures the agent's forecast *deviation* from (7a). Second, $(\lambda_g^z \eta_{t+1}^j(g_t^j), \lambda_g^z \eta_{t+1}^j(g_t^j))$ in the second and third equations measure the effect of j's belief on his forecast of the belief of *others* $(z_{t+1}^{j1}, z_{t+1}^{j2})$ at $t+1$. Since prices depend on (z_{t+1}^1, z_{t+1}^2), the terms $\lambda_g^z \eta_{t+1}^j(g_t^j)$ and $\lambda_g^z \eta_{t+1}^j(g_t^j)$ contribute to diversity of $t+1$ price forecasts. These are the key forces which generate market volatility.

(11a)–(11e) show that given the assumed symmetry, the case of $\lambda_g = (\lambda_g^x, \lambda_g^{z1}, \lambda_g^{z2}) = (0,0,0)$ fully characterizes an economy where all agents believe the empirical distribution is the truth. This case has the essential property of an REE since there is no diversity of beliefs. In such an economy the volatility of equilibrium quantities is determined only by the volatility of x_t in (2).

Summary of belief parameters. Given symmetry, parameters specifying belief of j are then (λ_g, Ω^j) and parameters defining the variable $\eta_{t+1}^j(g_t^j)$ explained next. A theory with unrestricted beliefs would specify these parameters and some may consider it to be Bounded Rationality. Our models below assume agents satisfy the RB rationality conditions and we explain in Section 4.4 the restrictions imposed by RB. Note that *anonymity* requires the idiosyncratic component of an agent's belief not to be correlated with market beliefs. This is translated to the requirement that

$$\Omega_{z^1 g^j} = Cov(\tilde{\rho}_{t+1}^{z^1}, \tilde{\rho}_{t+1}^{g^j}) = 0$$

$$\Omega_{z^2 g^j} = Cov(\tilde{\rho}_{t+1}^{z^2}, \tilde{\rho}_{t+1}^{g^j}) = 0.$$

Hence, anonymity restricts two components of Ω_{wg^j} even in a theory without restrictions on belief.

4.3 Modeling tractable and computable functions $\Psi_{t+1}(g_t^j)$

We model $\Psi_{t+1}(g_t^j)$ so as to permit agents to be *over confident* by assigning to some events higher probability than the empirical frequency. Evidence from the psychological literature (e.g. Svenson, 1981; Camerer and Lovallo, 1999; and references) shows agents exhibit such behavior.

In some cases this behavior may be irrational but this is not generally true. Institutional and technical changes are central to an economy and past statistics do not provide the best forecasts for the future. Deviations from empirical frequencies reflect views based on limited *recent* data about changed conditions. Agents have financial incentive to make such judgments since major financial gains are available to those who bet on the correct market changes. We now discuss the asymmetries built into modeling $\Psi_{t+1}(g_t^j)$.

Asymmetry and intensity of fat tails in $\Psi_{t+1}(g_t^j)$. Asymmetry and "fat" tails, reflecting over confidence, is introduced into the computational model through $\eta_{t+1}^j(g_t^j)$. We define $\eta_{t+1}^j(g_t^j)$ by its density, conditional on g_t^j, as a step function

$$p(\eta_{t+1}^j|g^j) = \begin{cases} \phi_1(g^j)f(\eta_{t+1}^j) & \text{if} \quad \eta_{t+1}^j \geq 0 \\ \phi_2(g^j)f(\eta_{t+1}^j) & \text{if} \quad \eta_{t+1}^j < 0 \end{cases} \tag{12}$$

where η_{t+1}^j and $\tilde{\rho}_{t+1}^{g^j}$ (in (10)) are independent and where $f(\eta) = [1/\sqrt{2\pi}]e^{-\frac{\eta^2}{2}}$. The functions $(\phi_1(g), \phi_2(g))$ are defined by a logistic function with two parameters a and b

$$\phi(g^j) = \frac{1}{1 + e^{b(g^j - a)}}, \quad \text{and define} \quad G \equiv E_g \phi(g^j), \tag{13}$$

$$a < 0, b < 0 \ \text{ and } \phi_1(g^j) = \frac{\phi(g^j)}{G}, \phi_2(g^j) = 2 - \phi_1(g^j). \tag{14}$$

The parameter a measures *asymmetry* and the parameter b measures *intensity of fat tails* in beliefs. Details of this construction and the implied moments are discussed in Appendix A.

To explain (12)–(14), note that for g_t^j large, $\phi(g^j)$ goes to one, implying that $\phi_1(g^j)$ goes to 1/G. Hence, by (12) large $g_t^j > a$ implies high probability that $\eta_{t+1}^j > 0$. Similarly, small $g_t^j < a$ implies high probability that $\eta_{t+1}^j < 0$. To interpret what $g_t^j > a$ means requires *a model convention.* A convention must specify what $\eta_{t+1}^j > 0$ means and this depends upon $\Psi_{t+1}(g_t^j)$. In an economy with $\lambda_g^x > 0, \eta_{t+1}^j > 0$ raises j's forecast of x_{t+1}^j but in an economy with $\lambda_g^x < 0, \eta_{t+1}^j > 0$ lowers j's forecast of x_{t+1}^j. We have thus motivated a formal definition of bull and bear states:

Definition 2. *Let Q^j be the probability belief of agent j. Then g_t^j is said to be*
a bear state *for agent j if $E_{Q^j}[x_{t+1}^j | g_t^j, H_t] < E_m(x_{t+1}|H_t)$;*
a bull state *for agent j if $E_{Q^j}[x_{t+1}^j | g_t^j, H_t] > E_m(x_{t+1}|H_t)$.*

Now recall that $x_{t+1}^j = (1-\lambda_x)x^* + \lambda_x x_t + \lambda_g^x \eta_{t+1}^j(g_t^j) + \tilde{\rho}_{t+1}^{xj}$. From (12) we know (see Appendix A for detail) that if $g_t^j > a$ then $E_j[\eta_{t+1}^j(g_t^j)|g_t^j] > 0$ hence we have two cases. If $\lambda_g^x < 0$ then $g_t^j > a$ means agent j is in a *bear state* and expects profit growth below normal at date $t+1$. Since $a < 0$ this also means that bear states occur with frequency higher than 50%. If $\lambda_g^x > 0$ then $g_t^j > a$ means agent j is in a *bull state* and is optimistic about $t+1$ profit growth being above normal. Again, since $a < 0$ this also implies that bull states occur with frequency higher than 50%. "Normal" *is defined relative to the empirical forecast.* In sum, we have two basic economies:

Economy 1. for all agents $\lambda_g^x < 0$ *and bear states are more frequent*, hence

$g_t^j > a$ means agent j is pessimistic about profit growth and excess stock returns at $t+1$
$g_t^j < a$ means agent j is optimistic about profit growth and excess stock returns at $t+1$.

Economy 2. for all agents $\lambda_g^x > 0$ *and bull states are more frequent*, hence

$g_t^j > a$ means agent j is optimistic about profit growth and excess stock returns at $t+1$
$g_t^j < a$ means agent j is pessimistic about profit growth and excess stock returns at $t+1$.

What are the beliefs in bull and bear states? Consider Economy I. As $(g_t^j - a)$ increase, $\phi_1(g_t^j)$ rises and $\phi_2(g_t^j)$ declines. Hence when $g_t^j > a$ an agent increases the positive part of a normal density in (12) by a factor $\phi_1(g_t^j) > 1$ and decreases the negative part by $\phi_2(g_t^j) < 1$. When $g_t^j < a$ the opposite occurs: the *negative* part is shifted up by $\phi_2(g_t^j) > 1$ and the positive part is shifted down by $\phi_1(g_t^j) < 1$. The amplifications $(\phi_1(g^j), \phi_2(g^j))$ are defined by g^j, by a and by the"fat tails" parameter b. The parameter b measures the degree by which the distribution is shifted per unit of $(g_t^j - a)$. In Figure 1 we draw densities of $\eta^j(g^j)$ for $g^j > a$ and for $g^j < a$. These are not normal densities. As g_t^j varies, the densities of $\eta_{t+1}^j(g_t^j)$ change. However, the empirical distribution of g_t^j is normal with zero unconditional mean and hence the empirical distribution of $\eta_{t+1}^j(g_t^j)$, averaged over time (including over g_t^j), also has these same properties.

The parameter a measures asymmetry. It determines the frequency at which agents are bears. To see why note that if $a = 0$, (13) is symmetric around 0 and the probability of $g^j > 0$ is 50%. When $a < 0$ the probability of $g^j > a$ is more than 50% and if $\lambda_g^x < 0$, the *frequency* at which j is pessimistic, is more than 50%. In a

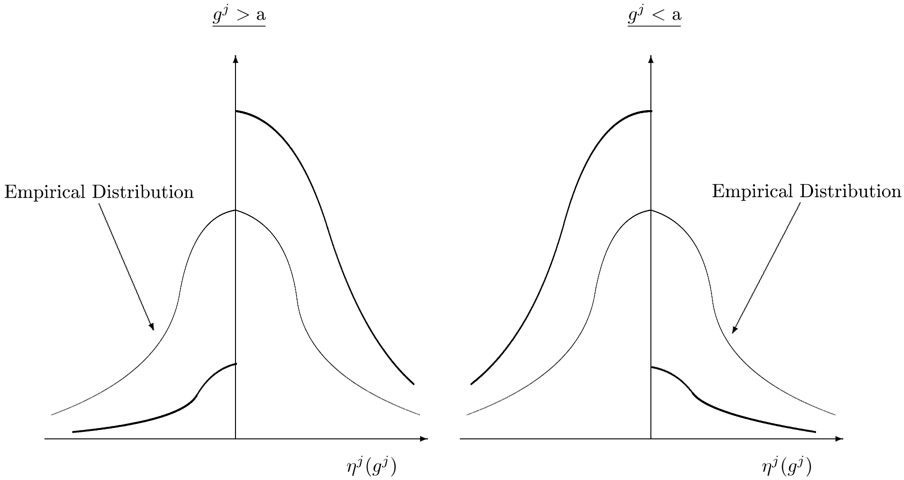

Figure 1. Non-normal belief densities

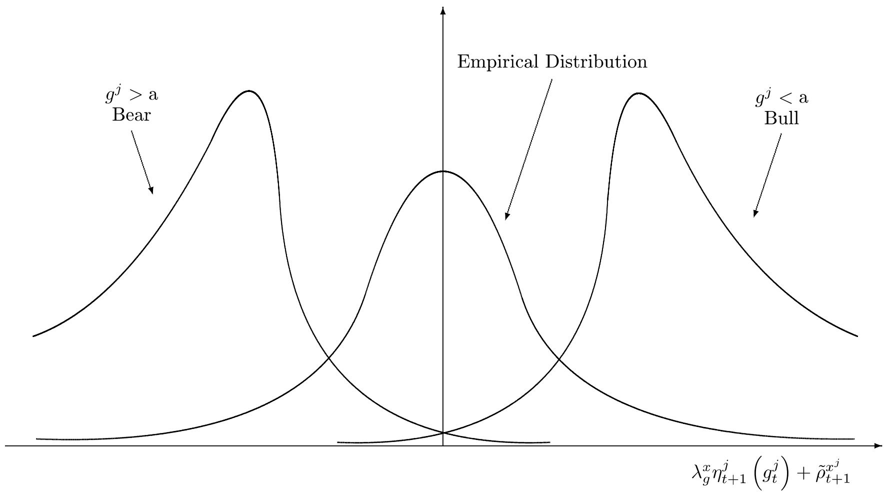

Figure 2. Density $\Psi(g_t^j)$ with fat tails

large market this would mean that, on average, a majority of traders are bears about $t+1$ excess returns. Similarly, when $\lambda_g^x > 0$, *bulls are in the majority.*

Each component of $\Psi_{t+1}(g_t^j)$ is a sum of two random variables: one as in Figure 1 and the second is normal. In Figure 2 we draw two densities of $\Psi_{t+1}(g_t^j)$, each being a convolution of the two constituent distributions with $\lambda_g^x < 0$. One density for $g^j > a$ and a second for $g^j < a$, showing both have "fat tails." Since b measures intensity by which the positive portion of the distribution in Figure 1 is shifted, it measures the degree of fat tails in the distributions of $\Psi_{t+1}(g_t^j)$.

4.4 Restrictions on beliefs under the rational belief principle

We now define a Rational Belief (due to Kurz, 1994, 1996) and discuss the restrictions which the theory imposes on the belief parameters.

Definition 3. *A perception model as defined in (11a)–(11e) is a Rational Belief if the agent's model* $w^j_{t+1} = Aw_t + \Psi_{t+1}(g^j_t)$ *together with (5) has the same empirical distribution as* $w_{t+1} = Aw_t + \rho_{t+1}$ *in (8).*

Definition 3 implies that $\Psi_{t+1}(g^j_t)$ together with (5) must have the same empirical distribution as ρ_{t+1} in (8), i.e. $N(0,\Sigma)$. An RB is a model which cannot be rejected by the data as it matches *all moments of the observables*. Agents holding RB may exhibit over confidence by deviating from the empirical frequencies but their behavior is rationalizable if the time average of the probabilities of an event equals it's empirical frequency. What are the restrictions implied by the RB principle?

Theorem *Let the beliefs of an agent be a Rational Belief. Then the belief is restricted as follows:*

(i) For any feasible vector of parameters $(\lambda^x_g, \lambda^z_g, a, b)$ *the Variance-Covariance matrix* Ω^j *is fully defined and is not subject to choice;*
(ii) The condition that Ω^j *is a positive definite matrix establishes a feasibility region for the vector* $(\lambda^x_g, \lambda^z_g, a, b)$. *In particular it requires* $|\lambda^x_g| \le \sigma_x$, $|\lambda^z_g| \le 1$.
(iii) $\Psi_{t+1}(g^j_t)$ *cannot exhibit serial correlation and this restriction pins down the vector*

$$\Omega_{wg^j} = [Cov(\tilde{\rho}^x_{t+1}, \tilde{\rho}^{g^j}_{t+1}), Cov(\tilde{\rho}^{z^1}_{t+1}, \tilde{\rho}^{g^j}_{t+1}), Cov(\tilde{\rho}^{z^2}_{t+1}, \tilde{\rho}^{g^j}_{t+1})].$$

The proof is in Appendix B. As to implications of (iii), since $\{g^j_t, t = 1,2,\dots\}$ exhibit serial correlation, to isolate the subjective component of belief we exclude from g^j_t information which is in the market at t. Define a *pure belief* index $u^j_t(g^j_t)$ as follows. Recall that $r_j = Cov(w, g^j)$ is agent j's covariance vector and, keeping in mind (8), define $u^j_t(g^j_t)$ by a standard regression filter

$$u^j_t(g^j_t) = g^j_t - r'_j V^{-1} w_t. \tag{15}$$

The index $u^j_t(g^j_t)$ now replaces g^j_t everywhere and is uncorrelated with public information. In all equations we replace $\Psi_{t+1}(g^j_t)$ with $\Psi_{t+1}(u^j_t)$ and show in Appendix B that it is serially uncorrelated.

Under the RB restrictions we can thus select only $(\lambda^x_g, \lambda^z_g, a, b)$ subject to feasibility conditions imposed by the Theorem. In practice these restrictions imply that

- $\sigma_x = 0.03256$ implies $|\lambda^x_g| < 0.03$. The covariance structure further restricts $|\lambda^x_g| < 0.028$.
- The covariance structure implies that $|\lambda^z_g| < 0.30$.
- The parameter b has a feasible range between 0 and -16.

We finally offer some some additional considerations to restrict the parameter λ_g^z.

4.4.1 Selecting λ_g^z: the principle of maintaining relative market position

We assumed nothing regarding belief of agents about the beliefs of others and we have only very limited data on it. To examine this question let $\overline{z}_t = \frac{1}{N}\sum_{j=1}^{N} z_t^j$ be the mean market belief and we ask the following question. Suppose an agent is a bear about profit growth. What would be his position about the mean belief of others? In principle we need a second belief index to define a separate belief about "others." Thus, suppose that, in addition, agent j is more bearish than the average so that $g_t^j > \overline{z}_t^j$. How would his bearish outlook about profit growth alter the expected relative position of his belief in relation to the mean market belief? There is no uniform answer to this question but the data suggests *a relative inertia* which can be expressed by the following:

Definition 4. *Agent j expects to Enhance his Relative Position within the belief distribution given his current state of belief if his belief about others takes the form*

$$E_t^j(g_{t+1}^j - \overline{z}_{t+1}^j) > \lambda_z(g_t^j - \overline{z}_t^j) \quad \textit{if} \quad g_t^j > a; \tag{16a}$$

[Note: if $\lambda_g^x < 0$ then j is in a bear state]

$$E_t^j(g_{t+1}^j - \overline{z}_{t+1}^j) < \lambda_z(g_t^j - \overline{z}_t^j) \quad \textit{if} \quad g_t^j < a; \tag{16b}$$

[Note: if $\lambda_g^x < 0$ then j is in a bull state].

(11a)–(11e) says $E_t^j(g_{t+1}^j - \overline{z}_{t+1}^j) - \lambda_z(g_t^j - \overline{z}_t^j) = \lambda_g^z E_t^j[\eta_{t+1}^j(g_t^j)|g_t^j]$ and Appendix A shows that $E_t^j[\eta_{t+1}^j(g_t^j)|g_t^j] > 0$ if $g_t^j > a$ and $E_t^j[\eta_{t+1}^j(g_t^j)|g_t^j] < 0$ if $g_t^j < a$.

Hence (16a)–(16b) say that if agent j is bear and past market norms predict his relative position at $t+1$ to be $\lambda_z(g_t^j - \overline{z}_t^j)$, his bearish outlook today will motivate him to predict a persistence of this position and this is what (16a) says. But then the implication of Definition 4 is that agents who adapt their beliefs in this manner must satisfy the condition $\lambda_g^z > 0$. However if $\lambda_g^x > 0$ the reasoning is reversed, leading to the conclusion that $\lambda_g^z < 0$. It then follows that under the condition of Enhancing Relative Belief Position in the distribution of beliefs our two possible economies are

Economy I: $(\lambda_g^x < 0, \lambda_g^z > 0)$ in which bear states are more frequent

Economy II:$(\lambda_g^x > 0, \lambda_g^z < 0)$ in which bull states are more frequent.

We shall conduct a computational test to identify the economy which matches U.S. volatility data.

4.4.2 Model parameters and a note on computed equilibria

Our question is now simple: are there feasible parameter values so the model replicates the empirical record in the U.S.? Moreover, what is the economic interpretation of the behavior implied by these parameter values? Our discussion above shows we need to study the effect of the belief parameters $(\lambda_g^x, \lambda_g^z, a, b)$ on volatility and test which of the two economies (Economy 1 vs. Economy 2) fits the data. For Economy 1 we select $\lambda_g^x = -0.027, \lambda_g^z = 0.25$ with absolute values close to the feasible boundary.[8] By setting them as high as compatible with rationality we focus on examining the effect of (a, b). Thus, given a limited choice of $(\lambda_g^x, \lambda_g^z)$ we search for values of a and b with which the model generates volatility which matches the data. We then vary$(\lambda_g^x, \lambda_g^z)$ in order to understand the *qualitative properties* of such economies, aiming for general conclusions.

We compute equilibria with perturbation methods using a program developed by Hehui Jin (see Jin and Judd, 2002; Jin, 2003). A solution is declared to be an equilibrium if: (i) a model is approximated by at least second order derivatives; (ii) errors in market clearing conditions and Euler equations are less than 10^{-3}. Appendix C provides details on the computational model.

5 Characteristics of volatility I: moments

Intensity of fat tails and asymmetry in beliefs are key components of our theory. But, what is the role of intensity and asymmetry in propagating volatility and which of the two economies are compatible with the data?

5.1 The role of intensity is pure volatility

We start with an experiment disabling the asymmetry parameter by setting $a = 0$ while varying b. Keeping in mind that parameters of the real economy imply *riskless* steady state values $q^{s*} = 16.58, R^* = r^* = 7.93$, we vary b from $b = -1$ up to a value at which the mean riskless rate reaches 0.66%. Table 2 reports the results. These show that the market exhibits substantial non - linearity in response to rising volatility as q^s and σ_{q^s} do not change monotonically with market volatility. However, as market volatility increases the following change in a monotonic manner:

- The risky rate R rises from 8.00% to 9.27%
- The standard deviation σ_R of the risky rate rises from 4.60% to 20.53%
- The riskless rate r declines from 7.88% to 0.66%

[8] The feasible set is open as it requires the covariance matrix to be positive definite hence no maximal values can be taken. The objective is to set the parameters so they are close to the boundary but do not destabilize the computational procedure, generating error in the Euler Equations.

- The correlation coefficient $\rho(x, R)$ declines from 0.83 to 0.20
- The Sharpe Ratio (Shrp) rises from 0.02 to 0.42.

Without a detailed demonstration we add the fact that the pattern observed in Table 2 *remains the same for all feasible belief parameters* $(\lambda_g^x, \lambda_g^z)$ *of the model.*

Table 2. The effect of pure intensity (all moments are annualized)

$b \Rightarrow$	-1.00	-5.00	-10.00	-11.00	-12.50	-14.00	-15.50	
$a=0$								Record
q^s	16.66	17.59	19.91	20.26	20.51	20.22	19.12	25.00
σ_{q^s}	0.53	2.06	3.68	3.96	4.29	4.45	4.38	7.10
R	8.00%	8.14%	8.21%	8.26%	8.42%	8.72%	9.27%	7.00%
σ_R	4.60%	10.36%	15.78%	16.66%	17.92%	19.17%	20.53%	18.00%
r	7.88%	7.73%	6.20%	5.75%	4.82%	3.28%	0.66%	1.00%
σ_r	2.53%	6.94%	6.78%	6.26%	5.27%	4.39%	5.21%	5.70%
e_p	0.11%	0.42%	2.00%	2.51%	3.60%	5.44%	8.62%	6.00%
$\rho(x, R)$	0.83	0.38	0.25	0.24	0.22	0.21	0.20	0.10
Shrp	0.02	0.04	0.13	0.15	0.20	0.28	0.42	0.33

Table 2 shows the riskless rate declines towards 1% simply because the RBE becomes more volatile. This affects both the volatility of individual consumption growth rates as well as their correlation with the growth rate of aggregate consumption. In an REE this correlation is close to 1 but not in an RBE where variability of individual consumption growth depends upon the agents' beliefs. The low observed riskless rate has been a central issue in the equity premium puzzle debate (see Weil, 1989) and the effect of the parameter b goes to the heart of this issue. Our theory offers the intuitive explanation *that non normal belief densities with fat tails and high intensity propagate high market volatility hence risk, making financial safety costly.*

Now compare Table 2 with the empirical record. As volatility increases and the riskless rate r reaches 0.66% we see that (i) q^s declines below 25, (ii) R rises above 7.0%, (iii) σ_R rises above 18.0%, (iv) the Sharpe Ratio rises above 0.33. Some reflection shows that these are reasonable conclusions *for a symmetric volatile economy* in which bear and bull states mirror each other: each bull state has an exact opposite bear state. Such symmetry implies that as risk level increases, the riskless rate should decline since the cost of safety increases and the risky rate should rise for analogous reasons. These are exactly the results in Table 2! If the riskless rate declines but the risky rate remains the same as volatility increases, there must be some belief asymmetry to induce a *rise* in the price/dividend ratio. The asymmetry parameter a measures the frequency of bear states over time, which is $1-\Phi(\frac{a}{\sigma_u})$. Since $a<0$, we conclude $1-\Phi(\frac{a}{\sigma_u}) > 50\%$ (Φ is a cumulative standard normal). When $a<0$ the tail on the bull side must compensate for the higher frequency on the bear side hence *the bull distribution must be more skewed than the bear distribution.* To examine asymmetry we return to the two asymmetric economies we considered. In Economy 1 bear states are more frequent while in Economy 2 bull states are more frequent.

5.2 Which asymmetry?

In Table 3 we test for asymmetry with a simple experiment. We fix $b = -9.00$, $a = -0.40$ and pick three random pairs of values $\lambda_g^x < 0, \lambda_g^z > 0$ for Economy 1. We then simulate the equilibria for these pairs and for the negative values of the same pairs, defining Economy 2.

The results for $\lambda_g^x > 0, \lambda_g^z < 0$, on the left of Table 3, are counter-factual. Asymmetry according to which bull states are more frequent imply too low Price/Dividend ratio and too high risky and riskless rates. These *qualitative* conclusions remain the same for all $a < 0, b < 0$ and $\lambda_g^x < 0, \lambda_g^z > 0$ *for which this experiment is feasible.* Hence we reject the hypothesis that bull states are more frequent. However, we now give an intuitive explanation for the higher Price/Dividend ratio and the lower returns on the right hand side of Table 3.

Table 3. Which asymmetry? (all moments are annualized)

Economy 2				Economy 1		
$\lambda_g^x = 0.015$	$\lambda_g^x = 0.012$	$\lambda_g^x = 0.012$		$\lambda_g^x = -0.012$	$\lambda_g^x = -0.012$	$\lambda_g^x = -0.015$
$\lambda_g^z = -0.100$	$\lambda_g^z = -0.120$	$\lambda_g^z = -0.140$		$\lambda_g^z = 0.140$	$\lambda_g^z = 0.120$	$\lambda_g^z = 0.100$
7.55	8.20	6.83	q^s	21.96	22.10	24.00
1.55	1.33	1.15	σ_{q^s}	2.27	2.37	3.26
17.34%	15.50%	18.25%	R	6.87%	6.87%	6.74%
19.13%	15.24%	16.58%	σ_R	9.28%	9.59%	11.17%
19.42%	18.14%	20.86%	r	5.53%	5.36%	4.50%
26.00%	22.35%	26.22%	σ_r	5.19%	5.84%	8.18%
−2.08%	−2.63%	−2.61%	e_p	1.34%	1.51%	2.24%
0.19	0.24	0.23	$\rho(x, R)$	0.42	0.41	0.33
−0.11	−0.17	−0.16	Shrp	0.14	0.16	0.19

Agents in Economy 1 are in bear states at a majority of dates. Hence, at more than 50% of the time they do not expect *to make excess stock returns* in the next date. They expect dividends to grow slower than normal and their stock portfolio to produce lower than normal returns. The question is: what is the resulting "normal"equilibrium Price/Dividend ratio? An answer needs to distinguish between the price of the stock and the Price/Dividend ratio q^s. The non - normalized stock Euler equation is $(C_t^j)^{-\gamma}\tilde{q}_t^s = \beta E_{Q_t{}^j}[(C_{t+1}^j)^{-\gamma}(D_{t+1} + \tilde{q}_{t+1}^s)]$. Hence, if an agent is bullish about future dividend growth his demand for the stock increases and if most agents are optimistic, the stock price rises. If a majority is bearish about future dividend growth the price declines. The normalized Euler equation (4a) $(c_t^j)^{-\gamma}q_t^s = \beta E_{Q_t^j}[(c_{t+1}^j)^{-\gamma}(1 + q_{t+1}^s)e^{(1-\gamma)x_{t+1}}]$ shows the situation is different for q^s. Suppose an agents' perceived conditional distribution of x_{t+1} shifts downward. The effect on equilibrium q^s depends upon the elasticity of substitution between c_t and c_{t+1} which is $-\frac{1}{\gamma}$. Being pessimistic about returns is the same as considering present consumption as cheaper. Since γ = 2, for a pessimistic agent a 1% decreased relative cost of today's consumption leads to a 0.5% increase in today's consumption. Equivalently, in an economy where agents are, on average, more frequently bearish about x_{t+1} the time average of q^s is higher. This implies that in an economy where agents are, on average, bearish more than 50% of the time the mean price/dividend ratio is higher and the mean rate of return is lower than in an economy where agents are, on average, bullish more than 50% of the time. This is exactly what happens in Table 3.

We do not have direct evidence to support the conclusion that the frequency of bear states is higher than 50%. It is, however, supported by the fact that in the long run most above normal stock returns are made over *relatively small proportion of time* when asset prices rally strongly (see Shilling, 1992). The empirical frequencies show that agents experience large excess returns only a small proportion of time. Hence, on average, the proportion of time that one may expect to make excess returns is much less than 50%.[9] Additional indirect support comes from the psychological literature which suggests agents place heavier weight on losses than on gains. Under our interpretation this is indeed the case at majority of dates since on those dates agents believe abnormally lower return are more likely than abnormally higher. By the RB principle the higher frequency of bear states implies that when in bull states, an agent's intensity of optimism is higher than the intensity of pessimism. Hence, the average size of the positive tail in the belief densities is bigger than the average size of the negative tail. This has useful implications to the appearance of bubbles in an RBE. It is a fact that there is strong positive correlation in beliefs among agents. Since investors in optimistic states expect to make abnormal returns, the correlation among them generates correlated demand and price movements which look much like bubbles.

5.3 Matching the moments

We now combine the effects of intensity and asymmetry to exhibit a region of parameters where the simulated moments are close to the empirical record. Table 4 provides a summary. The table shows that for values of b around $(-14.0, -15.0)$[10] and a around $(-0.15, -0.25)$ all moments are close to those observed in the market. Most significant is the fact that around this region of the parameter space all model statistics match *simultaneously* the moments and premium in the empirical record. We show later that the model with these parameter values exhibit dynamic properties such as *forecastability of returns* and *stochastic volatility* which are similar to those observed in the U.S. data.

The fact our model matches simultaneously the moments and other phenomena associated with market volatility (see below, Section 5) provides strong support

[9] Shilling (1992) shows that during the 552 months from January 1946 through December of 1991 the mean real annual total return on the Dow Jones Industrials was 6.7%. However, if an investor missed the 50 strongest months the real mean annual return over the other 502 months was -0.8%. Hence the financial motivation to time the market is very strong, as is the case with the agents in our model.

[10] One might think that if b is in the range of $(-14, -15)$ the function $\phi(g^j)$ in (13) would take only values of 0 and 1. This is not the case due to the persistence in beliefs. b actually regulates *the speed* in which an agent moves from states where ϕ is close to 1 or ϕ is close to 0, ending up spending most of his time in transition between these extremes. Also, even when ϕ is close to extreme values, the agent is still uncertain since his transition functions (11a)–(11c) contain the white noise terms $\tilde{\rho}^j_{t+1}$.

Table 4. The combined effect of intensity and asymmetry (all moments are annualized)

	$b, a \Rightarrow$ $\Downarrow$	-0.15	-0.20	-0.25	
q^s		24.44	26.13	27.99	
σ_{q^s}		5.21	5.51	5.84	
R		7.69%	7.37%	7.07%	
σ_R		18.47%	18.24%	18.01%	
r	-14.0	2.73%	2.53%	2.32%	
σ_r		4.84%	5.00%	5.15%	
e_p		4.96%	4.84%	4.75%	
$\rho(x, R)$		0.22	0.22	0.22	
Shrp		0.27	0.27	0.26	Record
q^s		24.20	25.91	27.78	25.00
σ_{q^s}		5.21	5.51	5.83	7.10
R		7.79%	7.46%	7.14%	7.00%
σ_R		18.80%	18.53%	18.27%	18.00%
r	-14.5	2.07%	1.88%	1.67%	1.00%
σ_r		4.91%	5.11%	5.30%	5.70%
e_p		5.73%	5.58%	5.47%	6.00%
$\rho(x, R)$		0.21	0.22	0.22	0.10
Shrp		0.30	0.30	0.30	0.33
q^s		23.84	25.54	27.41	
σ_{q^s}		5.17	5.46	5.78	
R		7.92%	7.57%	7.23%	
σ_R		19.12%	18.81%	18.51%	
r	-15.0	1.26%	1.08%	0.88%	
σ_r		5.22%	5.44%	5.68%	
e_p		6.66%	6.49%	6.35%	
$\rho(x, R)$		0.21	0.21	0.22	
Shrp		0.35	0.34	0.34	

for the theory. Indeed, a property of simultaneous explanation of diverse phenomena *by a single model* rather than a specialized model for each phenomenon, is a crucial property any good theory of market volatility must have. However, our deeper conclusion is the results in Table 4 are due to two key factors: intensity of fat tails and asymmetry. We noted the evidence regarding asymmetry. We add the well documented fact that the distribution of asset returns exhibit fat tails (e.g. see Fame, 1965; Shiller, 1981). It is natural to ask where these tails come from. Our theory proposes that these fat tails in returns come from fat tails in the probability models of agents' beliefs.

We do not propose that the exact values of a or b have particular significance. Indeed, since $\lambda_g^x < 0, \lambda_g^z > 0$ multiply $\eta_{t+1}^j(g_t^j)$ they also measure intensity, there is some substitution between $(\lambda_g^x, \lambda_g^z)$ and b. Hence, *there is a manifold of parameter* *values for which the model simulations would approximate the empirical record.* We used specific values for $\lambda_g^x < 0, \lambda_g^z > 0$ and then found values of (a, b) to match the data. Other parameter configurations would arrive at similar results but these economies are qualitatively similar, leading to common qualitative results. The theoretical conclusion are general for all models that match the data and consist of three parts:

1. *Asset pricing is a non stationary process reflecting dynamic changes in our economy. The true underlying process is not known, giving rise to a wide diversity of beliefs about profitability of investments. This diversity is the main force for propagating volatility.*
2. *The first factor of volatility is the high intensity of fat tails of the agents' conditional densities: it is the crucial force which generates volatility and low riskless rate.*
3. *The second component of market volatility is asymmetry in the belief densities giving rise to markets in which the frequency of bear states is higher than 50%.*

We also add that we arrived at our conclusions without specifying the formal subjective belief formation models of the agents. The RB rationality principle is central in two ways. First, it provides the restrictions on the belief parameters. Second, it implies that asymmetry in frequency of bear or bull states is compensated by asymmetry in the size of the positive and negative fat tails of the belief densities. These factors turned out to be important for the way in which our model works. The available empirical evidence supports this asymmetry.

5.4 Why does the RBE resolve the equity premium puzzle?

Risk premia are compensations for risk perception by risk averse agents. In most single agent models, the volatility of aggregate consumption is exogenously set. In such models the market portfolio is identified with a security whose payoff is aggregate consumption. The Equity Premium Puzzle is an observation that the small volatility of aggregate consumption growth cannot justify a large equity premium. Our theory of risk premia takes a very different approach.

Heterogenous beliefs cause diverse individual consumption growth rates even if aggregate consumption is exogenous, which is the case in our model. Hence, individual consumption growth rates need not equal the aggregate rate. Since the agents' beliefs are as essential to them as the stochastic aggregate growth rate, they do not seek to own a portfolio whose payoff is aggregate consumption. Moreover, *they disagree on the riskiness of this hypothetical asset*. As a result, we do not focus on the relation between asset returns and aggregate consumption growth but instead, on the relation between asset returns and the volatility of individual consumption growth rates. We can thus sum up the factors contributing to the formation of risk premia in Table 4.

(i) *Low riskless rate.* We have already seen that low riskless rate is a direct consequence of the high volatility hence riskiness of the RBE. This added riskiness is called "Endogenous Uncertainty."

(ii) *Higher volatility of individual consumption growth rates and correlation with x which is less than 1*. It is often argued that the Equity Premium Puzzle arises because the model's consumption growth is not sufficiently volatile. Since this puzzle does not arise in our model, the question is then how volatile do individual consumption growth rates need to be in order to generate an equity premium of 6% and a riskless rate of 1%? The answer is: *not very much*. When $a = -0.20$ and $b = -15.00$ as in Table 4 we find that although $\sigma_x = 0.03256$, the standard deviation of individual consumption growth rates is only 0.039 (i.e. 3.9%) and the correlation between individual consumption growth rate and x is only 0.83 (compared to 1.00 in a representative household model). Both figures are compatible with survey data showing individual consumption growth are more volatile than the aggregate.

6 Characteristics of volatility II: predictability of returns and stochastic volatility

We turn now to other dimensions of asset price dynamics, aiming to compare predictions of our theory with the empirical evidence. We study the predictability of stock returns and stochastic volatility, or GARCH, properties of stock prices and returns. Results reported were computed for a sample of 20,000 data points generated by Monte Carlo simulation of the model with $a = -0.20$ and $b = -15.00$ in Table 4.

6.1 Predictability of stock returns

The problem of predictability of risky returns generated an extensive literature in empirical finance (e.g. Fame and French, 1988a, 1998b; Poterba and Summers, 1988; Campbell and Shiller, 1988; Paye and Timmermann, 2003). This debate is contrasted with the simple theoretical observation that under risk aversion asset prices and returns are not martingales, hence they contain a predictable component. It appears the disagreement is not about the empirical record but about the interpretation of the record and about the stability of the estimated forecasting models. Here we focus only on the empirical record.

We examine the following: (i) Variance Ratio statistic; (ii) autocorrelation of returns and of price/dividend ratios; (iii) regressions of cumulative returns, and (iv) the predictive power of the dividend yield. We first introduce notation. Let $\varrho_t = log[\frac{(q_t^s+1)e^{x_t}}{q_{t-1}^s}]$ be the log of gross one year stock return, $\varrho_t^k = \sum_{i=0}^{k-1} \varrho_{t-i}$ be the cumulative log-return of length k from $t-k+1$ to t, and $\varrho_{t+k}^k = \sum_{j=1}^{k} \varrho_{t+j}$ be the cumulative log-return over a k-year horizon from $t+1$ to $t+k$.

6.1.1 Variance ratio test

Let the variance-ratio be $VR(k) = \frac{var(\varrho_t^k)}{(kvar(\varrho_t))}$. As k rises it converges to one if returns are uncorrelated. However, if returns are negatively autocorrelated at some lags, the ratio is less than one. Our results show there exists a significant higher order autocorrelation in stock returns hence there is a long run predictability which is consistent with U.S. data on stock returns, as reported in Poterba and Summers

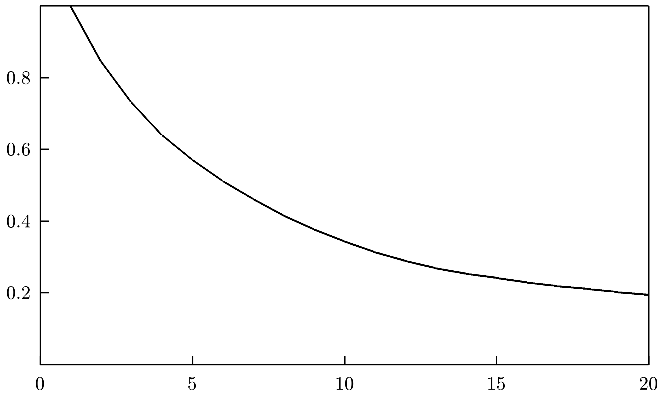

Figure 3. Variance ratios

Table 5. Variance ratios for NYSE 1926–1985

k	1	2	3	4	5	6	7	8	9	10
$VR(k)$	1.00	0.85	0.73	0.64	0.57	0.51	0.46	0.41	0.38	0.34
U.S.	1.00	0.96	0.84	0.75	0.64	0.52	0.40	0.35	—	—

(1988). In Figure 3 we present a plot of the variance ratios computed from our model. For $k > 1$ the ratio is less than 1 and declines with k.

In Table 5 we report the computed values of the ratios for $k = 1, 2, \ldots, 10$ and compare them with ratios computed for U.S. stocks by Poterba and Summers ((1988), Table 2, line 3) for $k = 1, 2, \ldots, 8$. Our model's prediction is very close to the U.S. empirical record.

6.1.2 The autocorrelation of log-returns and price-dividend ratios

In Table 6 we report the autocorrelation function of log annual returns. Our model predicts negatively autocorrelated returns at all lags. This implies a long horizon mean reversion of the kind documented by Poterba and Summers (1988), Fame and French (1998a) and Campbell and Shiller (1988). Thus, apart from the very short returns which exhibit positive autocorrelation, the model reproduces the empirical record reasonably well.

Table 6. Autocorrelation of log-returns

corr$(\varrho_t, \varrho_{t-i})$	Model	Empirical record
$i = 1$	−0.154	0.070
$i = 2$	−0.094	−0.170
$i = 3$	−0.069	−0.050
$i = 4$	−0.035	−0.110
$i = 5$	−0.040	−0.040

In Table 7 we report the autocorrelation function of the price-dividend ratio. The table shows the model generates a highly autocorrelated price/dividend ratio which matches reasonably well the behavior observed in the U.S. stock market data. The empirical record in Tables 6 and 7 is for NYSE data covering the period 1926–1995 as reported in Barberis et al. (2001).

Table 7. Autocorrelation of price-dividend ratio

corr(q_t^s, q_{t-i}^s)	Model	Empirical record
$i=1$	0.695	0.700
$i=2$	0.485	0.500
$i=3$	0.336	0.450
$i=4$	0.232	0.430
$i=5$	0.149	0.400

6.1.3 Mean reversion of log-returns

Mean reversion of stock returns was studied under several models. We compare our results with the results of Fama and French (1988a). Thus, consider the regression model

$$\varrho_{t+k}^k = \alpha_k + \delta_k \varrho_t^k + \upsilon_{t,k}. \tag{17}$$

The evidence suggests that stock prices have a random-walk and a stationary component (see Fama and French, 1988a), depending upon the horizon k. If there was no stationary component the slopes δ_k in (17) would be 0 for all k. If there was no random-walk component the slopes would approach -0.5 for large values of k. When prices have both a random walk and a slowly decaying stationary components, the slopes δ_k in (17) should form a U-shaped pattern, starting around 0 for short horizons, becoming more negative as k increases, and then moving back to 0 as the random walk component begins to dominate at very long horizons. Our model produces exactly this pattern which is reported in Figure 4.

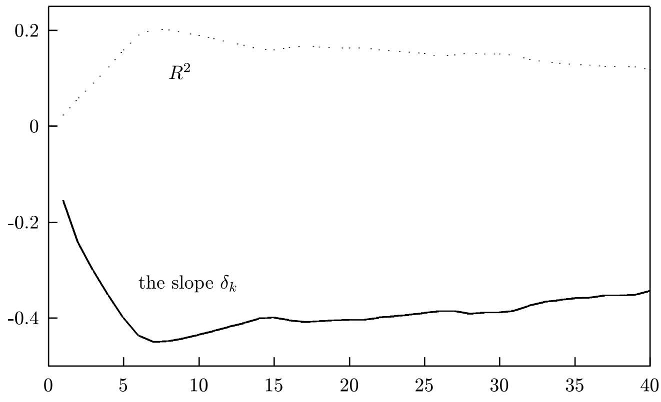

Figure 4. Properties of returns from regression (17)

6.1.4 Dividend yield as a predictor of future stock returns

The papers cited above show that the price/dividend ratio is the best explanatory variable of long returns. To test this fact in our model we consider the following regression model

$$\varrho_{t+k}^{k} = \zeta_k + \chi_k(\mathrm{D}_t/\tilde{q}_{t-1}^{s}) + \vartheta_{t,k}. \tag{18}$$

Fama and French (1988b) report that the ability of the dividend yield to forecast stock returns, measured by regression coefficient R^2 of (18), increases with the return horizon. We find that our model captures the main features of the empirical evidence and we report it in Table 8.

Table 8. The behavior of the regression slopes in (18)

Time horizon	Model		Empirical record	
k	χ_k	R^2	χ_k	R^2
1	5.03	0.08	5.32	0.07
2	8.66	0.14	9.08	0.11
3	11.16	0.18	11.73	0.15
4	13.10	0.21	13.44	0.17

To conclude the discussion of predictability, we observe that the empirical evidence reported by Fama and French (1998a, b) Campbell and Shiller (1988), Poterba and Summers (1998), and others is consistent with asset price theories in which time-varying expected returns generate predictable, mean-reverting components of prices (see Summers, 1986). The important question left unresolved by these papers is what drives the predictability of returns implied by such mean-reverting components of prices? Part of the answer is the persistence of the dividend growth rate via the equilibrium map (6). Our theory offers a second and stronger persistent mechanism which is also seen in (6). It shows these results are primarily driven by the dynamics of market state of beliefs which exhibit correlation across agents and persistence over time. Agents go through bull and bear states causing their perception of risk to change and expected returns to vary over time. Equilibrium asset prices depend upon states of belief which then exhibit memory and mean reversion. Hence both prices and returns exhibit these same properties.

6.2 GARCH behavior of the price-dividend ratio and of the risky returns

Stochastic volatility in asset prices and returns is well documented (e.g. Bollerslev, Engle and Nelson, 1994; Brock and LeBaron, 1996). In partial equilibrium finance it is virtually standard to model asset prices by stochastic differential equations, *assuming* an exogenously driven stochastic volatility. But where does stochastic volatility come from? Dividends certainly do not exhibit stochastic volatility. One of the most important implication of our theory is that *it explains why asset prices and returns exhibit stochastic volatility*. We start by presenting in Figures 5 and 6 the results of simulated 500 observations: in Figure 5 we report price/dividend

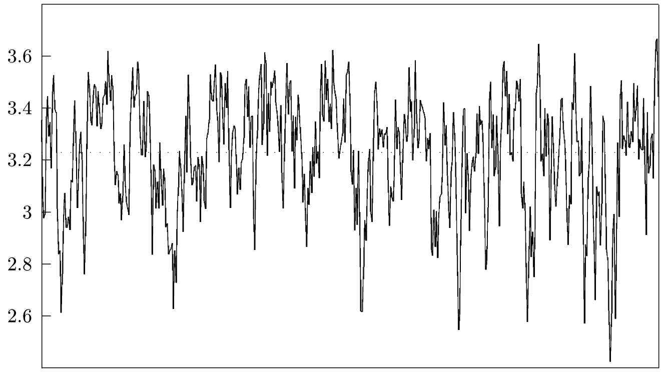

Figure 5. Log of price-divided ratios

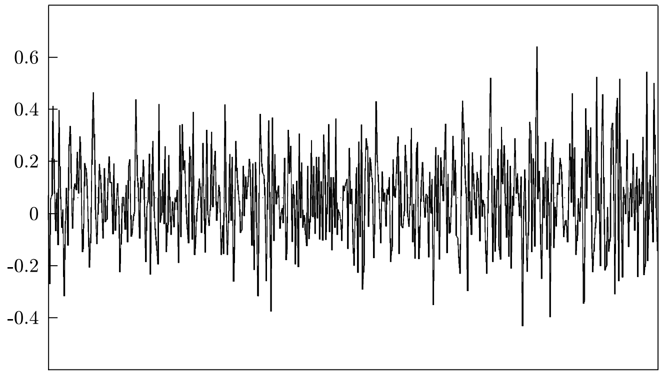

Figure 6. Log of risky rates of return

ratios and in Figure 6 the associated risky rates of return. These figures reveal time varying volatility reflected in time variability of the variance of prices and returns. However, GARCH behavior is more subtle than just volatility clustering; it requires volatility to be persistent and this requires a formal test.

To formally test the GARCH property of the price/dividend ratio and of the risky returns we used the 20,000 simulated observations discussed in the previous section. Using these data we estimated the following econometric model of the dynamics of the log of the price/dividend ratio

$$\begin{aligned} &\log(q_t^s) = \kappa^q + \mu^q \log(q_{t-1}^s) + \varsigma_t^q \\ &\varsigma_t^q \sim N(0, h_t^q) \\ &h_t^q = \xi_0^q + \xi_1^q (\varsigma_{t-1}^q)^2 + \nu_1^q h_{t-1}^q. \end{aligned} \tag{19}$$

Since the price dividend ratio is postulated to be an AR(1) process, the process in (19) is GARCH(1,1). Similarly, for the risky rates of return we postulated the econometric model

$$\begin{aligned} &\varrho_t^s = \kappa^\varrho + \mu^\varrho \log(q_t^s) + \varsigma_t^\varrho \\ &\varsigma_t^\varrho \sim N(0, h_t^\varrho) \\ &h_t^\varrho = \xi_0^\varrho + \xi_1^\varrho (\varsigma_{t-1}^\varrho)^2 + \nu_1^\varrho h_{t-1}^\varrho. \end{aligned} \tag{20}$$

For a specification of (19) and (20) we have also tested ARCH(1) and GARCH (2,1) but have concluded that the proposed GARCH(1,1) as in (19)–(20), describes best the behavior of the data over time. Due to the large sample we ignore standard error and report our model implies that the estimated model for the log of the price-dividend ratio satisfies the GARCH(1,1) specification

$$\begin{aligned}
&\log(q_t^s) = 0.99001 + 0.69384 \log(q_{t-1}^s) + \varsigma^{q_t} \\
&\varsigma_t^q \sim N(0, h_t^q) \\
&h_t^q = 0.00592 + 0.02370(\varsigma_{t-1}^q)^2 + 0.73920 h_{t-1}^q, \quad R^2 = 0.481.
\end{aligned}$$

For the risky rates of return the estimated model satisfies the GARCH(1,1) specification

$$\begin{aligned}
&\varrho_t = 1.13561 - 0.33355 \log(q_t^s) + \varsigma_t^\varrho \\
&\varsigma_t^\varrho \sim N(0, h_t^\varrho) \\
&h_t^\varrho = 0.00505 + 0.01714(\varsigma_{t-1}^\varrho)^2 + 0.77596 h_{t-1}^q, \quad R^2 = 0.180.
\end{aligned}$$

Stochastic volatility in our model is a direct consequence of the dynamics of the market beliefs $z_t = (z_t^1, z_t^2)$. It is clear that persistence of beliefs and correlation across agents introduce similar patterns into prices and returns. When agents disagree (i.e $z_t^1 z_t^2 < 0$) they offset the demands of each other and as that pattern persists, prices do not need to change by very much for markets to clear. During such periods prices exhibit low volatility hence persistence of belief states induce persistence of low volatility. When agents agree (i.e $z_t^1 z_t^2 > 0$) they compete for the same assets and prices are determined by difference in belief levels. Changes in the levels of bull or bear states (i.e. values of (g_t^1, g_t^2)) generate high volatility in asset prices and returns. Persistence of beliefs cause such high volatility regimes to exhibit persistence. Market volatility is then time dependent. It changes with the market state of beliefs and hence it has a predictable component as in (19)–(20). These results extend the earlier and similar result in Kurz and Motolese (2001).

The virtue of the above argument is that it explains stochastic volatility *as an endogenous consequence of equilibrium dynamics*. Some "fundamental" shocks (i.e. an oil shock) surely cause market volatility, but it has been empirically established that market volatility cannot be explained *consistently* by repeated "fundamental" exogenous shocks (see Pesaran and Timmermann, 1995). Our explanation of stochastic volatility is thus consistent with the empirical evidence.

7 Concluding remarks

This paper presents a unified paradigm which proposes that market volatility is driven primarily by market expectations. This conclusion extends our previous work (cited earlier) on the subject. The central new development is the formal introduction of the state of belief as a key tool of General Equilibrium analysis and a corresponding requirement for agents to forecast the market state of belief. Agents forecasting the state of belief of "others" is a precise mathematical structure which embodies the Keynesian intuitive "Beauty Contest" aspect of asset pricing. On a deeper level diverse individual forecasts of future market states of belief is equivalent to heterogenous forecasts of future asset prices. Hence diverse price forecasting is the essence of our theory. In a general equilibrium context the tool of a *market state of belief* is a formal method for allowing agents to be rational yet make heterogenous price forecasts. Given the general structure of equilibria with diverse and time dependent beliefs, we introduced the RB rationality principle and explored the restrictions it imposes on the beliefs of the agents in the economy.

We have then constructed a simple model and tested it on all aspects of market volatility and found the model to match well the empirical record. The structure of the model is applicable to any phenomenon associated with market volatility since our main results are qualitative rather than quantitative. The volatility results are driven by two crucial characteristics of the distribution of market beliefs. These are (i) over confidence of an agent's forecasts described by amplification of the agent's probabilities which, in turn, generate densities with fat tails, and (ii) asymmetry in the frequency of bull or bear states.

Appendixes

Appendix A: Construction of the random variables $\eta^j_{t+1}(u^j_t)$

In the text we defined $\phi(u^j_t) = 1/(1+e^{b(u^j_t - a)})$. Since computations use standard normal variables we normalize *u* as follows

$$\phi(u^j_t) = \frac{1}{1+\mathrm{e}^{b(u^j_t - a)}} = \phi(\tilde{u}^j_t) = \frac{1}{1+\mathrm{e}^{\bar{b}(\tilde{u}^j_t - \bar{a})}}$$
$$\text{where } \bar{b} = b\sigma_{u^j_t},\ \bar{a} = a/\sigma_{u^j_t} \text{ and } \tilde{u}^j_t \sim N(0,1). \tag{A1}$$

A.1 The density function

The variables $\eta^j_{t+1}(\tilde{u}^j_t)$ are now defined by specifying their density, conditional on $\tilde{u}^j_t$:

$$p(\eta^j_{t+1}|\tilde{u}^j_t) = \begin{cases} \phi_1(\tilde{u}^j_t) f(\eta^j_{t+1}) & \text{if} \quad \eta^j_{t+1} \geq 0 \\ \phi_2(\tilde{u}^j_t) f(\eta^j_{t+1}) & \text{if} \quad \eta^j_{t+1} < 0 \end{cases}$$

where $f(\eta^j_{t+1}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(\eta^j_{t+1})^2}{2}}$ is the standard normal density function and

$$\phi_1(\tilde{u}^j_t) = \frac{\phi(\tilde{u}^j_t)}{G},\ \phi_2(\tilde{u}^j_t) = 2 - \phi_1(\tilde{u}^j_t),\ G = \int_{-\infty}^{\infty} \phi(\tilde{u}^j_t) \frac{1}{\sqrt{2\pi}} e^{\frac{(\tilde{u}^j_t)^2}{2}} d\tilde{u}^j_t.$$

Note that $\mathrm{E}\phi_1(\tilde{u}^j_t) = \mathrm{E}\phi_2(\tilde{u}^j_t) = 1$, which ensures that the rationality conditions be satisfied, i.e. let $\varphi(\tilde{u})$ be the empirical density of $\tilde{u}$ then

$$\int_{-\infty}^{\infty} p(\eta^j_{t+1}|\tilde{u}^j_t)\varphi(\tilde{u}^j_t) d\tilde{u}^j_t = f(\eta^j_{t+1}). \tag{A2}$$

We need another condition to make sure that $p(\eta^j_{t+1}|\tilde{u}^j_t)$ is a density function for all $\tilde{u}^j_t$. This requires $\phi_2(\tilde{u}^j_t) \geq 0$ for all $\tilde{u}^j_t$, which implies $\mathrm{G} \geq 1/2$.

A.2 The moments of $\eta^j_{t+1}(\tilde{u}^j_t)$

A direct computations of the moments of this random variable leads to

$$\mathrm{E}[\eta^j_{t+1}|\tilde{u}^j_t] = \frac{\phi_1(\tilde{u}^j_t)}{\sqrt{2\pi}}\int_0^\infty x\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x + \frac{\phi_2(\tilde{u}^j_t)}{\sqrt{2\pi}}\int_{-\infty}^0 x\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x$$
$$= \left(\frac{\phi(\tilde{u}^j_t)}{G} - 1\right)\frac{2}{\sqrt{2\pi}} \tag{A3}$$
$$\mathrm{E}[(\eta^j_{t+1})^2|\tilde{u}^j_t] = \frac{\phi_1(\tilde{u}^j_t)}{\sqrt{2\pi}}\int_0^\infty x^2 e^{\frac{-x^2}{2}}dx + \frac{\phi_2(\tilde{u}^j_t)}{\sqrt{2\pi}}\int_{-\infty}^0 x^2 e^{\frac{-x^2}{2}}dx$$
$$= \phi_1(\tilde{u}^j_t)0.5 + \phi_2(\tilde{u}^j_t)0.5 = 1. \tag{A4}$$

Since $\mathrm{E}\phi_1 = \mathrm{E}\phi_2 = 1$ and $\mathrm{E}\tilde{u}^j_t = 0$ we have $\mathrm{E}[\eta^j_{t+1}] = 0, \mathrm{E}[(\eta^j_{t+1})^2] = 1$ and $\mathrm{E}[\eta^j_{t+1}\tilde{u}^j_t] = \frac{2}{\sqrt{2\pi}}\frac{\mathrm{E}[\phi(\tilde{u}^j_t)\tilde{u}^j_t]}{G}$.

Note that, given (A1) and the parameter choice $b < 0$, $E[\eta^j_{t+1}|\tilde{u}^j_t]$ defined in (A3) is a monotone increasing function in $\tilde{u}^j_t$. It is straightforward to show that $\phi(\tilde{u}^j_t) > G$ for any $\tilde{u}^j_t > a$ and $\phi(\tilde{u}^j_t) < G$ for any $\tilde{u}^j_t < a$ hence $E[\eta^j_{t+1}|\tilde{u}^j_t] > 0$ if $\tilde{u}^j_t > a$ and $E[\eta^j_{t+1}|\tilde{u}^j_t] < 0$ if $\tilde{u}^j_t < a$.

We now reformulate the random variable $\eta^j_{t+1}(u^j_t)$ as used in the computational model. Let

$$\mu(\tilde{u}^j_t) \equiv E[\eta^j_{t+1}|\tilde{u}^j_t],$$
$$s(\tilde{u}^j_t) \equiv \sqrt{E[(\eta^j_{t+1})^2|\tilde{u}^j_t] - (E[\eta^j_{t+1}|\tilde{u}^j_t])^2}, \nu^j_{t+1} \equiv \frac{\eta^j_{t+1} - \mu(\tilde{u}^j_t)}{s(\tilde{u}^j_t)}$$

and define

$$\hat{\eta}^j_{t+1}(\tilde{u}^j_t) = \mu(\tilde{u}^j_t) + s(\tilde{u}^j_t)\nu^j_{t+1}. \tag{A5}$$

Appendix B: Statement of the rationality conditions

The rationality of belief principle requires that

$$\Psi_{t+1}(u^j_t) = \begin{pmatrix} \lambda^x_g \eta^j_{t+1}(u^j_t) + \tilde{\rho}^{x^j}_{t+1} \\ \lambda^z_g \eta^j_{t+1}(u^j_t) + \tilde{\rho}^{z^{j1}}_{t+1} \\ \lambda^z_g \eta^j_{t+1}(u^j_t) + \tilde{\rho}^{z^{j2}}_{t+1} \end{pmatrix} \text{ has the same joint empirical distribution as}$$
$$\rho_{t+1} = \begin{pmatrix} \rho^x_{t+1} \\ \rho^{z^1}_{t+1} \\ \rho^{z^2}_{t+1} \end{pmatrix} \tag{B1}$$

when $\left\{u^j_t, t = 1, 2, \ldots\right\}$ is considered part of the variability of the term on the left.

To understand (B1) keep in mind the consistency conditions $g_t^j = z_t^j$ for all t. These are macroeconomic consistency condition (like market clearing conditions) but they do not hold in the agent's perception model who treats the z_t^j as exogenous variables. In the agent's perception model there is nothing to require the covariance between g^j and any state variable to be the same as the covariance implied by (7a)–(7c) between z^j and that variable. The presence of $\eta_{t+1}^j(u_t^j)$ in all equations of (11a)–(11c) generates perceived covariance between g^j and observed state variables that may not be in (7a)–(7c). Any covariance between an agent's own g^j and other variables in the economy *are strictly in the mind of the agent and no rationality conditions are imposed on them.*

We show first that (B1) fully pins down the covariance matrix $\Omega_{\rho\rho}^j$ of the three dimensional vector $\tilde{\rho}_{t+1}^j$ in (11a)–(11d). We write $\Omega_{\rho\rho}^j = \Omega_{\rho\rho}$ all j because of symmetry. To directly demonstrate why $\Omega_{\rho\rho}$ is pinned down by (B1), use it to rewrite (9a) in the form

$$w_{t+1}^j = Aw_t + \lambda_g \eta_{t+1}^j(u_t^j) + \tilde{\rho}_{t+1}^j, \quad \lambda_g = (\lambda_g^x, \lambda_g^z, \lambda_g^z)', \quad \text{and } (a,b) \text{ given.} \tag{B2}$$

Now define $\sigma_\eta^2 = E[(\eta_{t+1}^j(u_t^j))^2]$ and recall that V is the covariance matrix of w_t according to the empirical distribution defined in (9). Computing the covariance matrix in (B2) and equating the computed value to V leads to the equality $V = AVA' + \lambda_g(\lambda_g)'\sigma_\eta^2 + \Omega_{\rho\rho}$ which means that

$$\Omega_{\rho\rho} = V - AVA' - \lambda_g(\lambda_g)'\sigma_\eta^2. \tag{B3}$$

For any (a, b, λ_g), magnitudes on the right of (B3) are known and this pins down the matrix $\Omega_{\rho\rho}$.

The perception model of an agent includes the transition equation for g_{t+1}^j in (5) and this implies that the agent's model specifies a full joint distribution of *four* variables: three observables w_t and g_t^j. Hence, we specify a 4×4 covariance matrix Ω of the innovations which is written as.

$$\Omega = \begin{pmatrix} \Omega_{\rho\rho}, & \Omega_{wg} \\ \Omega_{wg}, & \sigma_g^2 \end{pmatrix}$$

where $\Omega_{wg} = [Cov(\tilde{\rho}_{t+1}^x, \tilde{\rho}_{t+1}^{g^j}), Cov(\tilde{\rho}_{t+1}^{z^1}, \tilde{\rho}_{t+1}^{g^j}), Cov(\tilde{\rho}_{t+1}^{z^2}, \tilde{\rho}_{t+1}^{g^j})]$ is a 3×1 covariance vector. In addition we now define the following

$r^j \equiv Cov(w, g^j)$ - unconditional covariance vector between g^j and the three observables w;
$d^j = (\lambda_x^{z^j}, 0, 0)$- vector of w parameters in the g^j equation (5).

By symmetry, both terms are the same for all j. To compute r we multiply (5) by the first three equations in (11a)–(11c) and compute the three equations to define the unconditional covariance as

$$r = \lambda_z A r + AVd + \lambda_z \lambda_g r_{\eta u} \sigma_u + \Omega_{wg}$$

hence the 3×1 covariance vector of the innovations of the agent's belief must satisfy

$$\Omega_{wg} = r - \lambda_z A r - AVd + \lambda_z \lambda_g r_{\eta u} \sigma_u. \tag{B4}$$

$r_{\eta u} = E[\frac{\eta_{t+1}(u)g}{\sigma_u}] = E[\frac{\eta_{t+1}(u)u}{\sigma_u}]$. By (B4) Ω_{wg} is pinned down when we specify $r \equiv Cov(w, g^j)$. A simpler approach is to treat (B4) as a system of 3 equations in the 6 unknowns (r, Ω_{xg}). For now we have only 3 restrictions in (B4). We now show that our theory provides three additional restrictions to determined (r, Ω_{xg}). To explore the added implied restrictions we first utilize the definition of anonymity:

Covariance implications of anonymity. *Anonymity* requires the idiosyncratic component of an agent's belief not to be correlated with market beliefs. It implies that in a symmetric equilibrium the unconditional correlation between an agent's state of belief and the belief of "others' is the same across agents. We translate this to require that

$$\Omega_{z^1 g^j} = Cov(\tilde{\rho}^{z^1}_{t+1}, \tilde{\rho}^{g^j}_{t+1}) = 0 \tag{B5a}$$

$$\Omega_{z^2 g^j} = Cov(\tilde{\rho}^{z^2}_{t+1}, \tilde{\rho}^{g^j}_{t+1}) = 0. \tag{B5b}$$

(B5a)–(B5b) restricts the vector r to satisfy $(\Omega_{wg})_2 = (\Omega_{wg})_3 = 0$.

We need one more restriction and we show that it is deduced from the *rationality condition requiring $\Psi_{t+1}(u^j_t)$ to be serially uncorrelated*. We claim that for $\Psi_{t+1}(u^j_t)$ to exhibit no serial correlation *it is sufficient that it is uncorrelated with date t public information*. To see why, recall $\Psi_{t+1}(u^j_t) = w_{t+1} - Aw_t$ and suppose $\Psi_{t+1}(u^j_t)$ is uncorrelated with observables up to date t. Hence

$$\begin{aligned} E[\Psi_{t+1}(u^j_t)\Psi_t(u^j_{t-1})] &= E[\Psi_{t+1}(u^j_t)(w_t - Aw_{t-1})] \\ &= E[\Psi_{t+1}(u^j_t)w_t] - E[\Psi_{t+1}(u^j_t)Aw_{t-1}] = 0. \end{aligned}$$

To ensure these conclusions hold we put restrictions on r which imply that u^j_t are uncorrelated with any w_{t-i}, for all $i \geq 0$. Recall first that by the definition of the filter u^j_t we have

$$u^j_t \equiv u(g^j_t) = g^j_t - r'V^{-1}w_t.$$

The requirement $Cov(u^j_t, w_t) = 0$ is a simple implication of the filter since $r^j_t \equiv Cov(w_t, g^j_t)$. To examine the requirement $Cov(u^j_{t+1}, w_t) = 0$, recall that the agent's model (11a)–(11d) specifies

$$w^j_{t+1} = Aw_t + \lambda_g \eta^j_{t+1}(u_t) + \tilde{\rho}^j_{t+1}.$$

Hence we have

$$\begin{aligned} u^j_{t+1} &\equiv u(g^j_{t+1}) = g^j_{t+1} - r'V^{-1}w^j_{t+1} \\ &= \lambda_z g^j_t + d'w_t + \tilde{\rho}^{g^j}_{t+1} - r'V^{-1}(Aw_t + \lambda_g \eta^j_{t+1}(u_t) + \tilde{\rho}^j_{t+1}). \end{aligned}$$

Consequently, the condition $Cov(u_{t+1}^{j}, w_t) \equiv E[u_{t+1}^{j} w_t'] = 0$ would be satisfied if

$$\lambda_z r' + d'V - r'V^{-1}AV = 0. \tag{B6}$$

Although (B6) is a system of 3 equations, we now show that (B6) is actually the last restriction implied by the rationality of belief conditions. To see this fact note that since V is an invertible matrix, the equations (B6) can be solved for the covariance vector r, implying

$$[A' - \lambda_z I]V^{-1}r' = d. \tag{B7}$$

We study only the case $\lambda_{z^1} = \lambda_{z^2} = \lambda_z$. In this case the matrix A' takes the following form

$$A' = \begin{pmatrix} \lambda_x, \lambda_x^z, \lambda_x^z \\ 0, \lambda_z, 0 \\ 0, 0, \lambda_z \end{pmatrix}$$

hence $[A' - \lambda_z I]$ is singular with the last two rows being zero. This is compatible with the fact that $d = (\lambda_x^z, 0, 0)$ hence, the system (B7) consists of only one restriction.

We can conclude that (B4) together with the conditions $(\Omega_{wg})_2 = (\Omega_{wg})_3 = 0$ and (B7) completely determine (r, Ω_{wg}). Finally, when r is known, $\tilde{\sigma}_{g^j}^2$ is pinned down as follows. Since we know that $\sigma_{u^j}^2 = \text{var}(g^j) - r'V^{-1}r$, we use the condition $\text{var}(g^j) = \text{var}(z^j)$ to compute $\tilde{\sigma}_{g^j}^2 = (1 - \lambda_z^2)\text{var}(g^j) - d'Vd - 2\lambda_z d'r$.

Appendix C: The computational model

This appendix explores several computational issues which have not been discussed in the text. We want to provide here a complete description of the computed equilibrium conditions.

C.1 Equilibrium set and indeterminacy in the riskless economy

To ensure the existence of bounded solutions and hence exclude explosive solutions which violate transversality we impose quadratic utility penalties on deviation of asset holdings away from steady state. The penalty functions are: $D_t^{1-\gamma} \frac{\tau_s}{2} (\theta_t^j - 0.5)^2$ for stock holdings and $D_t^{-1-\gamma} \frac{\tau_B}{2} (B_t^j)^2$ for bond holdings with $\tau_s = \tau_B = 0.005$. The penalty functions are then subtracted from utility.

Indeterminacy of the optimal portfolio allocation at the riskless steady state is also a *problem* for perturbation models with a number of financial assets greater or equal to two. This is due to the fact that financial assets exhibit the same rate of return in the riskless economy hence are perfect substitutes. Penalty functions ensure the steady state solution $\theta_t^j = 0.5$ and $B_t^j = 0$ is unique.

C.2 The revised optimization problem and the system of Euler equations

Given the above, the optimization problem of agent j is then reformulated as follows:

subject to:

$$\underset{(C^j,\theta^j,B^j)}{\text{Max}} E_{Q^j}\left[\sum_{t=0}^{\infty}\beta^t \frac{1}{1-\gamma}(C_t^j)^{1-\gamma} - D_t^{1-\gamma}\frac{\tau_s}{2}(\theta_t^j-0.5)^2 - D_t^{-1-\gamma}\frac{\tau_B}{2}(B_t^j)^2|H_t,g_t^j\right]$$

subject to:

$$C_t^j + \tilde{q}_t^s\theta_t^j + q_t^b B_t^j = \Lambda_t^j + (\tilde{q}_t^s + D_t)\theta_{t-1}^j + B_{t-1}^j$$

The Euler equations are:

$$\begin{aligned}
&(C_t^j)^{-\gamma}\tilde{q}_t^s + D_t^{1-\gamma}\tau_s(\theta_t^j-0.5) = \beta E_{Q_t^j}[(C_{t+1}^j)^{-\gamma}(\tilde{q}_{t+1}^s + D_{t+1})|H_t,g_t^j]\\
&(C_t^j)^{-\gamma}q_t^b + D_t^{-\gamma-1}\tau_B(B_t^j) = \beta E_{Q_t^j}[(C_{t+1}^j)^{-\gamma}|H_t,g_t^j].
\end{aligned}$$

After normalization the budget constraint becomes:

$$c_t^j + q_t^s\theta_t^j + q_t^b b_t^j = \omega + (q_t^s+1)\theta_{t-1}^j + b_{t-1}^j e^{-x_t}$$

and the Euler equations become:

$$\begin{aligned}
&(c_t^j)^{-\gamma}q_t^s + \tau_s(\theta_t^j-0.5) = \beta E_{Q_t^j}[(c_{t+1}^j)^{-\gamma}(1+q_{t+1}^s)e^{(1-\gamma)x_{t+1}}|H_t,g_t^j]\\
&(c_t^j)^{-\gamma}q_t^b + \tau_B(b_t^j) = \beta E_{Q_t^j}[(c_{t+1}^j)^{-\gamma}e^{-\gamma x_{t+1}}|H_t,g_t^j].
\end{aligned}$$

C.3 The riskless steady state

In the riskless steady state quantities are as follows:

$$b^{1*} = b^{2*} = 0, \theta^{1*} = \theta^{2*} = 0.5, c^{1*} = c^{2*} = \omega + 0.5.$$

And prices are:

$$q^s = \frac{\beta e^{x^*(1-\gamma)}}{1-\beta e^{x^*(1-\gamma)}}, q^b = \beta e^{-\gamma x^*}.$$

C.4 The perturbation structure

We specify here how we formulated the perturbation model. Let ε be the perturbation variable then the perturbation structure of the agent's perception model is as follows:

$$\begin{aligned}
x_{t+1}^j &= (1-\lambda_x)x^* + \lambda_x x_t + \lambda_g^x \hat{\eta}_{t+1}^j(\tilde{u}_t^j,\varepsilon) + \varepsilon\tilde{\rho}_{t+1}^{x^j}\\
z_{t+1}^{j1} &= \lambda_z z_t^1 + \lambda_x^z(x_t - x^*) + \lambda_g^z\hat{\eta}_{t+1}^j(\tilde{u}_t^j,\varepsilon) + \varepsilon\tilde{\rho}_{t+1}^{z^j 1}\\
z_{t+1}^{j2} &= \lambda_z z_t^2 + \lambda_x^z(x_t - x^*) + \lambda_g^z\hat{\eta}_{t+1}^j(\tilde{u}_t^j,\varepsilon) + \varepsilon\tilde{\rho}_{t+1}^{z^j 2}\\
g_{t+1}^j &= \lambda_z g_t^j + \lambda_x^z(x_t - x^*) + \varepsilon\tilde{\rho}_{t+1}^{g^j}
\end{aligned}$$

where $\hat{\eta}_{t+1}^j(\tilde{u}_t^j,\varepsilon) = \mu(\tilde{u}_t^j) + \varepsilon s(\tilde{u}_t^j)\nu_{t+1}^j$ as defined in (A3)–(A5) of Appendix A.

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